Kosaraju's Algorithm for Strongly Connected Components

Directed Graphs

SCC

https://cs.pomona.edu/classes/cs140/

Outline

Topics and Learning Objectives

- Review topological orderings
- Discuss strongly connected components
- Cover Kosaraju's Algorithm

Exercise

- Work through Kosaraju's Algorithm

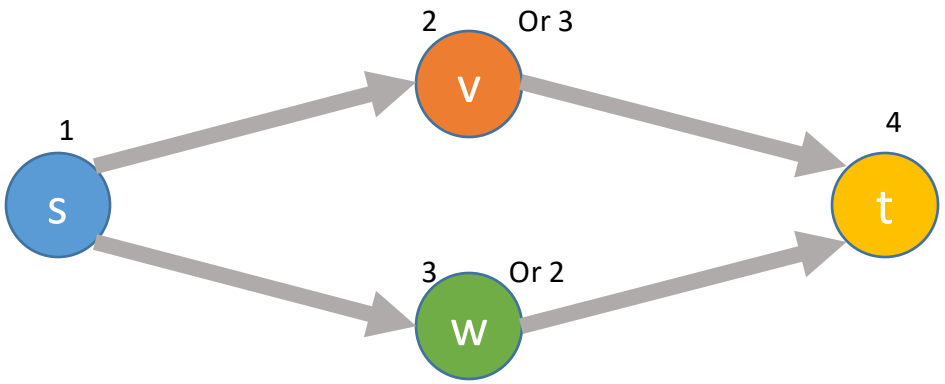
Extra Resources

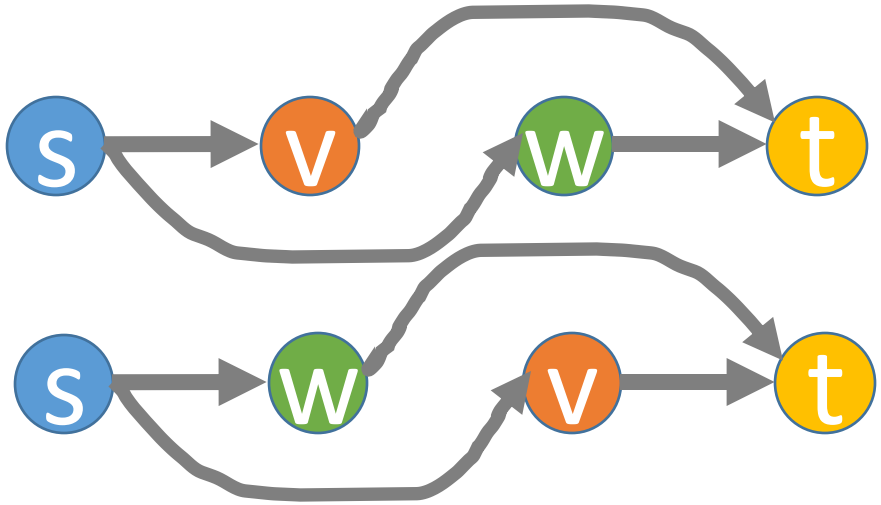
- Introduction to Algorithms, 3rd, chapter 22
- Algorithms Illuminated Part 2: Chapter 8

Topological Orderings

Definition: a topological ordering of a directed acyclic graph is a labelling f of the graph's vertices such that:

1. The f-values are of the set {1, 2, …, n}
2. For an edge (u, v) of G, f(u) < f(v)



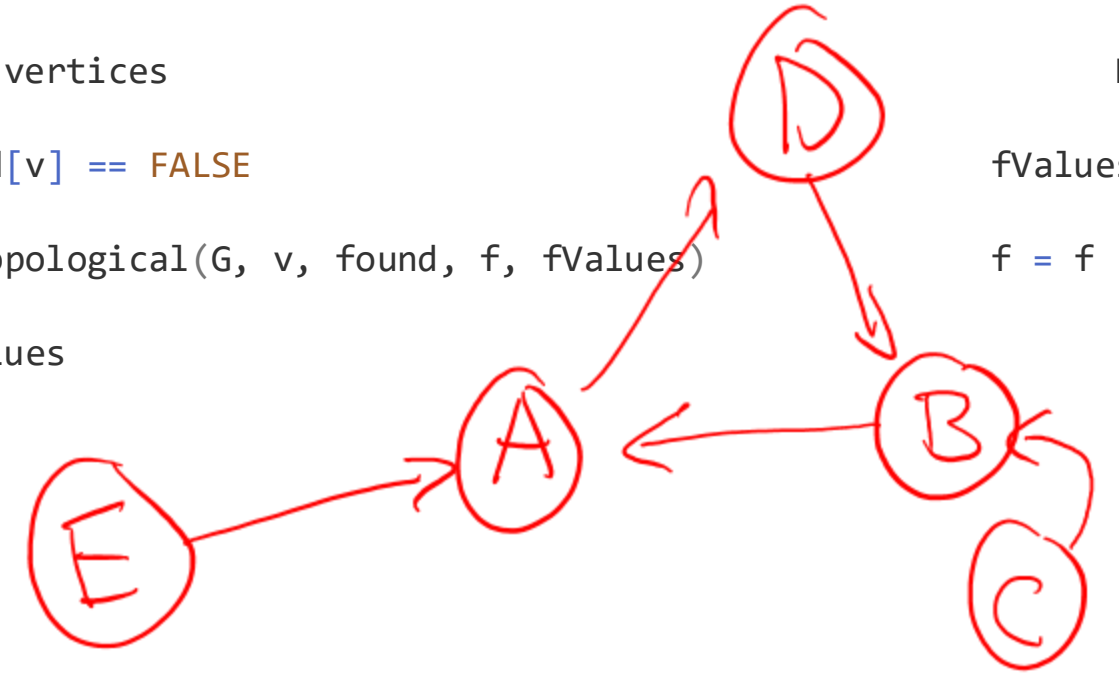


Solve with DFS

Are you guaranteed to find a "sink" in a DAG?

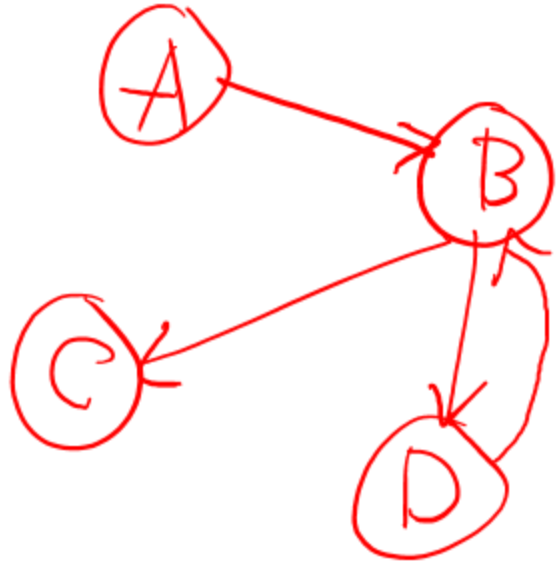
```
FUNCTION TopologicalOrdering(G)
    found = {v: FALSE FOR v IN G.vertices}
    fValues = {v: INFINITY FOR v IN G.vertices}
    f = G.vertices.length
    FOR v IN G.vertices
        IF found[v] == FALSE
            DFSTopological(G, v, found, f, fValues)
    RETURN fValues
```

```
FUNCTION DFSTopological(G, v, found, f, fValues)
    found[v] = TRUE
    FOR vOther IN G.edges[v]
        IF found[vOther] == FALSE
            DFSTopological(G, vOther, found, f, fValues)
    fValues[v] = f
    f = f - 1
```



≠ DAG

Strongly Connected Components



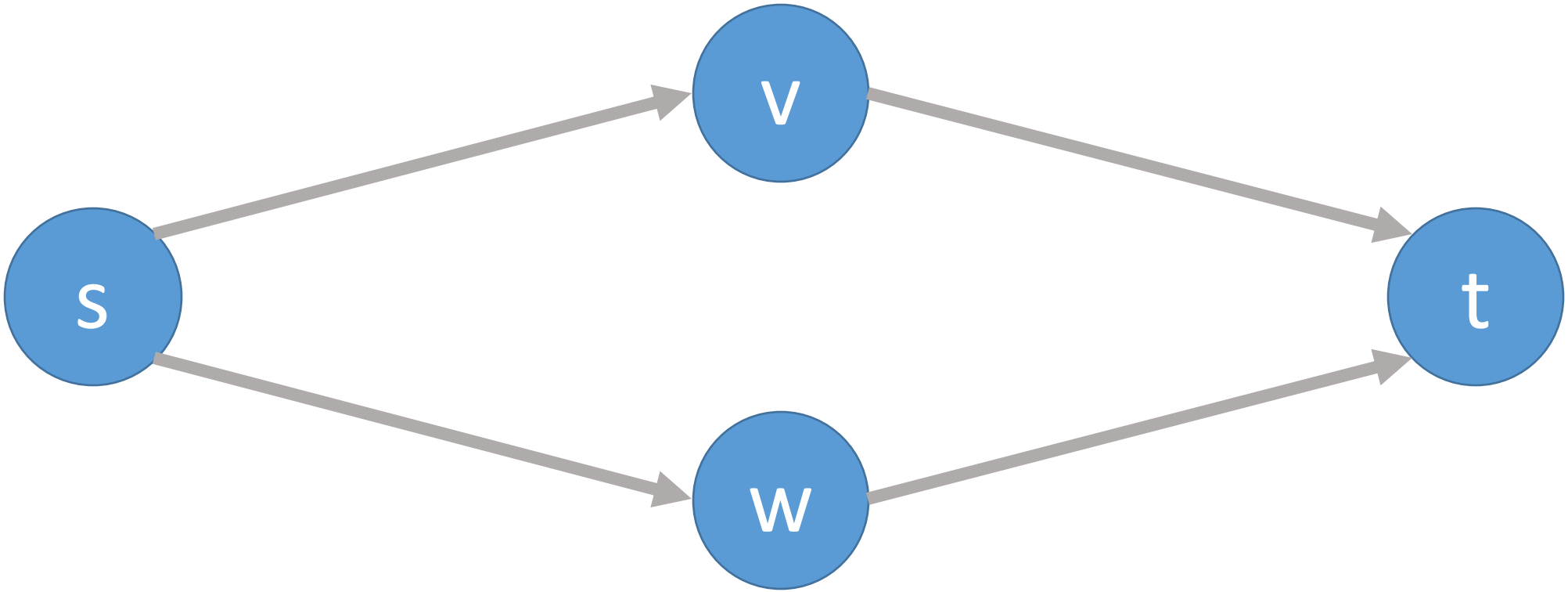
- Topological orderings are useful in their own right, but they also let us efficiently calculate the strongly connected components (SCCs) of a graph

- A component (set of vertices) of a graph is strongly connected if we can find a path from any vertex to any other vertex
- This is a concept for **directed** graphs only
- (just *connected components* for undirected graphs)

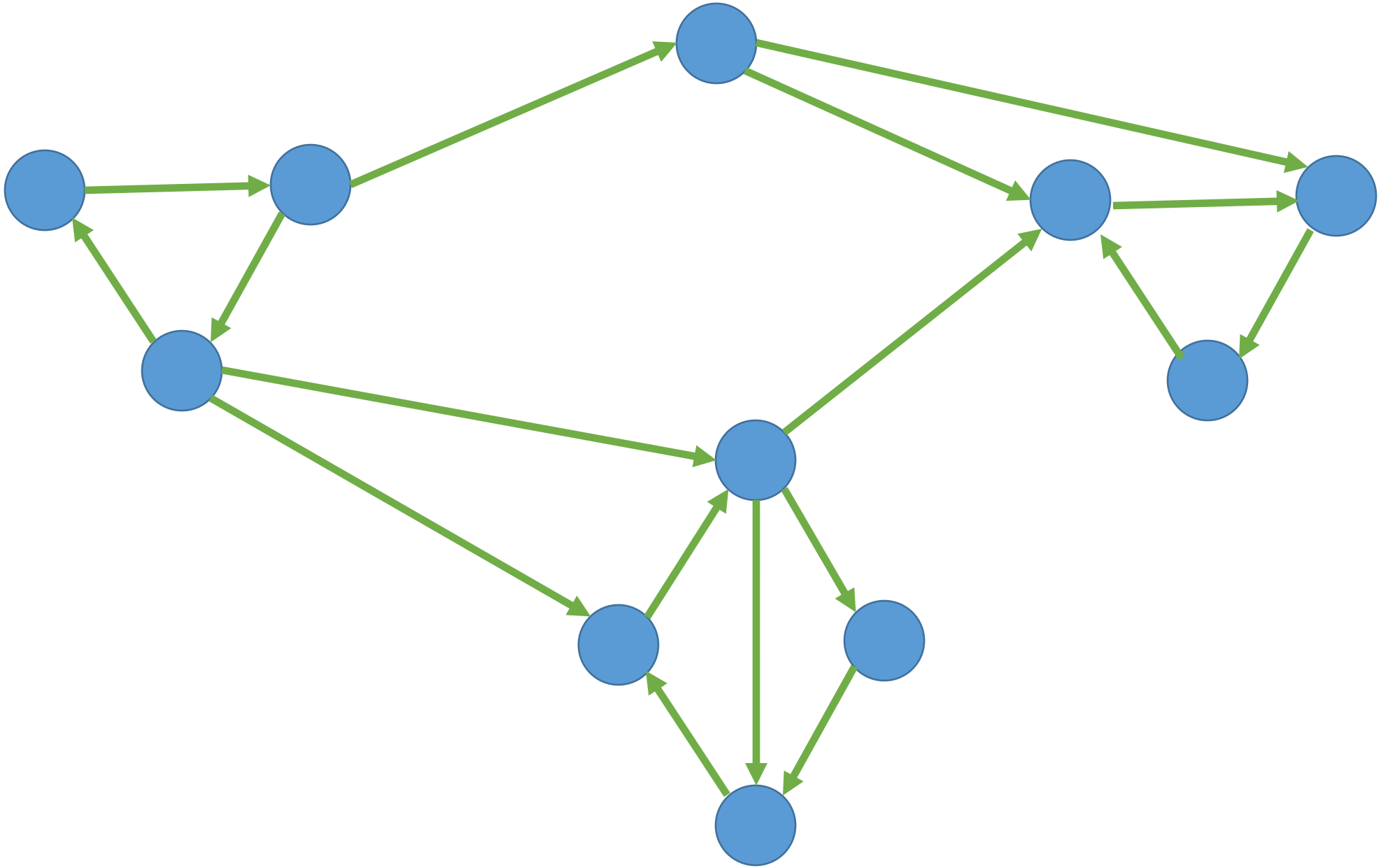
A

Why are SCCs useful?

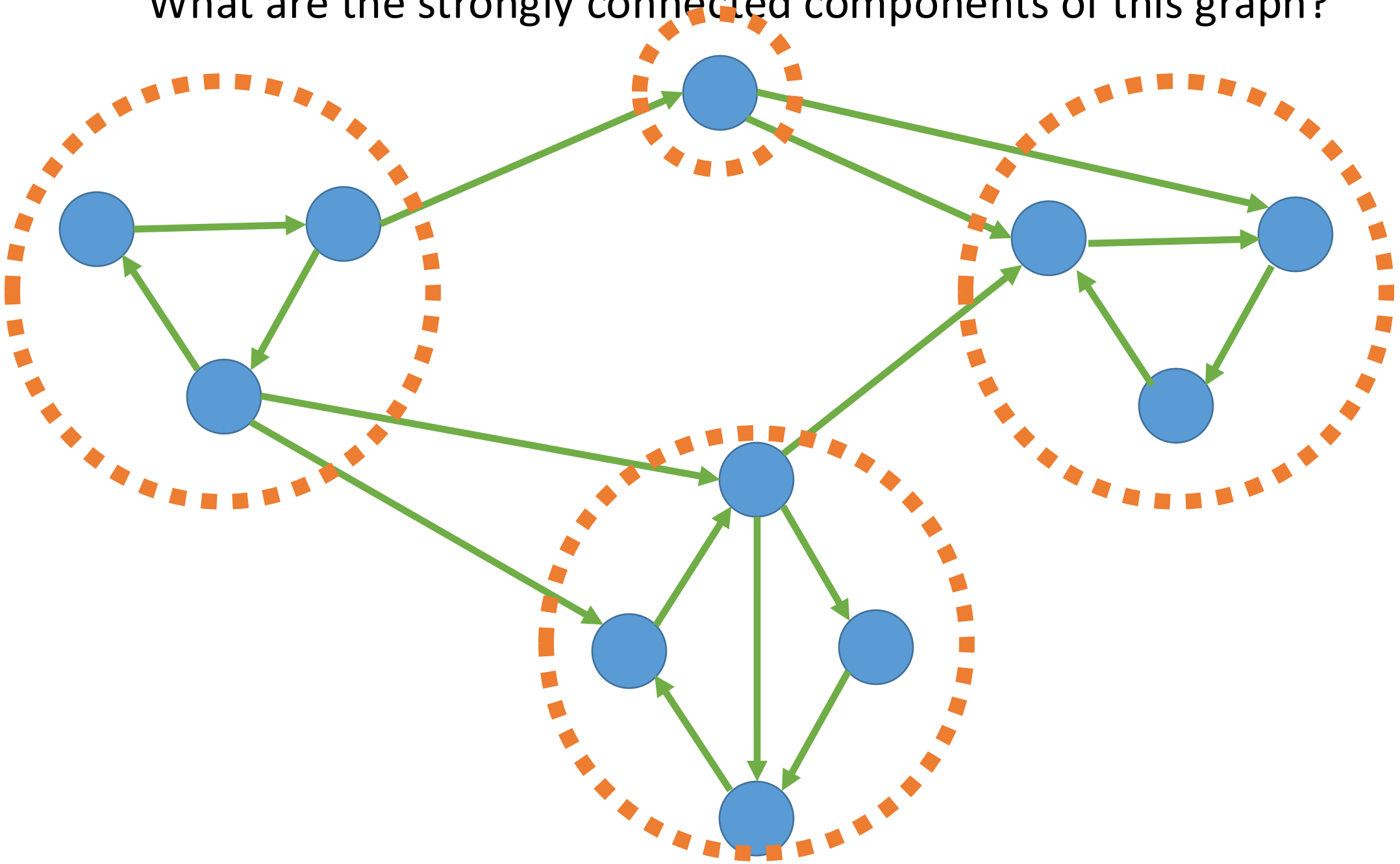
What are the strongly connected components of this graph?



What are the strongly connected components of this graph?



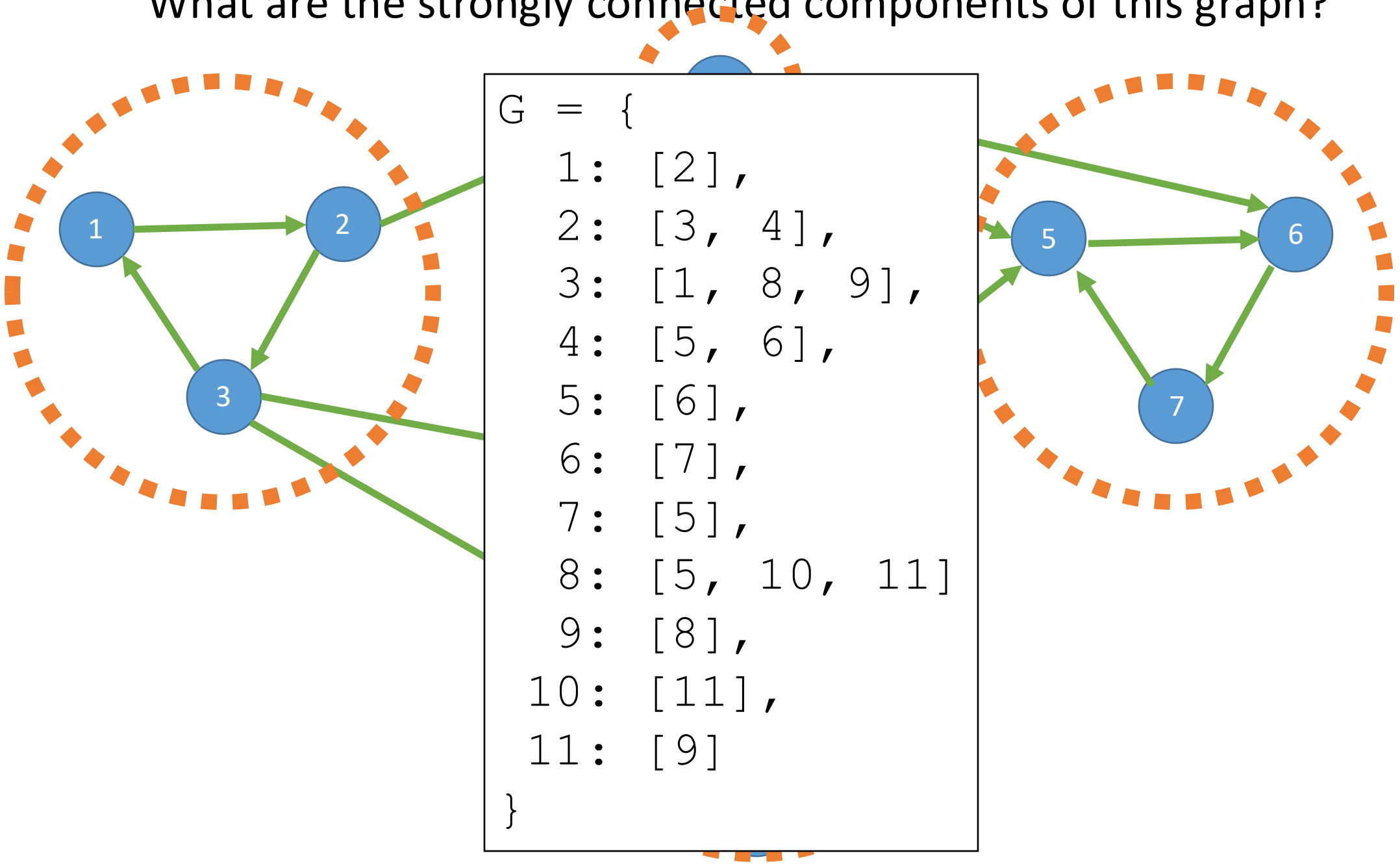
What are the strongly connected components of this graph?



What are the strongly connected components of this graph?

```
G = {
   1: [2],
   2: [3, 4],
   3: [1, 8, 9],
   4: [5, 6],
   5: [6],
   6: [7],
   7: [5],
   8: [5, 10, 11]
   9: [8],
  10: [11],
  11: [9]
}
```

What are the strongly connected components of this graph?



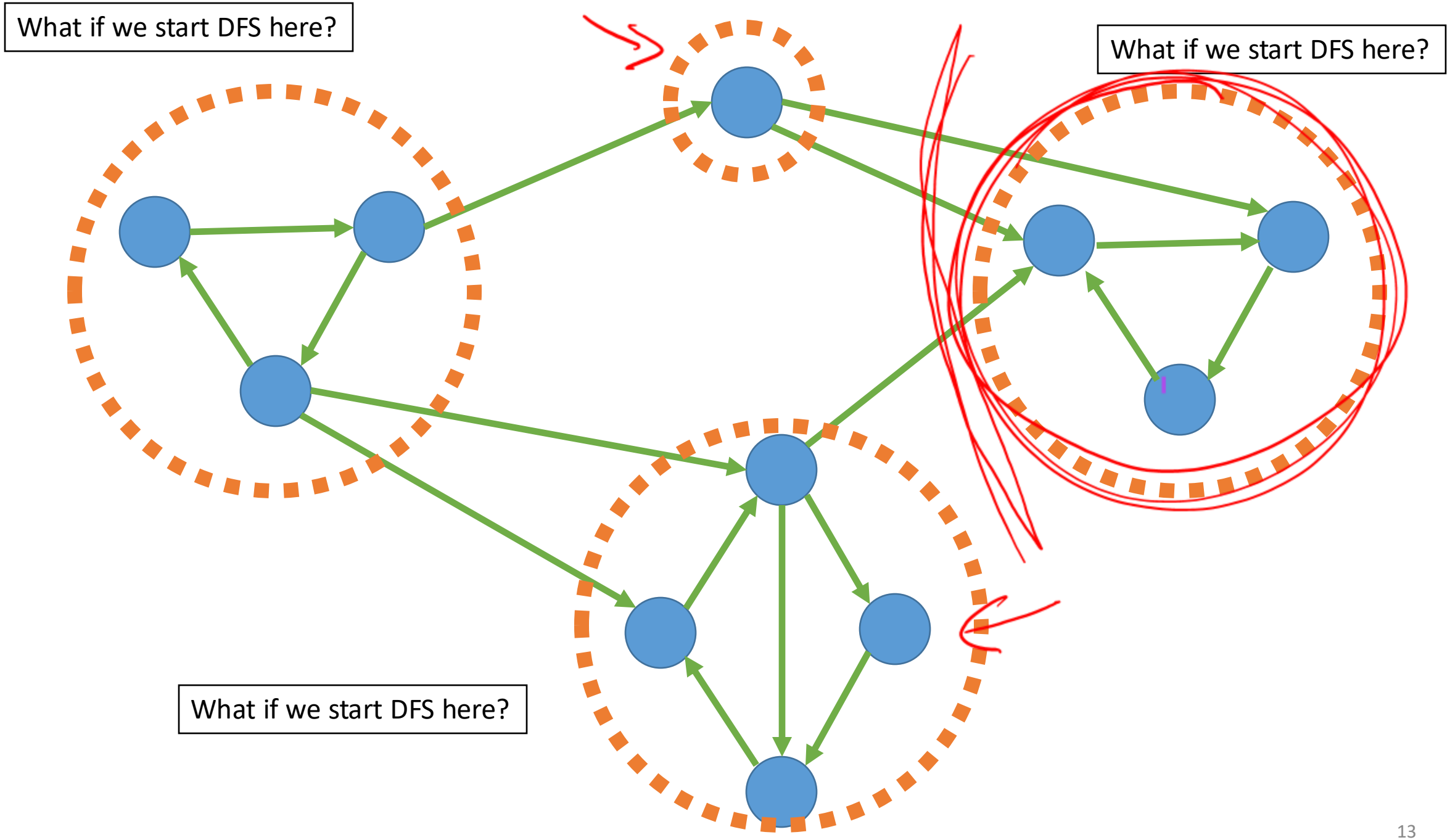

```
G = {
   1: [2],
   2: [3, 4],
   3: [1, 8, 9],
   4: [5, 6],
   5: [6],
   6: [7],
   7: [5],
   8: [5, 10, 11]
   9: [8],
  10: [11],
  11: [9]
}
```

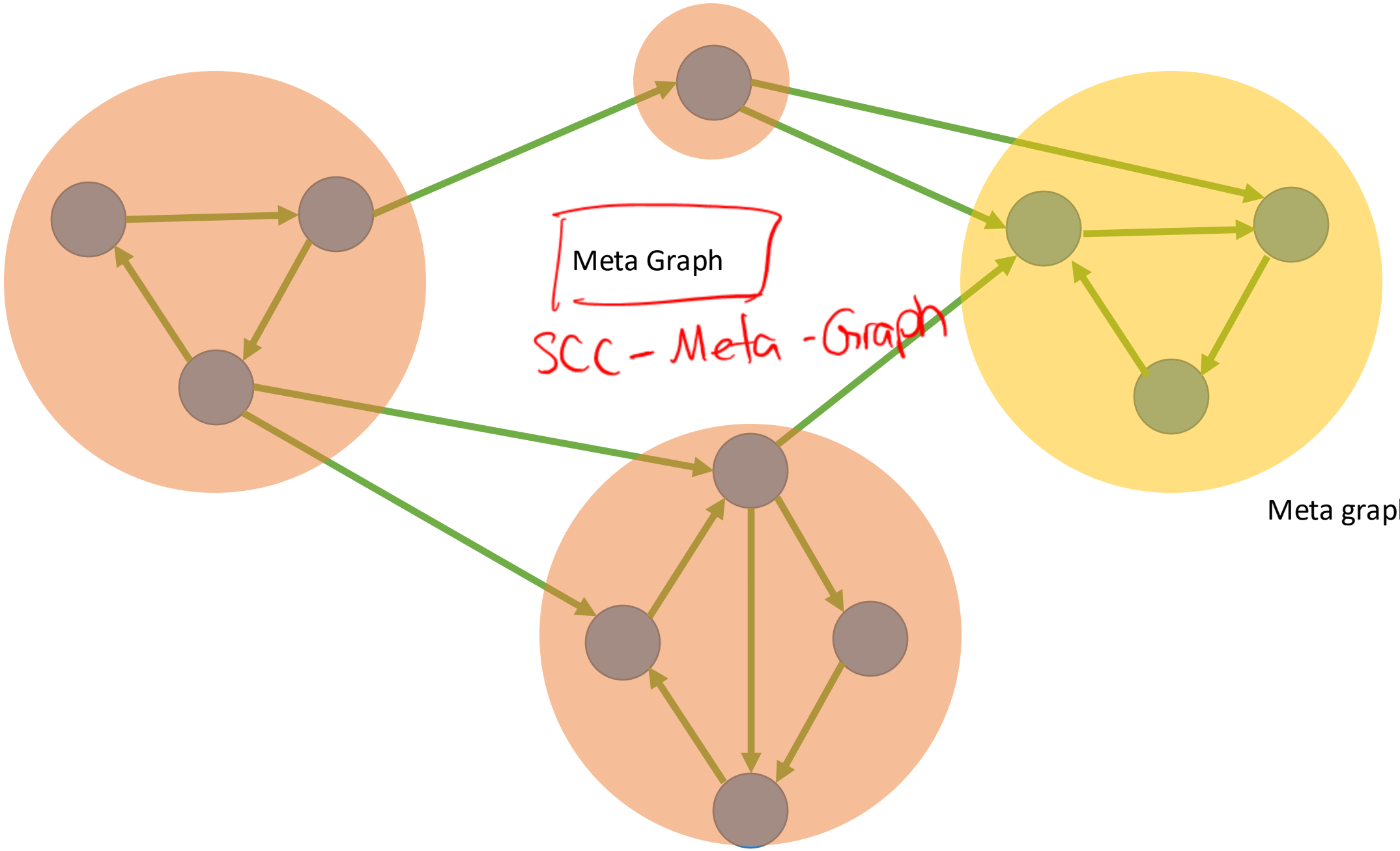
Can we use DFS?

What does a DFS do?

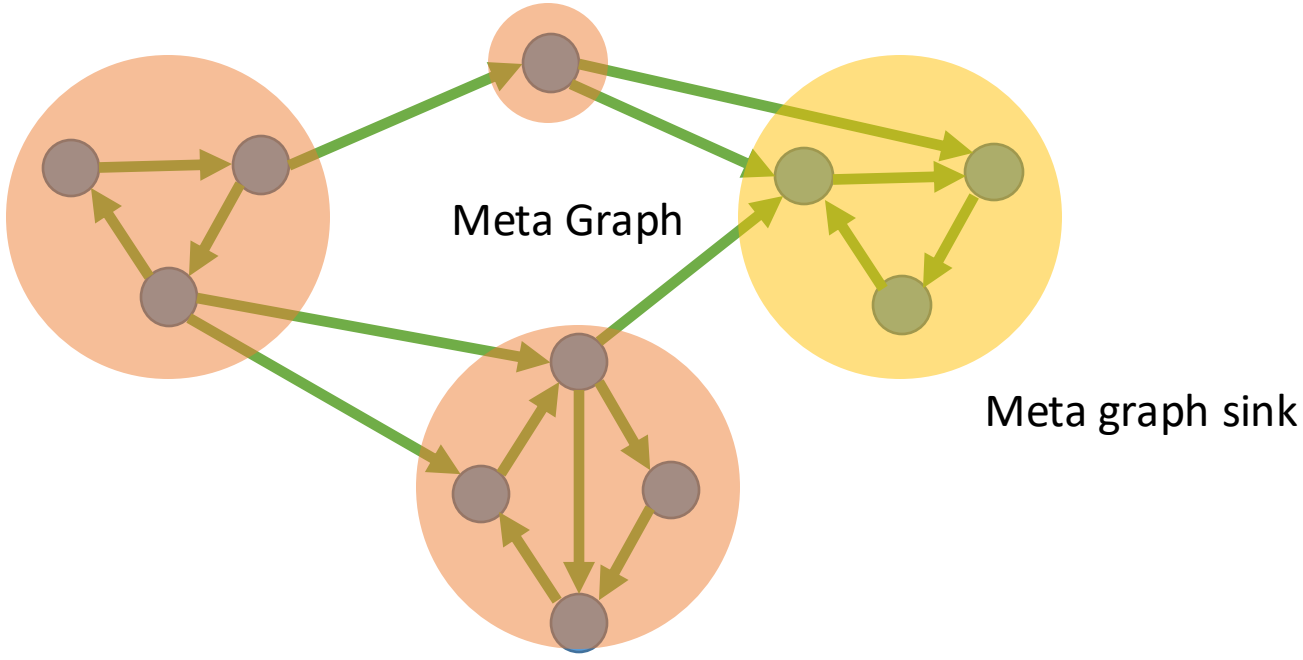
- Finds everything that is findable
- Does not visit any vertex more than once

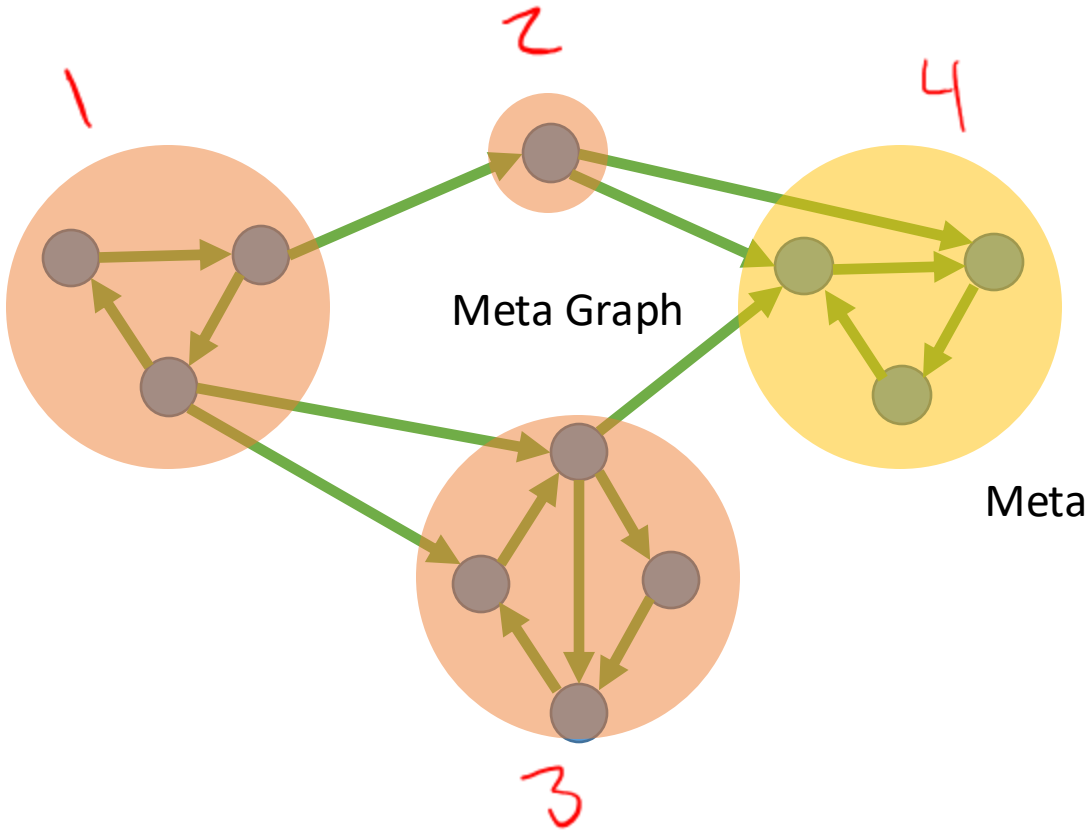
So, what can we find from each of the different nodes?


What if we start DFS here?
What if we start DFS here?
What if we start DFS here?

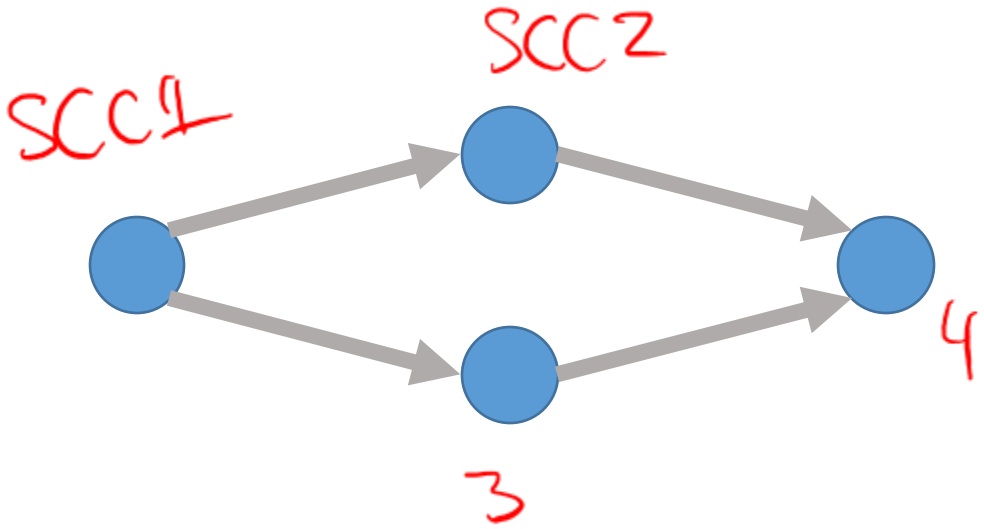


Meta graph sink


Meta Graph
Meta graph sink



Meta graph sink



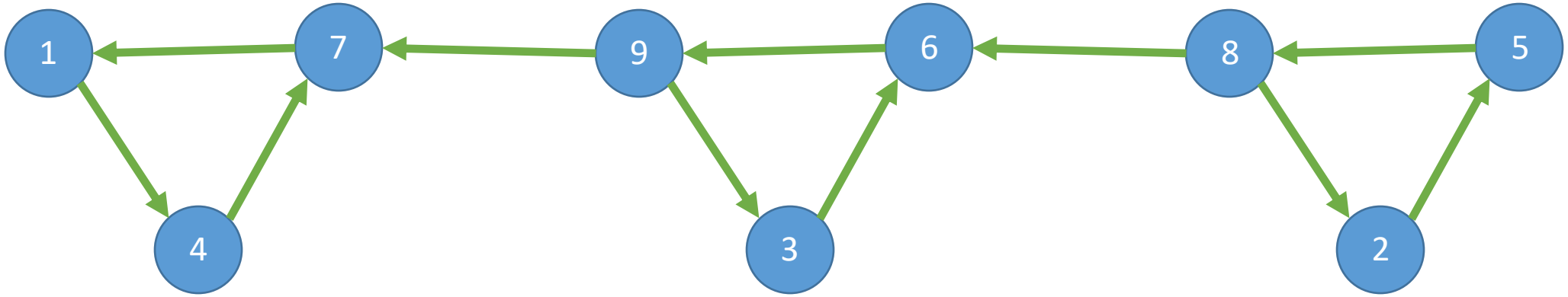
Kosaraju

Computes the SCCs in O(m + n) time **(linear!)**

1. Create a reverse version of the G called G_reversed

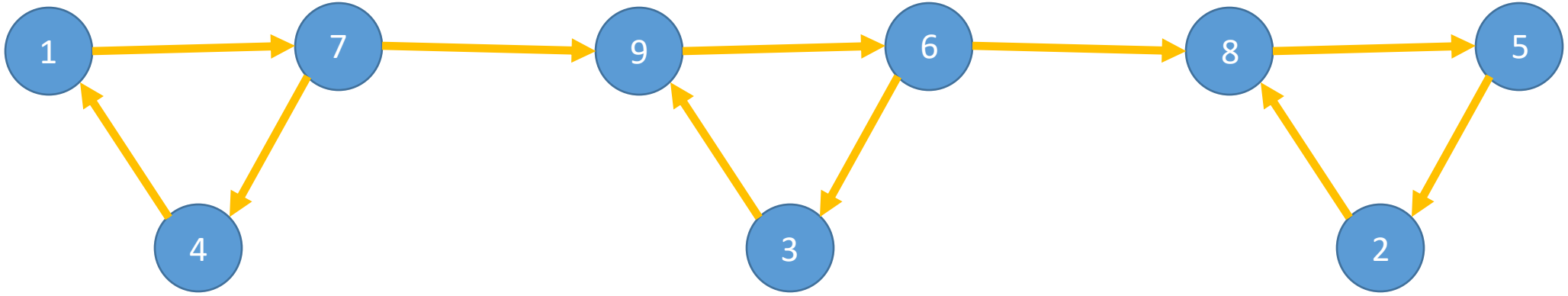
G

(7, 1)



G_reversed

(1, 7)



Kosaraju

Computes the SCCs in O(m + n) time **(linear!)**

1. Create a reverse version of the G called G_reversed

2. Run KosarajuLabels on G_reversed

   Compute a topological order of the meta graph

3. Create a relabeled version of the G called G_relabeled

4. Run KosarajuLeaders on G_relabeled

   Explore vertices in the new order

```
FUNCTION Kosaraju(G)
    G_reversed = reverse_graph(G)
    new_labels = KosarajuLabels(G_reversed)

    G_relabeled = relabel_graph(G, new_labels)
    leaders = KosarajuLeaders(G_relabeled)

    RETURN leaders
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    RETURN leaders
```

```
FUNCTION KosarajuLabels(G)
    found = {v: FALSE FOR v IN G.vertices}
    label = 0
    labels = {v: NONE FOR v IN G.vertices}

    FOR v IN G.vertices
        IF found[v] == FALSE
            DFSLabels(G, v, found, label, labels)

    RETURN labels
```

```
FUNCTION DFSLabels(G, v, found, label, labels)
    found[v] = TRUE
    FOR vOther IN G.edges[v]
        IF found[vOther] == FALSE
            DFSLabels(G, vOther, found, label, labels)
    label = label + 1
    labels[v] = label
```

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FUNCTION Kosaraju(G)
    G_reversed = reverse_graph(G)
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    RETURN leaders
```

```
FUNCTION KosarajuLeaders(G)
    found = {v: FALSE FOR v IN G.vertices}
    leaders = {v: NONE FOR v IN G.vertices}

    FOR v IN G.vertices.reverse_order
        IF found[v] == FALSE
            leader = v
            DFSLeaders(G, v, found, leader, leaders)

    RETURN leaders
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These are typically implemented in a single function

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    found[v] = TRUE
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        IF found[vOther] == FALSE
            DFSLeaders(G, vOther, found, leader, leaders)
```

These are typically implemented in a single function

```
FUNCTION KosarajuLoop(G)
    found = {v: FALSE FOR v IN G.vertices}
    label = 0
    labels = {v: NONE FOR v IN G.vertices}
    leaders = {v: NONE FOR v IN G.vertices}

    FOR v IN G.vertices.reverse_order
        IF found[v] == FALSE
            leader = v
            KosarajuDFS(G, v, found, label, labels, leader, leaders)

    RETURN labels, leaders

FUNCTION KosarajuDFS(G, v, found, label, labels, leader, leaders)
    found[v] = TRUE
    leaders[v] = leader
    FOR vOther IN G.edges[v]
        IF found[vOther] == FALSE
            KosarajuDFS(G, v, found, label, labels, leader, leaders)
    label = label + 1
    labels[v] = label
```

Does both labels and leaders.

```
FUNCTION Kosaraju(G)
    G_reversed = reverse_graph(G)
    new_labels = KosarajuLabels(G_reversed)

    G_relabeled = relabel_graph(G, new_labels)
    leaders = KosarajuLeaders(G_relabeled)

    RETURN leaders
```

```
FUNCTION Kosaraju(G)
    G_reversed = reverse_graph(G)
    new_labels, _ = KosarajuLoop(G_reversed)

    G_relabeled = relabel_graph(G, new_labels)
    _, leaders = KosarajuLoop(G_relabeled)

    RETURN leaders
```

Easier to maintain

Q5

Kosaraju

Computes the SCCs in O(m + n) time **(linear!)**

1. Create a reverse version of the G called G_reversed

2. Run KosarajuLoop on G_reversed

   Compute a topological order of the meta graph

3. Create a relabeled version of the G called G_relabeled

4. Run KosarajuLoop on G_relabeled

   Explore vertices in the new order

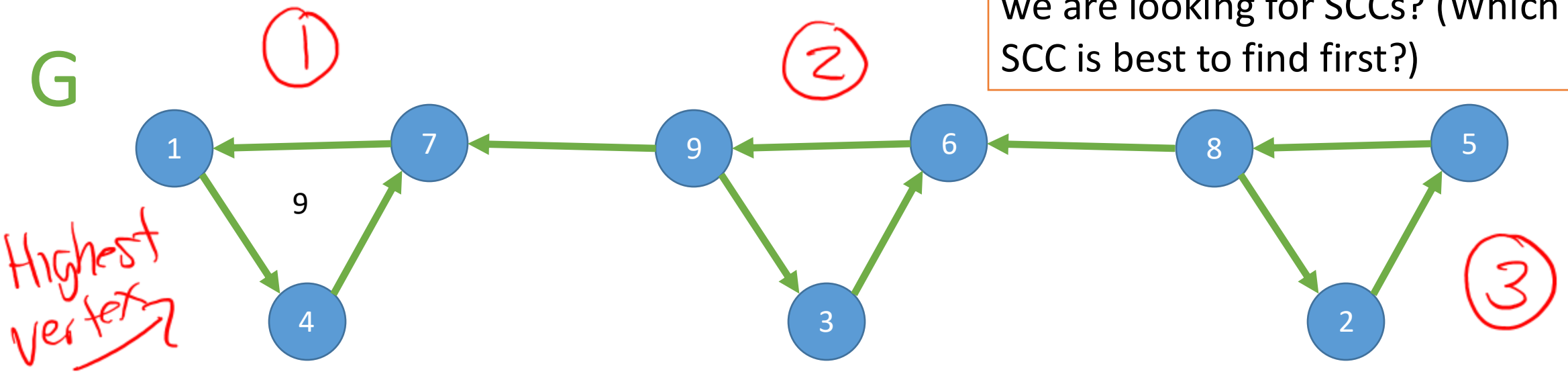
What are the SCCs?

```
FUNCTION Kosaraju(G)
    G_reversed = reverse_graph(G)
    new_labels, _ = KosarajuLoop(G_reversed)

    G_relabeled = relabel_graph(G, new_labels)
    _, leaders = KosarajuLoop(G_relabeled)

    RETURN leaders
```

Where do we want to start DFS if we are looking for SCCs? (Which SCC is best to find first?)



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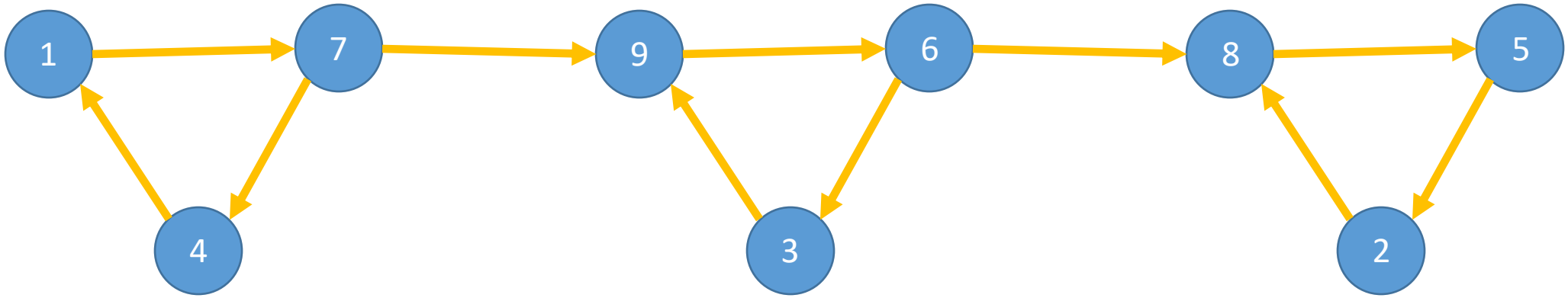
    RETURN leaders
```

Where do we want to start DFS if we are looking for SCCs? (Which SCC is best to find first?)

G_reversed



```
FUNCTION Kosaraju(G)
    G_reversed = reverse_graph(G)
    new_labels, _ = KosarajuLoop(G_reversed)

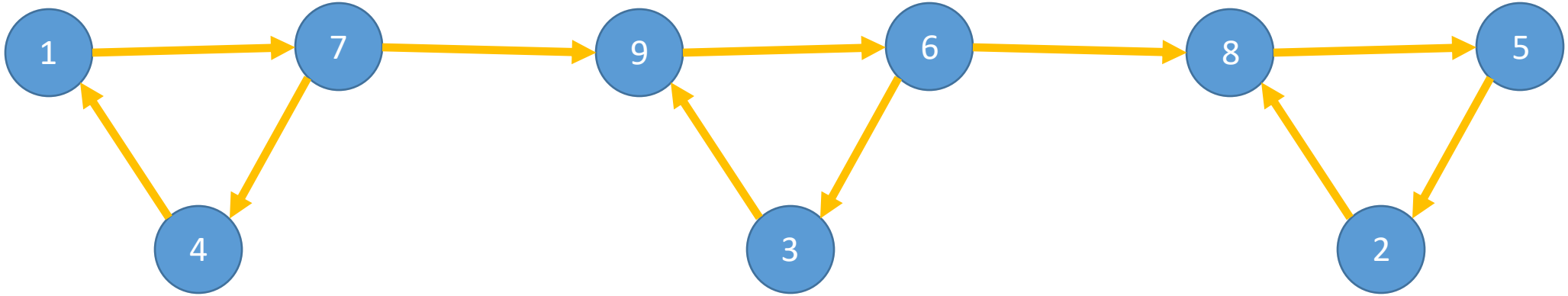
    G_relabeled = relabel_graph(G, new_labels)
    _, leaders = KosarajuLoop(G_relabeled)

    RETURN leaders
```

ignore

Where do we want to start DFS if we are looking for SCCs? (Which SCC is best to find first?)

G_reversed



```
FUNCTION KosarajuLoop(G)
    found = {v: FALSE FOR v IN G.vertices}
    label = 0
    labels = {v: NONE FOR v IN G.vertices}
    leaders = {v: NONE FOR v IN G.vertices}

    FOR v IN G.vertices.reverse_order
        IF found[v] == FALSE
            leader = v
            KosarajuDFS(…)

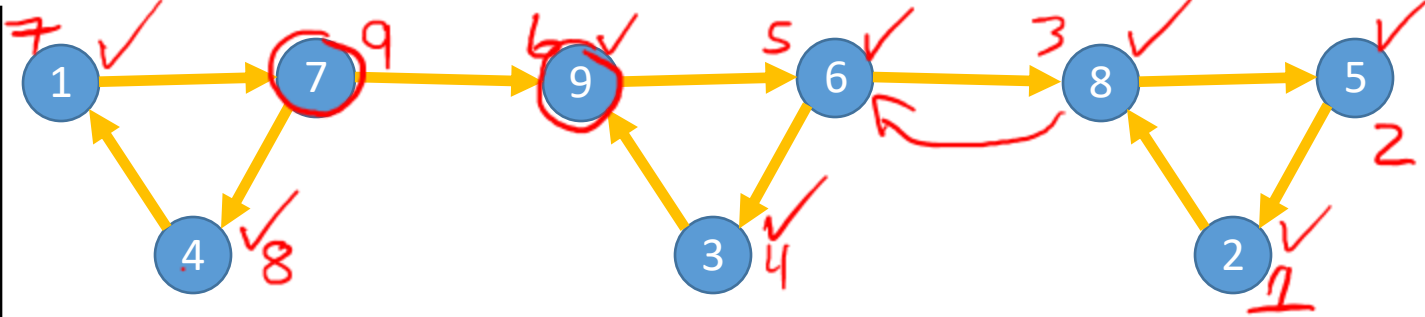
    RETURN labels, leaders
```

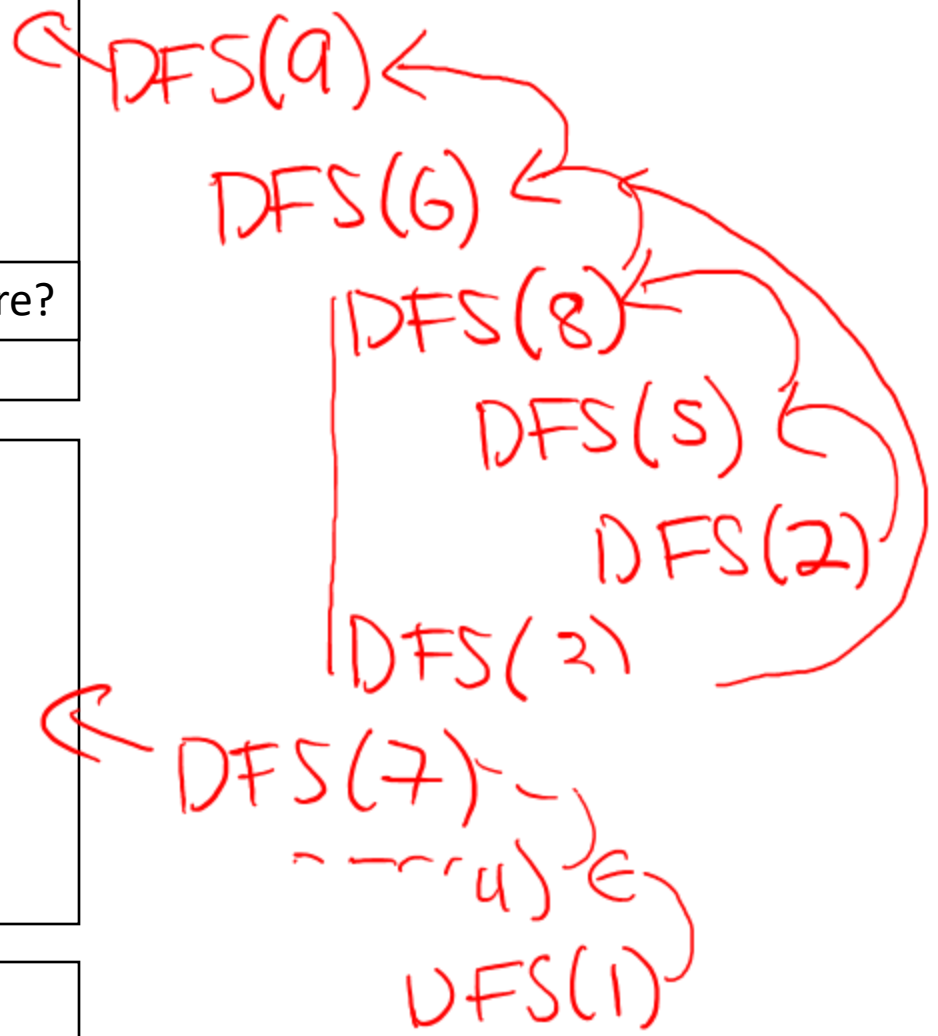
Can we start anywhere?

```
FUNCTION KosarajuDFS(…)
    found[v] = TRUE
    leaders[v] = leader
    FOR vOther IN G.edges[v]
        IF found[vOther] == FALSE
            KosarajuDFS(…)
    label = label + 1
    labels[v] = label
```

Ignore leaders the first pass

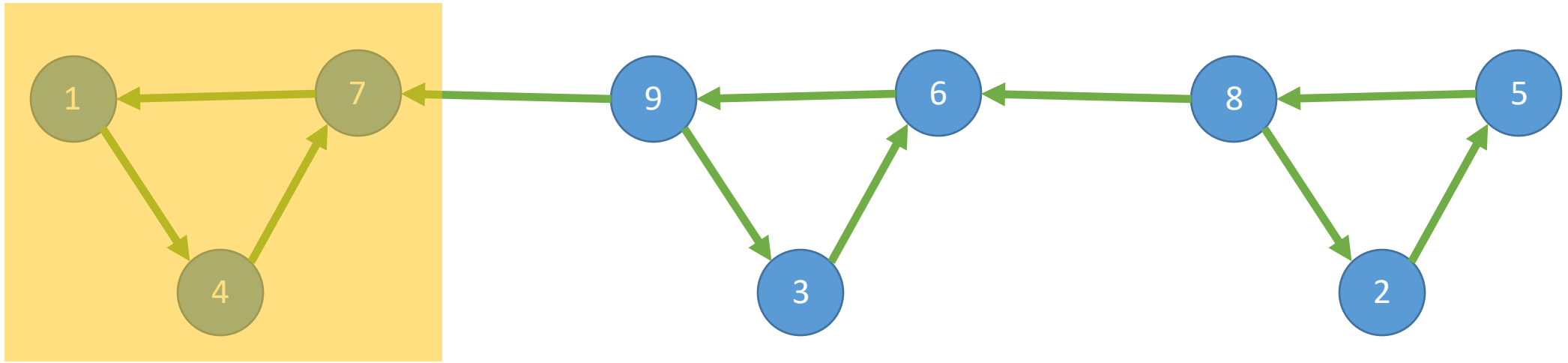
Ignore labels the second pass



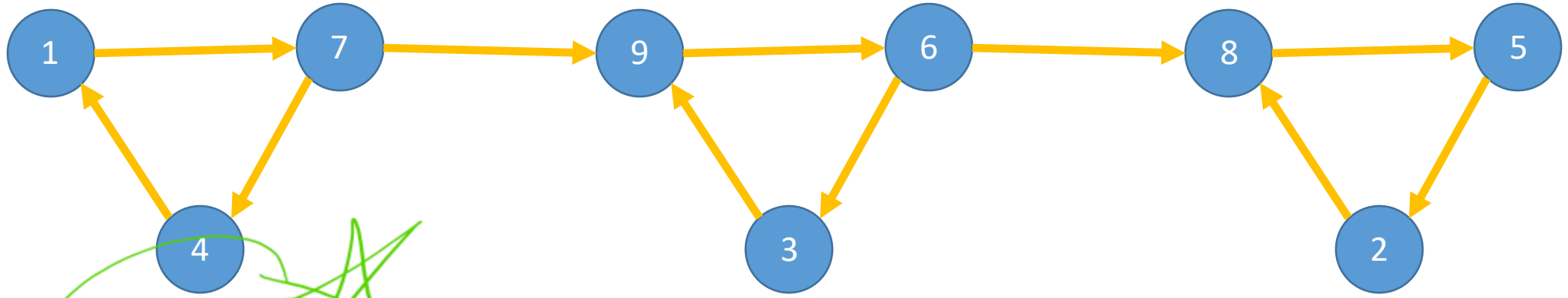


G

Sink SCC in
Meta Graph

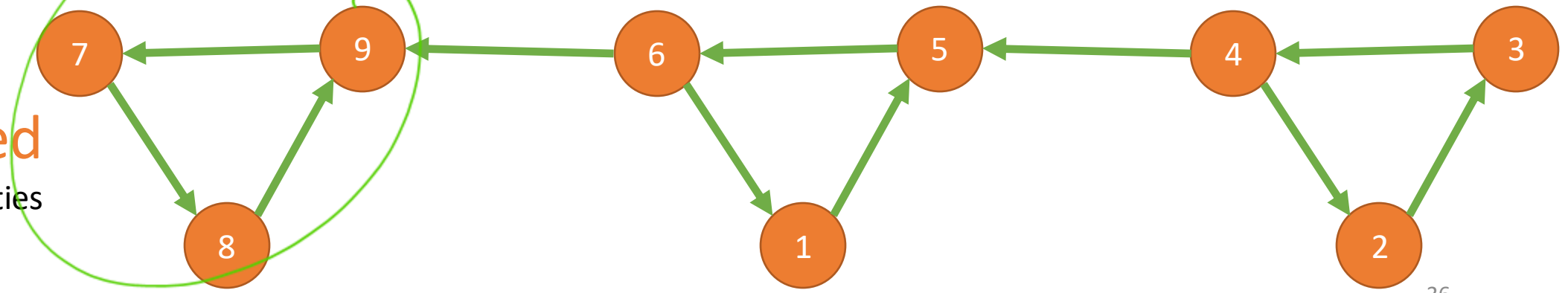


G_reversed



G_relabeled

Multiple possibilities

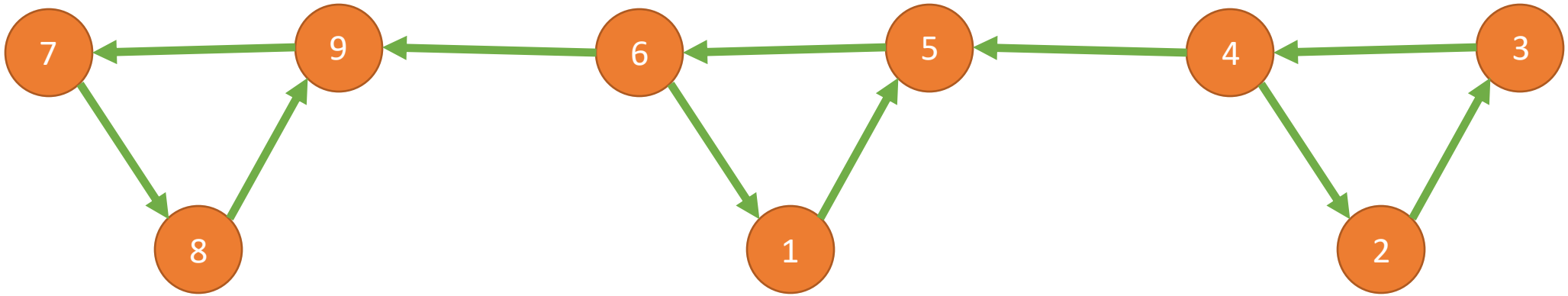


```
FUNCTION Kosaraju(G)
    G_reversed = reverse_graph(G)
    new_labels, _ = KosarajuLoop(G_reversed)

    G_relabeled = relabel_graph(G, new_labels)
    _, leaders = KosarajuLoop(G_relabeled)

    RETURN leaders
```

G_relabeled



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FUNCTION KosarajuLoop(G)
    found = {v: FALSE FOR v IN G.vertices}
    label = 0
    labels = {v: NONE FOR v IN G.vertices}
    leaders = {v: NONE FOR v IN G.vertices}

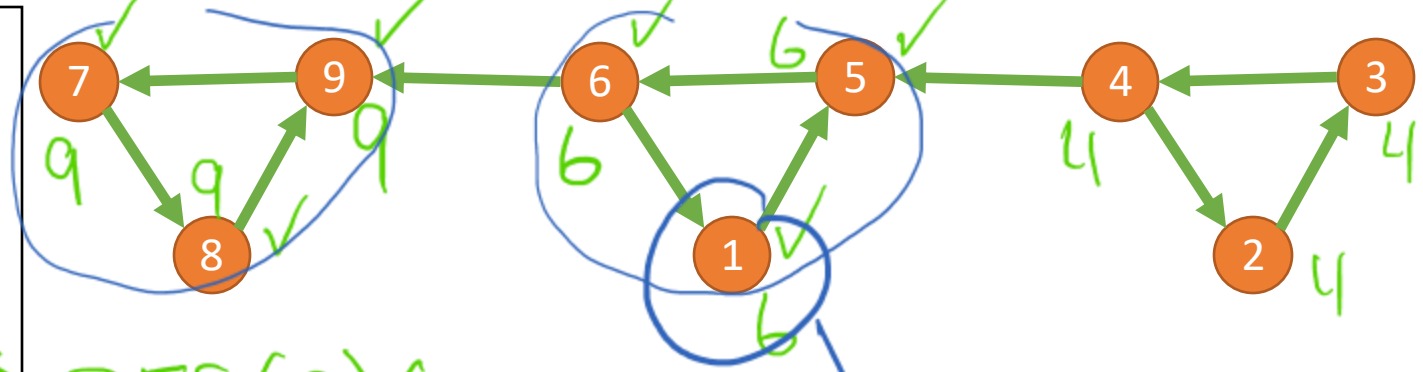
    FOR v IN G.vertices.reverse_order
        IF found[v] == FALSE
            leader = v
            KosarajuDFS(…)

    RETURN labels, leaders
```

```
FUNCTION KosarajuDFS(…)
    found[v] = TRUE
    leaders[v] = leader
    FOR vOther IN G.edges[v]
        IF found[vOther] == FALSE
            KosarajuDFS(…)
    label = label + 1
    labels[v] = label
```

Ignore leaders the first pass

Ignore labels the second pass



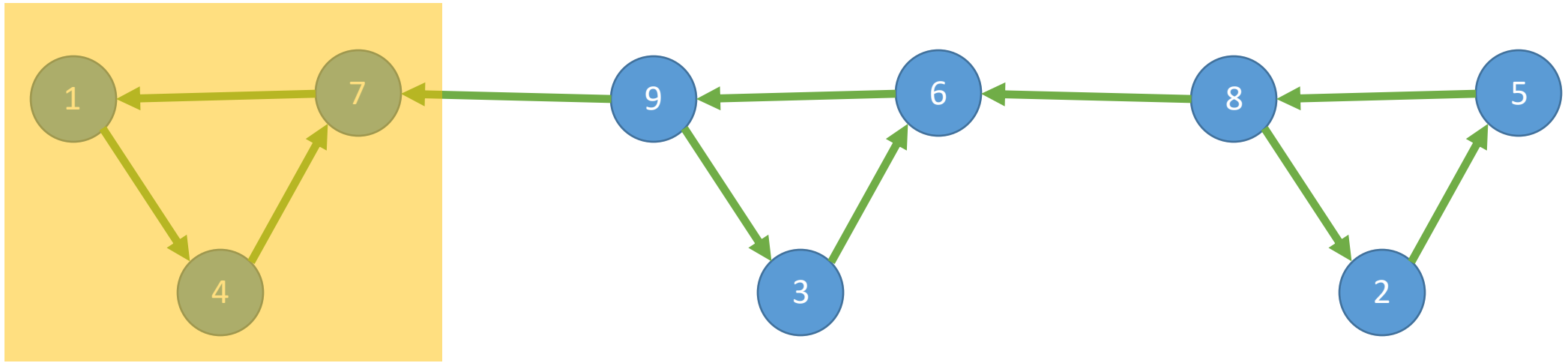
DFS(9)
DFS(7)
DFS(8)

DFS(6)
DFS(1)
DFS(5)
DFS(4)
⋮

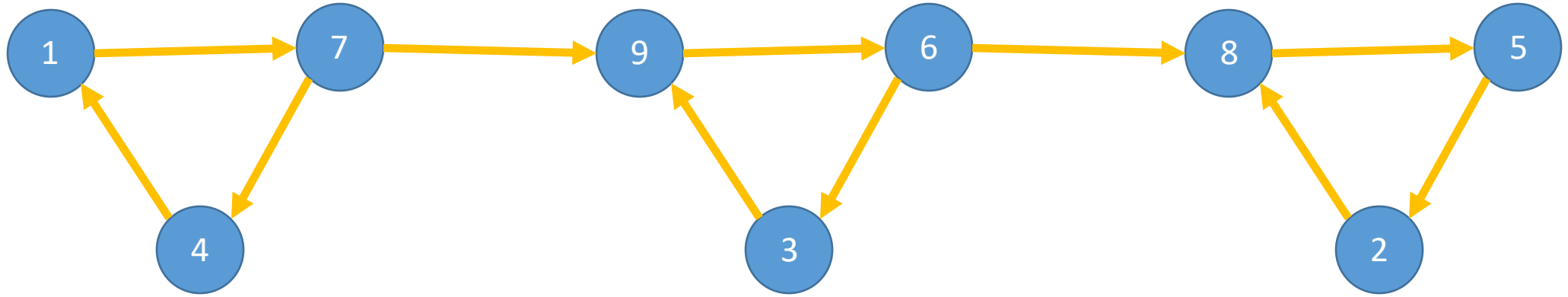
Is this a problem?

G

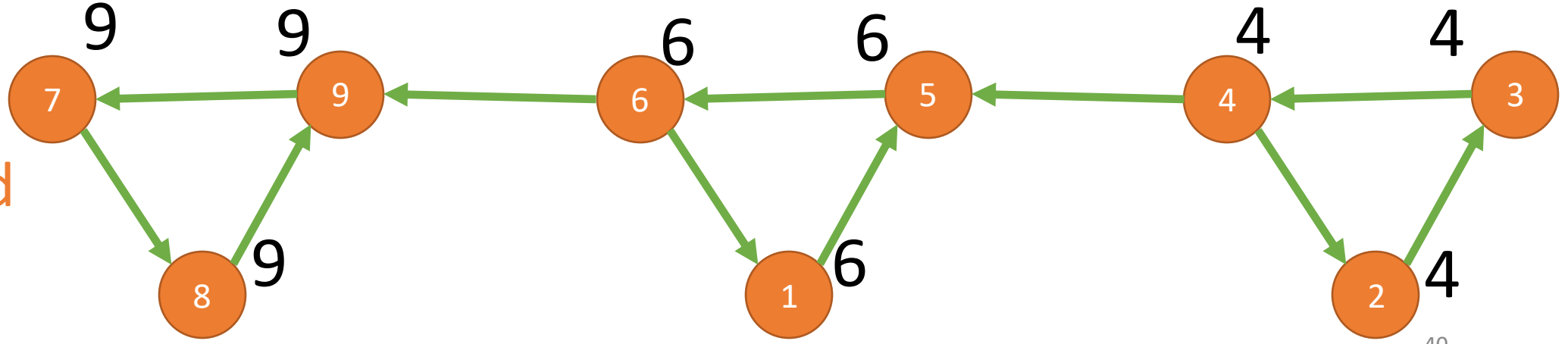
Sink SCC in
Meta Graph



G_reversed



G_relabeled



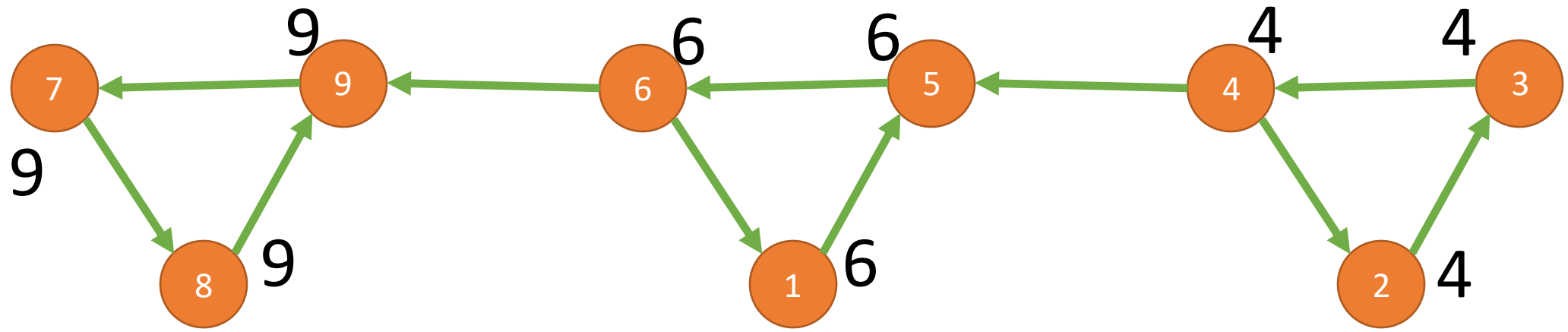
```
FUNCTION Kosaraju(G)
    G_reversed = reverse_graph(G)
    new_labels, _ = KosarajuLoop(G_reversed)

    G_relabeled = relabel_graph(G, new_labels)
    _, leaders = KosarajuLoop(G_relabeled)

    RETURN leaders
```

What could you do to make this API a bit nicer?

G_relabeled



Exercise

```
FUNCTION KosarajuLoop(G)
    found = {v: FALSE FOR v IN G.vertices}
    label = 0
    labels = {v: NONE FOR v IN G.vertices}
    leaders = {v: NONE FOR v IN G.vertices}

    FOR v IN G.vertices.reverse_order
        IF found[v] == FALSE
            leader = v
            KosarajuDFS(G, v, found, label, labels, leader, leaders)

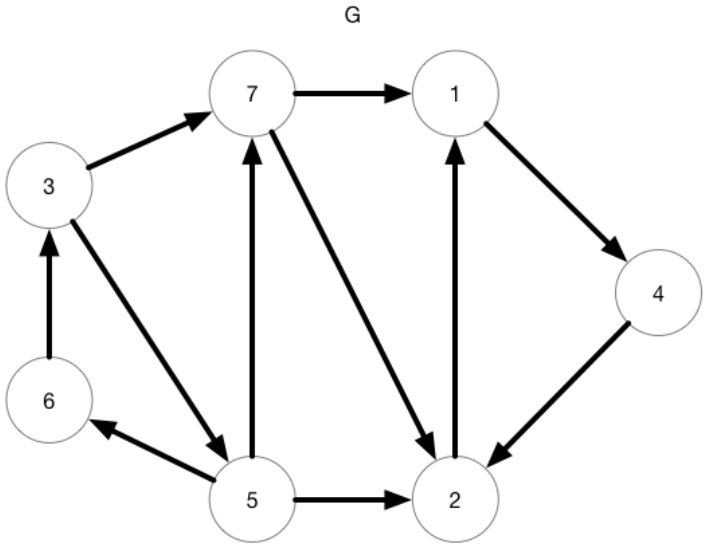
    RETURN labels, leaders
```

```
FUNCTION KosarajuDFS(G, v, found, label, labels, leader, leaders)
    found[v] = TRUE
    leaders[v] = leader
    FOR vOther IN G.edges[v]
        IF found[vOther] == FALSE
            KosarajuDFS(G, v, found, label, labels, leader, leaders)
    label = label + 1
    labels[v] = label
```

```
FUNCTION Kosaraju(G)
    G_reversed = reverse_graph(G)
    new_labels, _ = KosarajuLoop(G_reversed)

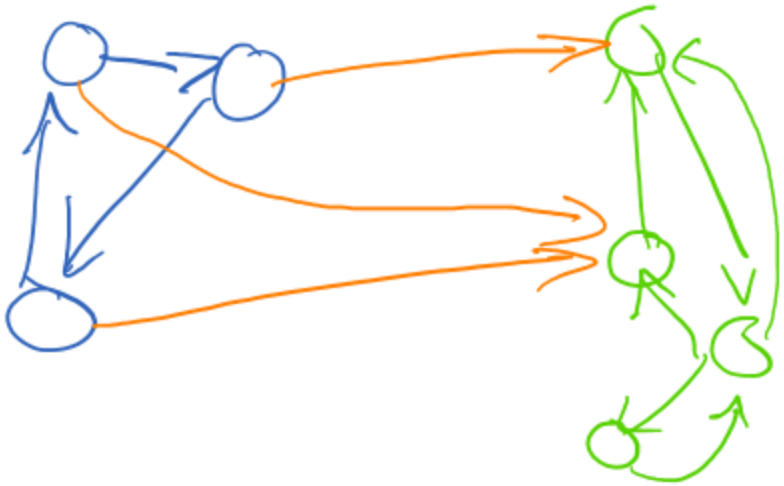
    G_relabeled = relabel_graph(G, new_labels)
    _, leaders = KosarajuLoop(G_relabeled)

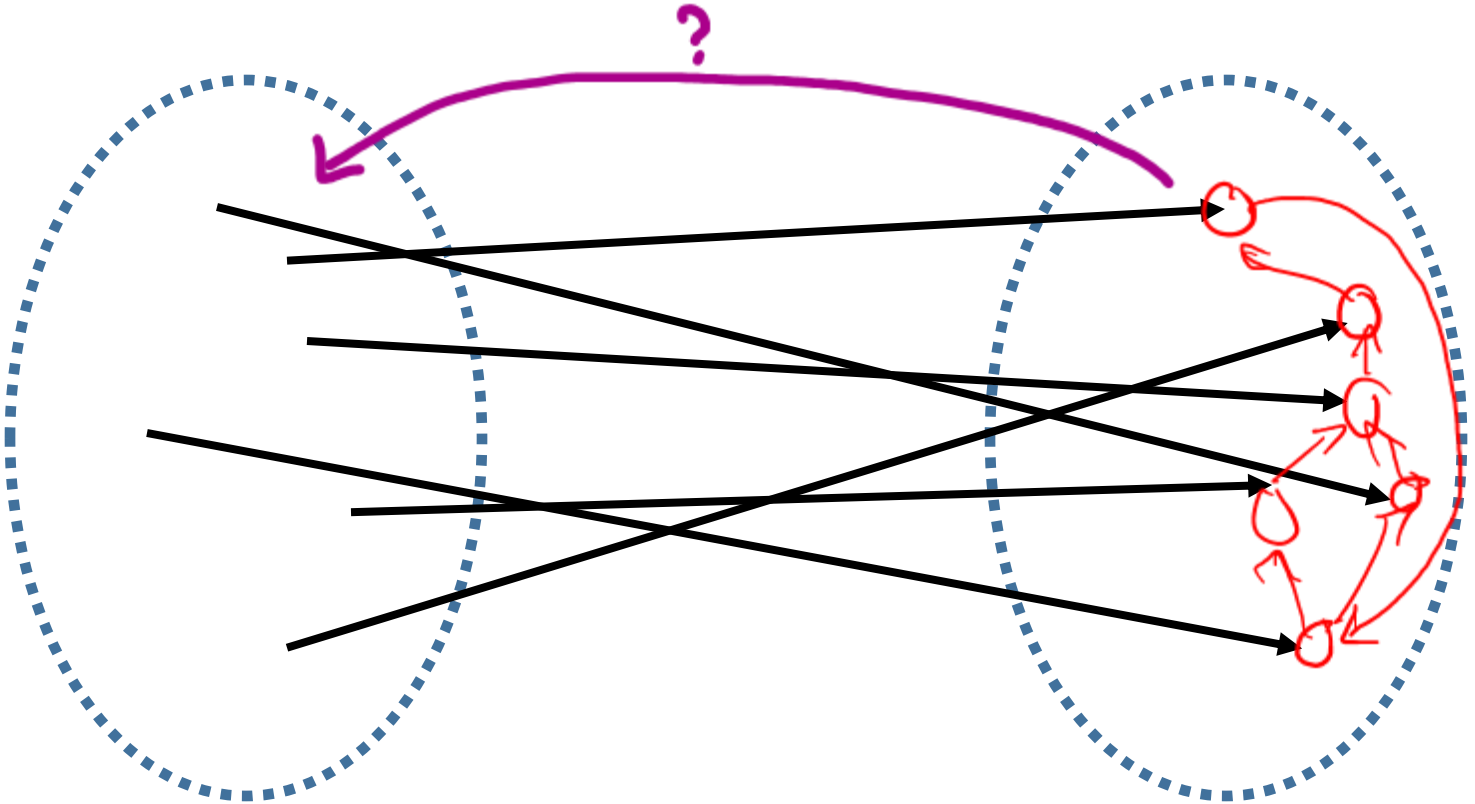
    RETURN leaders
```



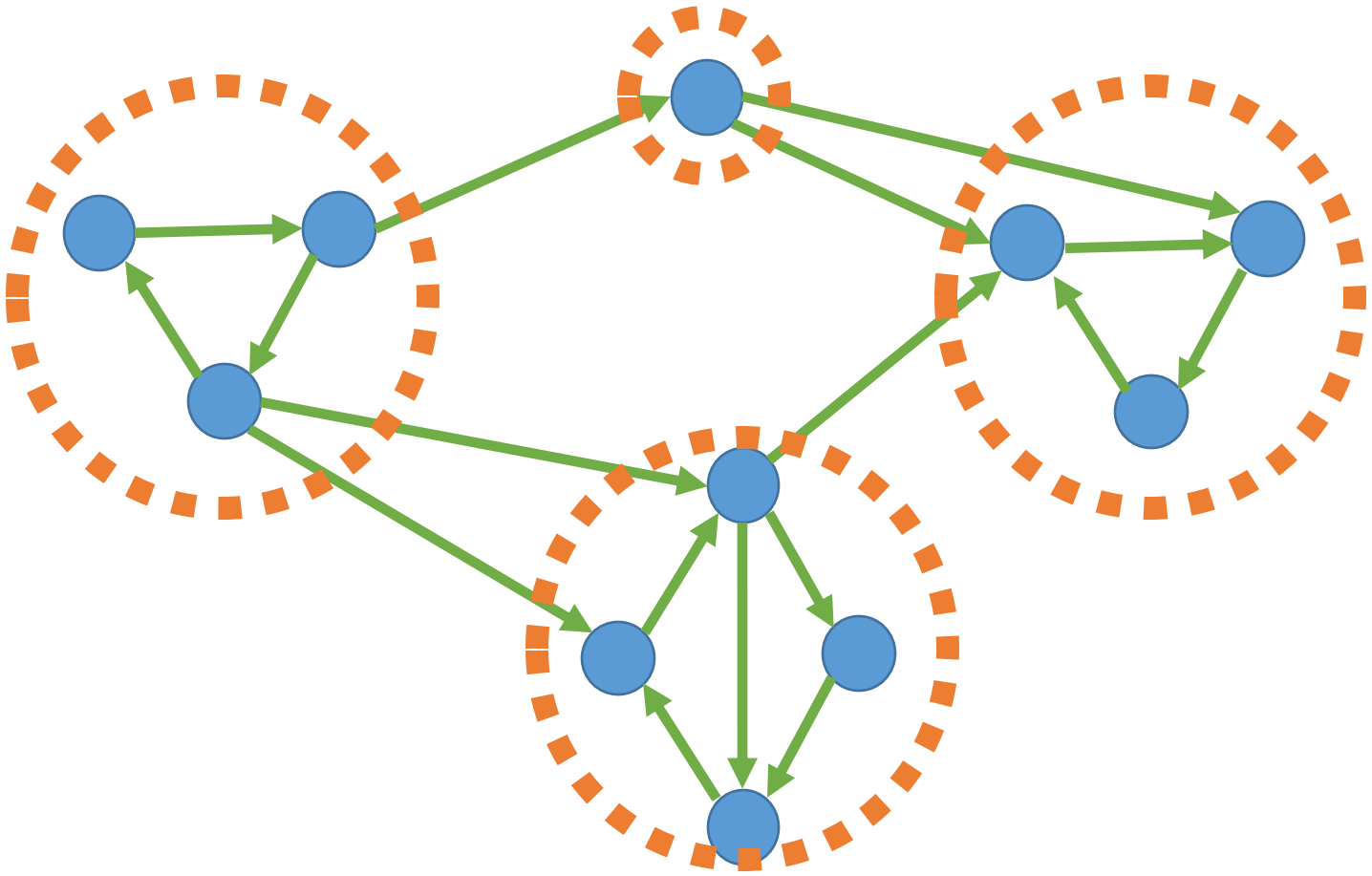
Why does this work?

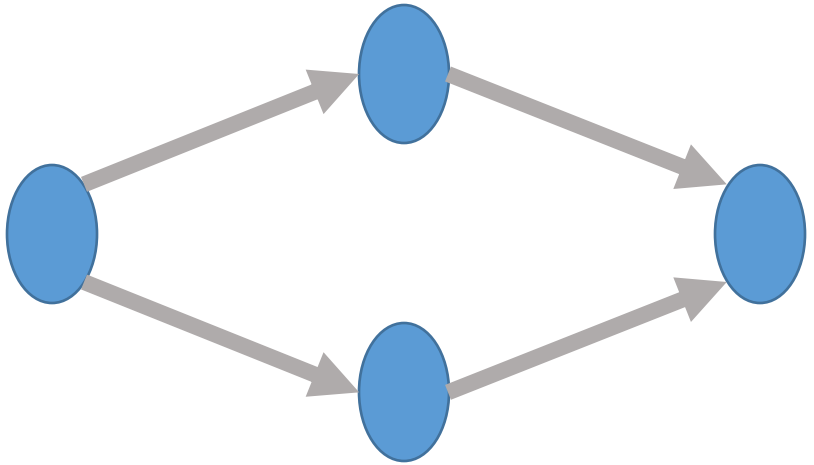
- Does this work for all graphs, or just this example?
- The SCCs of G create an acyclic “meta-graph”
- For the “meta-graph”
  - Vertices correspond to the SCCs
  - Edges correspond to paths among the SCCs

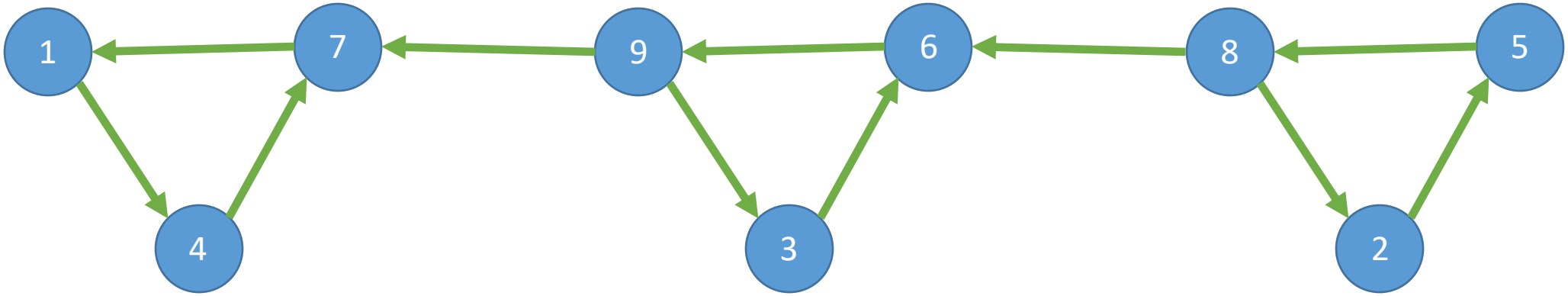



?
SCC1
SCC2


SCC1
SCC2






1
7
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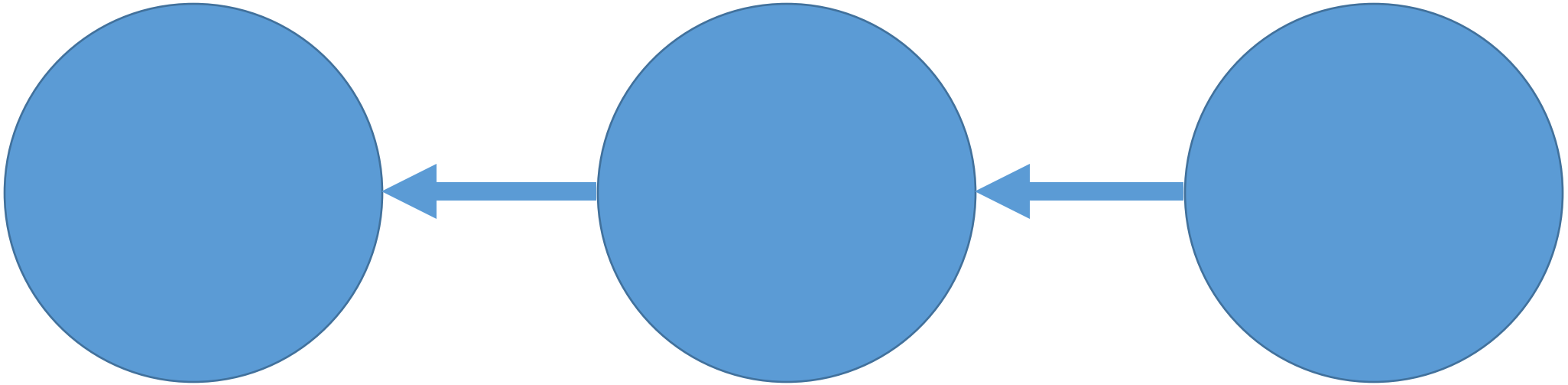
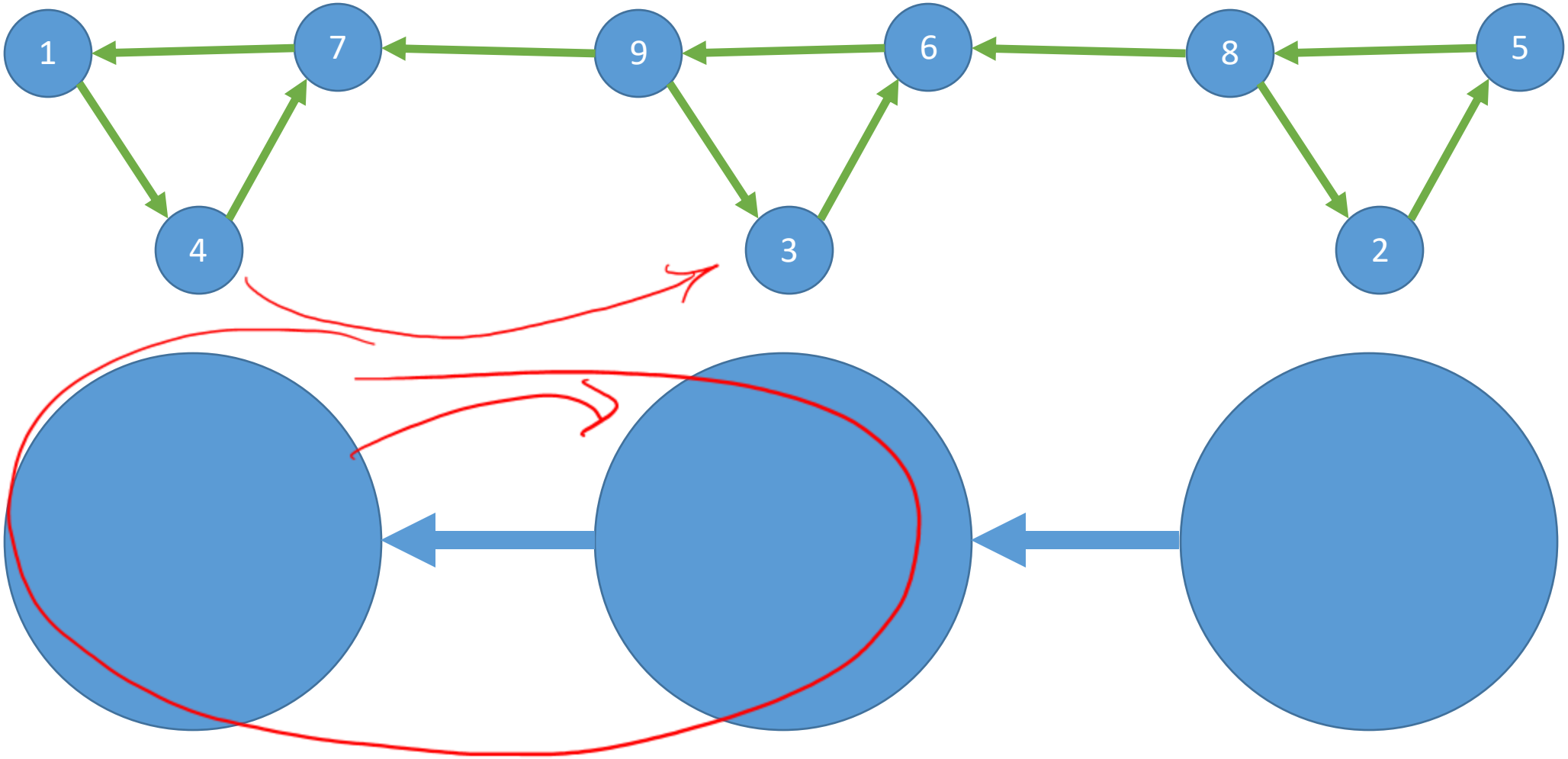
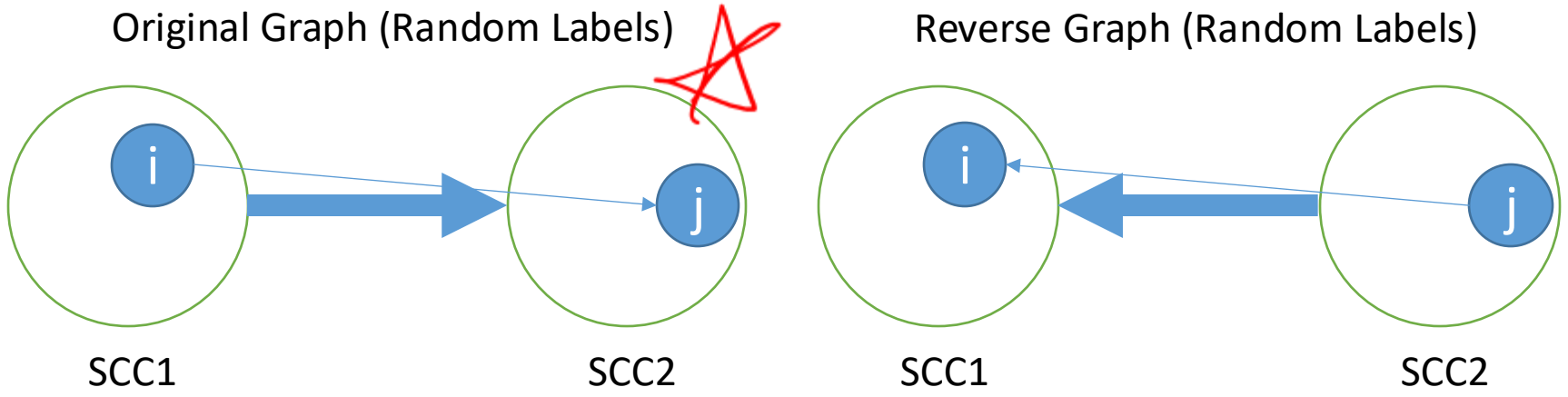


How do we know that the SCC based meta-graph is acyclic?



Key Lemma

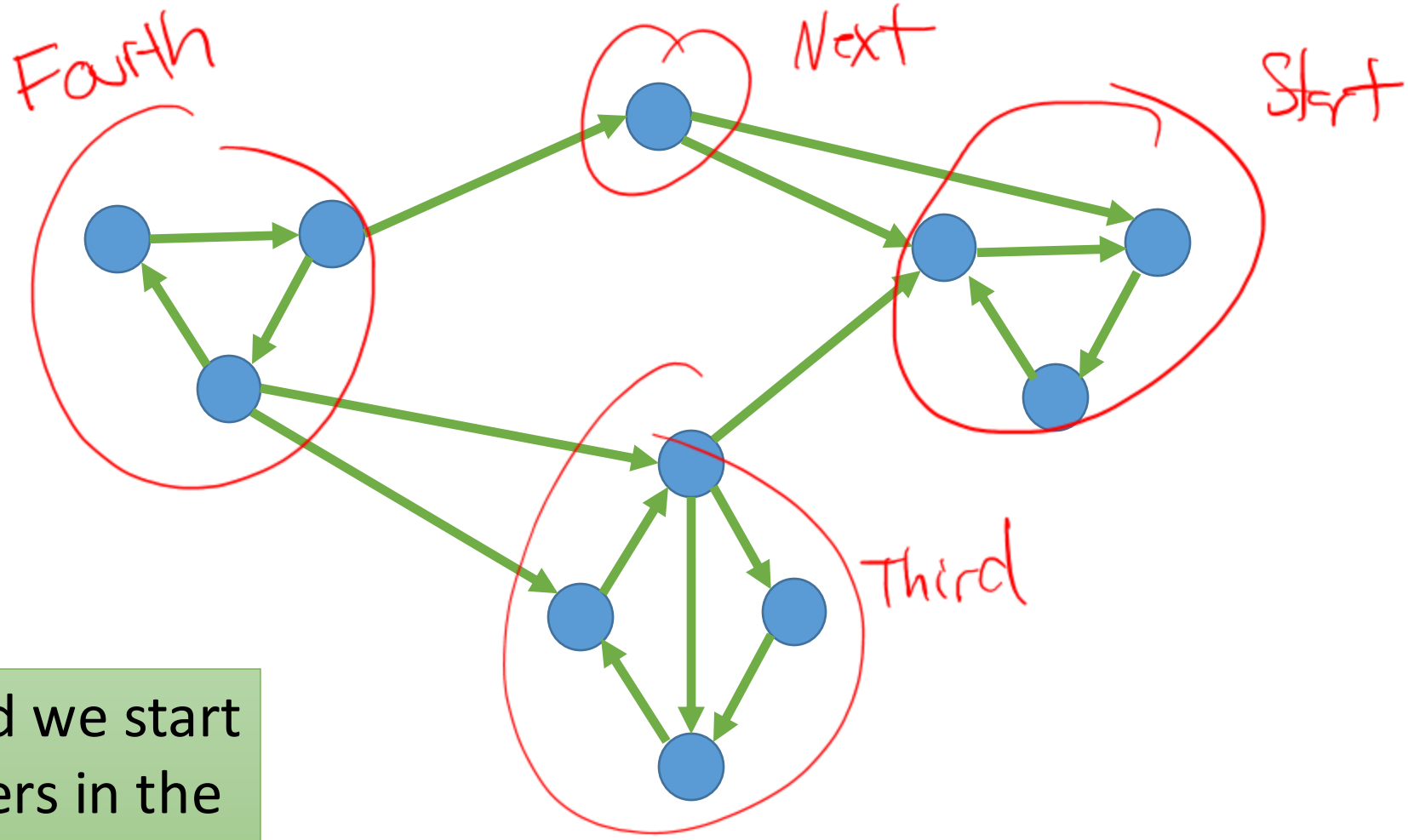


- Consider the two adjacent SCCs in the meta-graph above
- Now consider the re-labeling found from the reverse graph

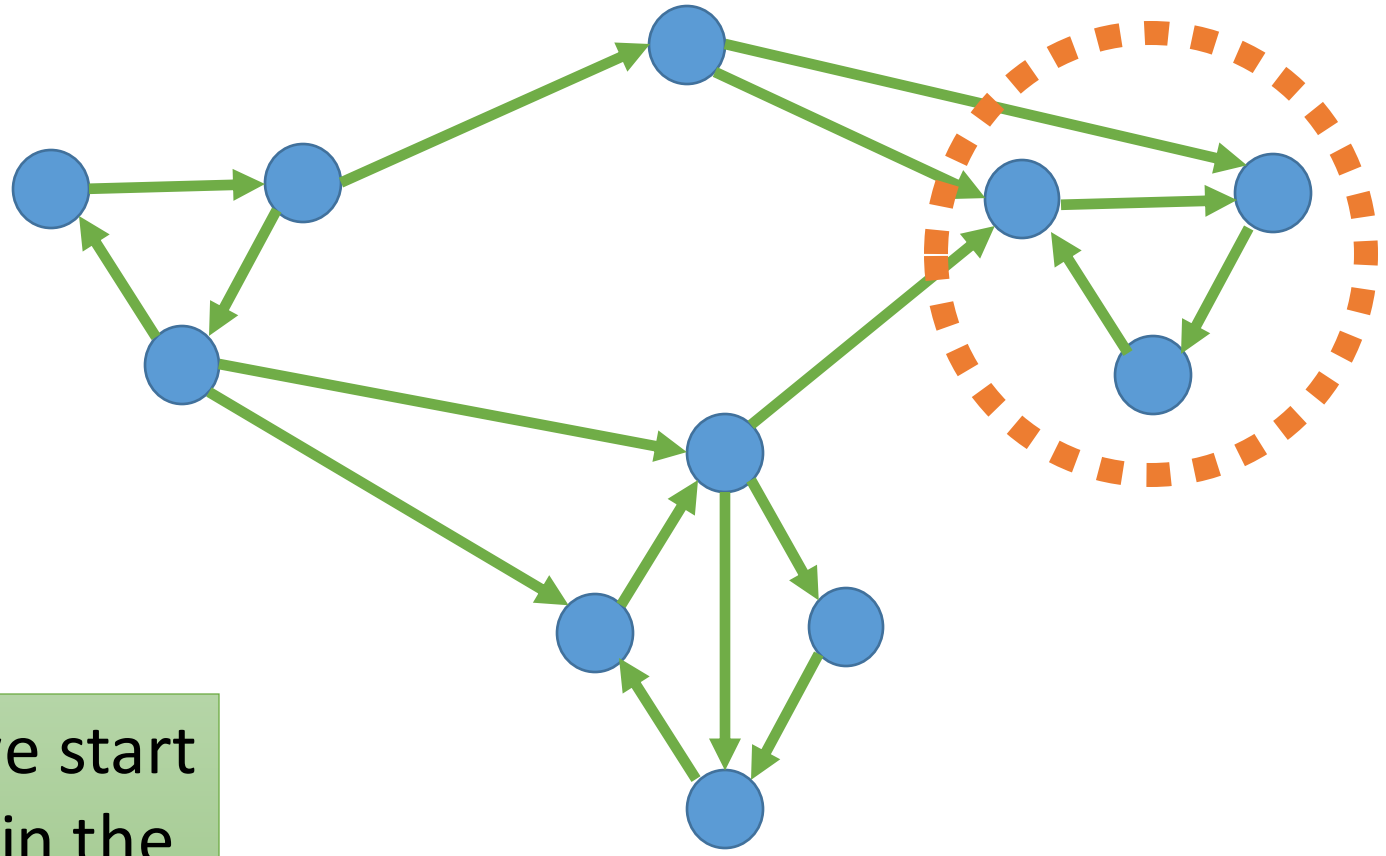
Where do we want to start DFS in the leaders pass?

- Let f(v) = the re-labeling resulting from `KosarajuLoop(G_reversed)`
- Then `max[f(.) in SCC1] < max[f(.) in SCC2]`
- Corollary: the maximum label must lie in a "sink SCC" of the original graph

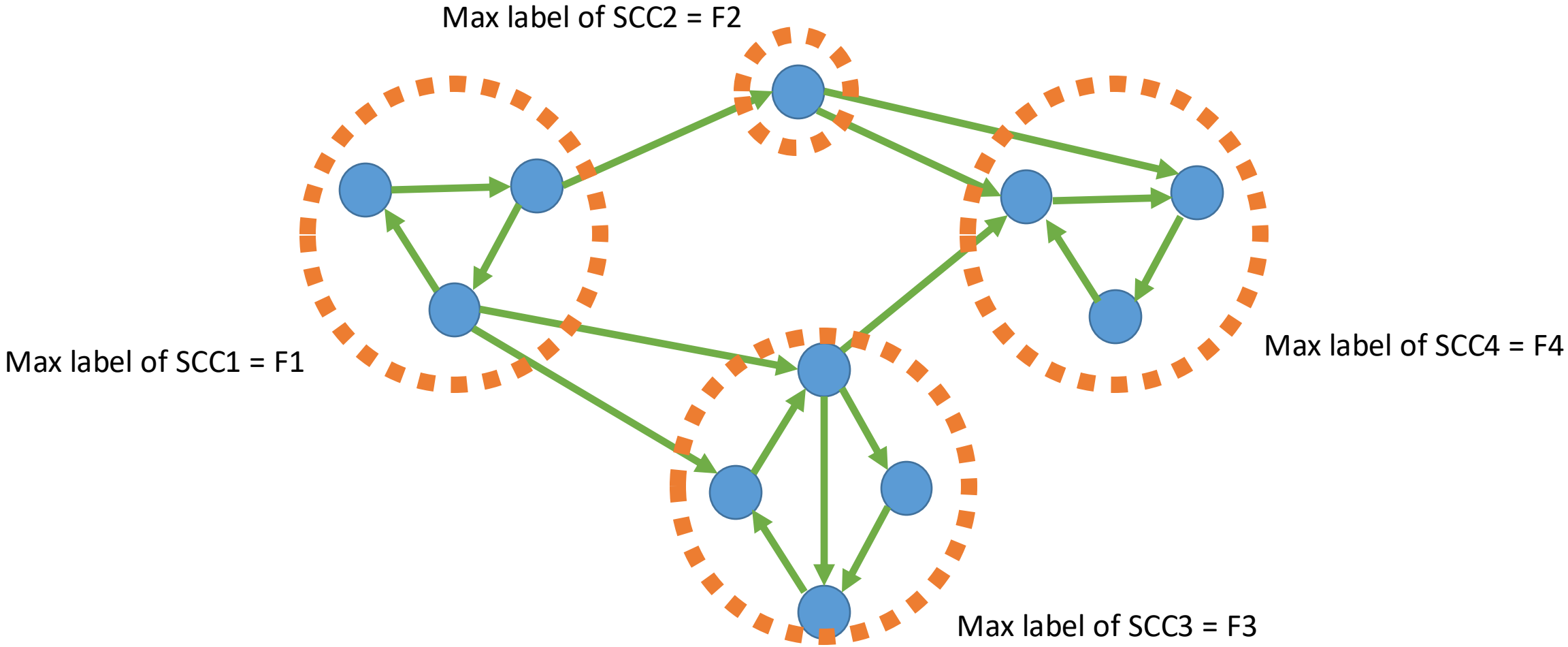
```
FUNCTION KosarajuDFS(…)
   found[v] = TRUE
   leaders[v] = leader
   FOR vOther IN G.edges[v]
      IF found[vOther] == FALSE
         KosarajuDFS(…)
   label = label + 1
   labels[v] = label
```



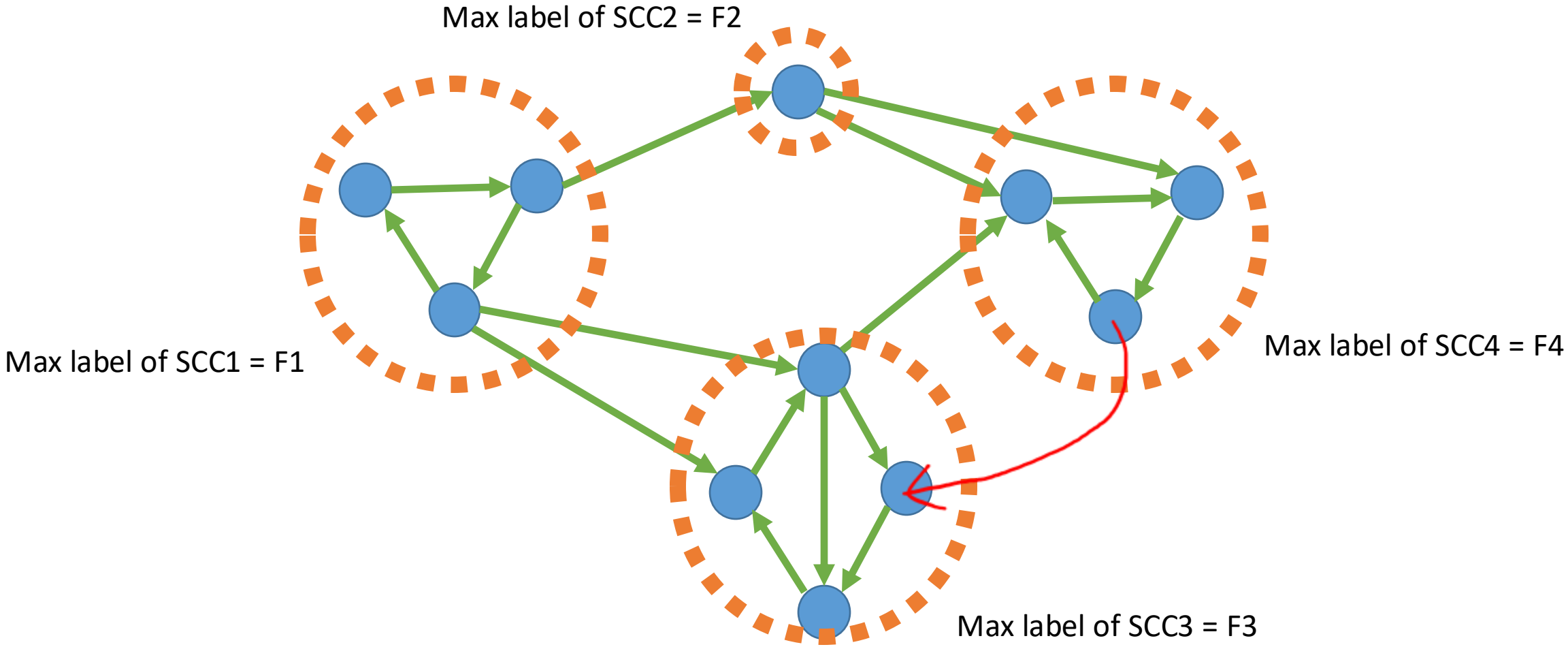
Where should we start labeling leaders in the second pass?



Where should we start labeling leaders in the second pass?



Then F1 < {F2, F3} < F4

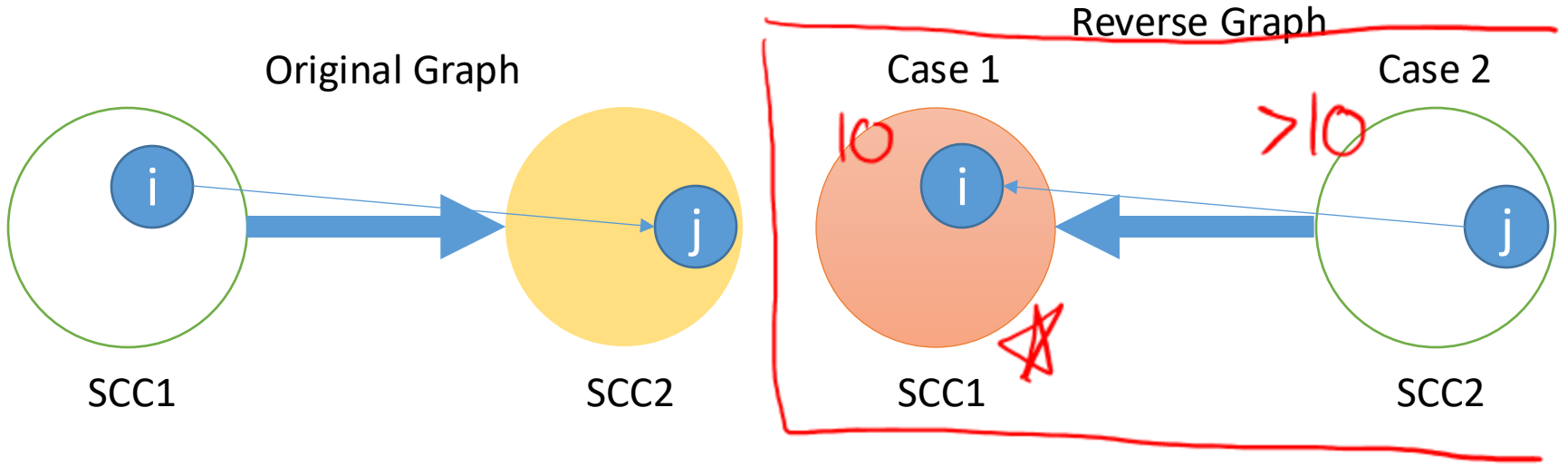


Then F1 < {F2, F3} < F4

What would happen if SCC4 had a link back to SCC3?

Proof of Lemma

Finding New labels



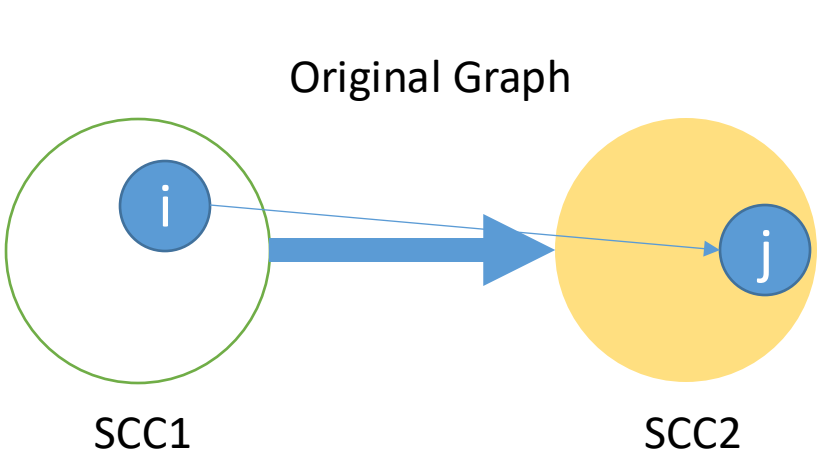
Case 1: consider the case when the first vertex that we explore is in SCC1

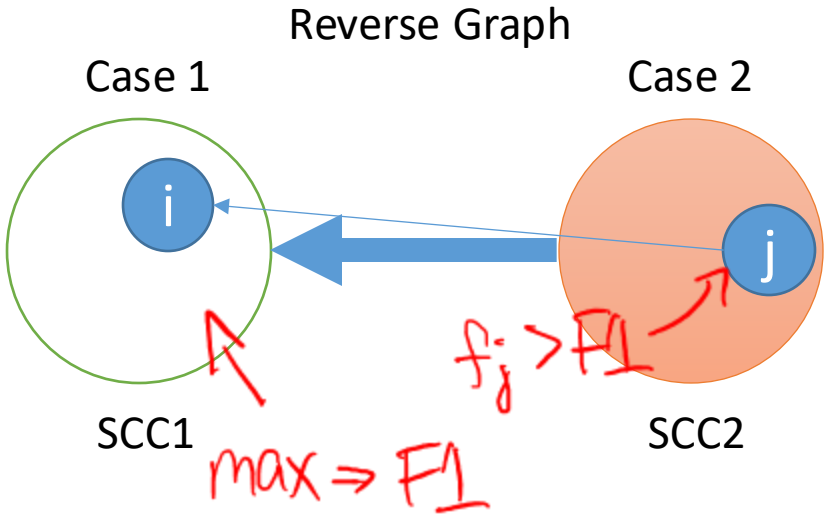
labeled

- Then all SCC1 is ~~explored~~ before SCC2
- Therefore, all labels in SCC1 are less than all labels in SCC2
- So, in the original graph we will start in SCC2 (the sink)

```
FUNCTION KosarajuDFS(…)
   found[v] = TRUE
   leaders[v] = leader
   FOR vOther IN G.edges[v]
      IF found[vOther] == FALSE
         KosarajuDFS(…)
   label = label + 1
   labels[v] = label
```

Proof of Lemma





Case 2: consider the case when the first vertex that we explore is in SCC2

- All other vertices in SSC2 are explored before vertex j
- All vertices in SSC1 are ~~explored~~ labeled before vertex j
- Therefore, all labels in SSC1 and SSC2 are less than the label of vertex j
- So, in the original graph we will start at vertex j in SSC2 (the sink)

```
FUNCTION KosarajuDFS(…)
   found[v] = TRUE
   leaders[v] = leader
   FOR vOther IN G.edges[v]
      IF found[vOther] == FALSE
         KosarajuDFS(…)
   label = label + 1
   labels[v] = label
```

What does this mean?

- We'll start the second `KosarajuLoop` at an "SCC sink"
- That sink will then be *removed* (by marking all vertices in the SCC as explored) and we'll next move to the newly created sink
- And so on

Why reverse

Kosaraju's Algorithm Summary

Computes the SCCs in O(m + n) time **(linear!)**

1. Create a reverse version of the G called G_reversed

2. Run KosarajuLoop on G_reversed
   - Create a topological ordering on the meta graph

3. Create a relabeled version of the G called G_relabeled

4. Run KosarajuLoop on G_relabeled
   - Find all nodes with the same "leader"