Asymptotic Notation (Big O)

https://cs.pomona.edu/classes/cs140/

Outline

Topics and Learning Objectives

- Discuss total running time
- Discuss asymptotic running time
- Learn about asymptotic notation

Exercise

- Running time

Extra Resources

- Chapter 3: asymptotic notation

Comparing Algorithms and Data Structures

We like to compare algorithms and data structures

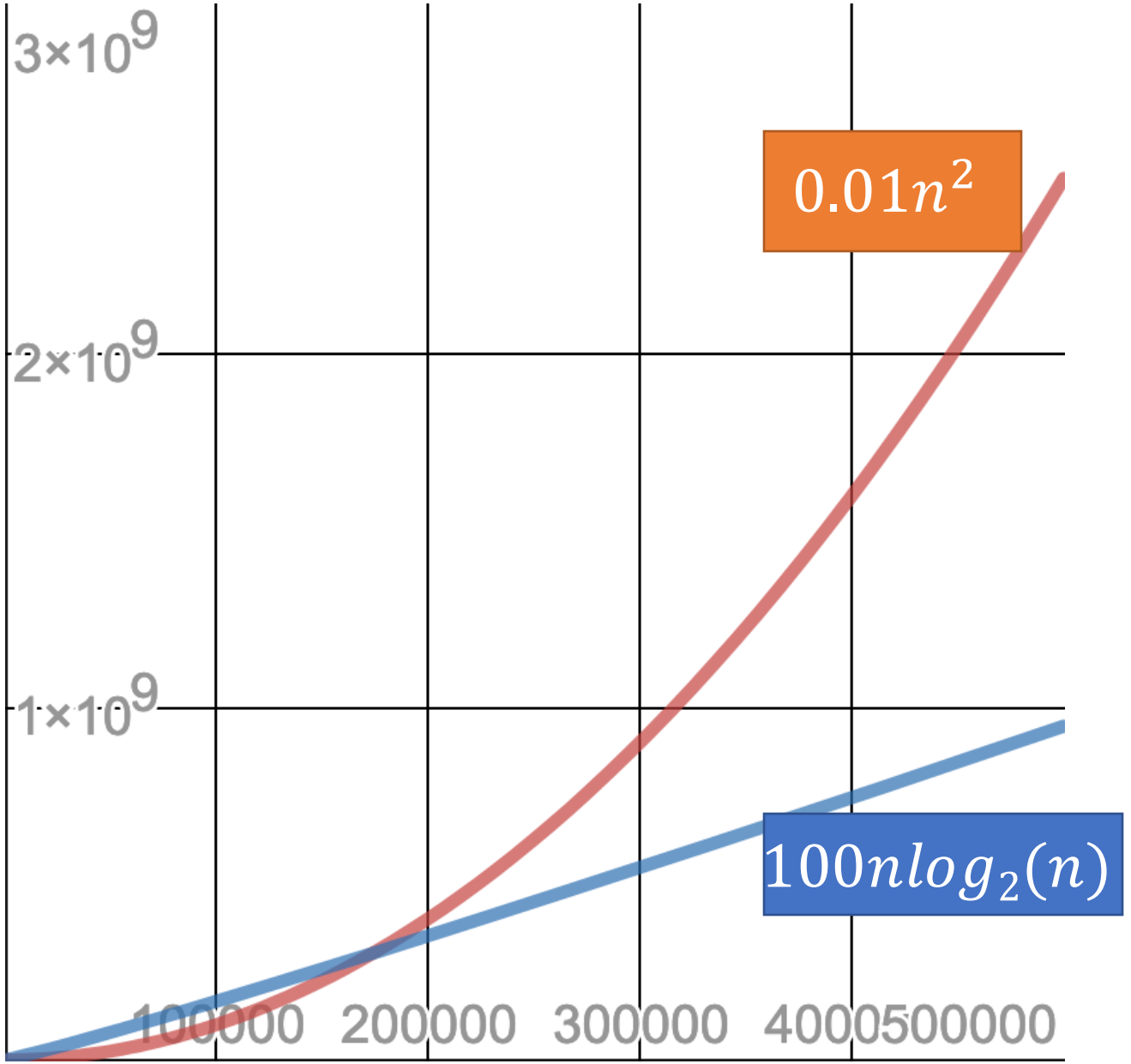
- Speed
- Memory usage

We don’t always need to care about little details

We ignore some details anyway

- Data locality
- Differences among operations

Constants


3×10^9
2×10^9
1×10^9
100000
200000
300000
400000
500000
$0.01n^2$
$100nlog_2(n)$

Big-O Example Code (ODS 1.3.3)

```
# function_one has a total running time of 2nlogn + 2n - 250
a = function_one(input_one)

# function_two has a total running time of 3nlogn + 6n + 48
b = function_two(input_two)
```

- The total running time of the code above is:

$$2n \log n + 2n - 250 + 1 + 3n \log n + 6n + 48 + 1$$

$$5n \log n + 8n - 200$$

Big-O Example Math (ODS 1.3.3)

$$5n \log n + 8n - 200$$

- We don't care about most of these details
- We want to be able to quickly glance at the running time of an algorithm and know how it compares to others
- So we say the following

$$5n \log n + 8n - 200 = O(n \log n)$$

Big-O (**Asymptotic Running Time**)

$$T(n) = O(f(n))$$

If and only if (iff) we can find values for c, $n_0 > 0$, such that

$$T(n) \leq c\ f(n), \text{ where } n \geq n_0$$

Note: c, n_0 <u>cannot</u> depend on n

Searching an array for a given number?

Write an algorithm (in pseudocode): **What is the total running time?**

```
Function FindNum(array, num)
{For val In array
  If val == num
    {Return True
{Return False
```

what is the total running time?

$T(n) = 2n + 1$

$T(n) \leq c\ f(n)$, where $n \geq n_0$

Searching an array for a given number?

What is the asymptotic running time? T(n) = 2n + 1

$T(n) = O(?)$

$T(n) = O(n)$

$f(n)$

$3n \leq cn$

$\forall\ n \geq n_0$

$\star\ T(n) \leq c\,n$

$\forall\ n \geq n_0$

$2n + 1 \leq cn \quad \forall\ n \geq n_0$

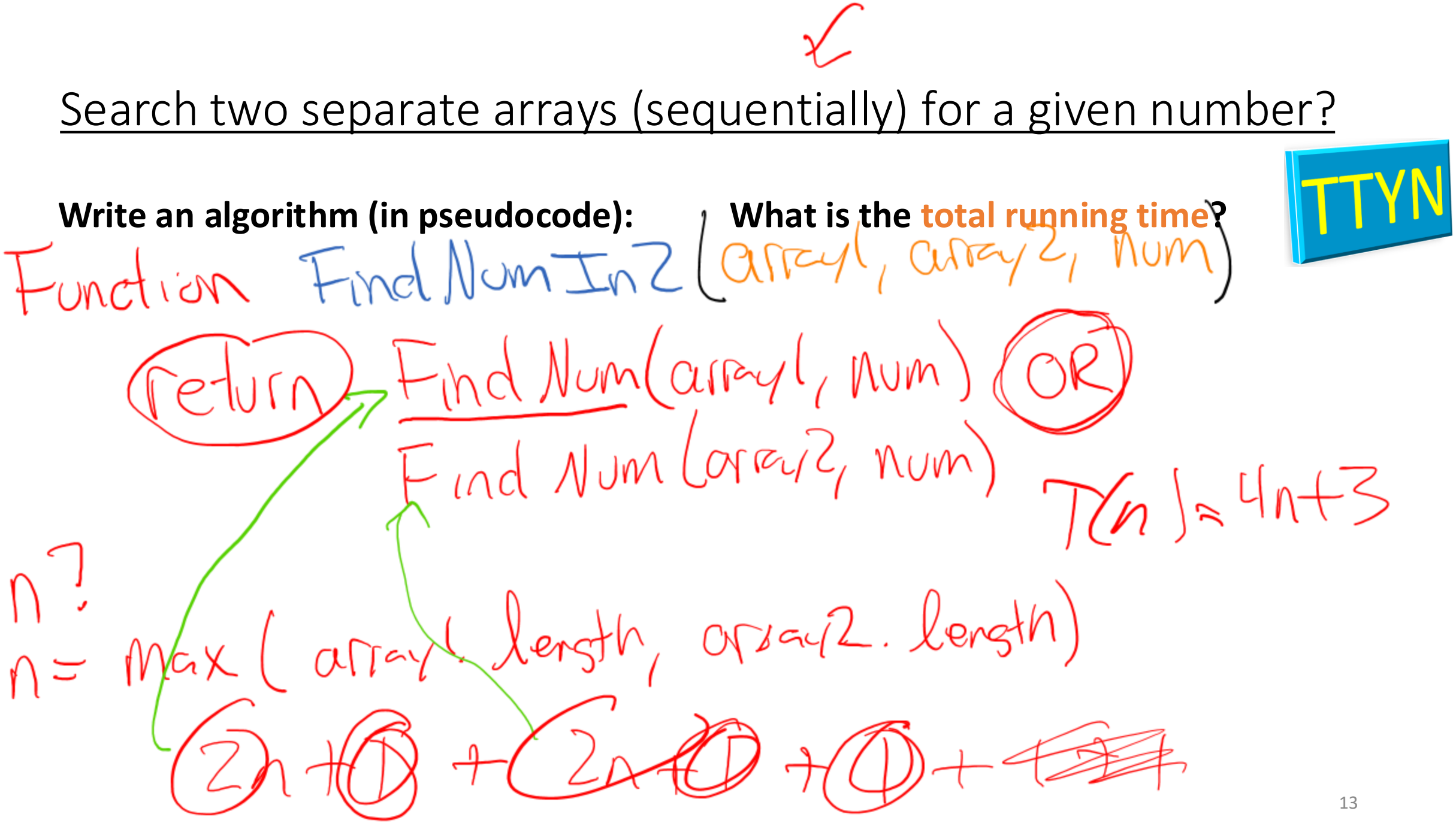
$2n + 1 \leq 2n + n \quad n \geq 1$

$c = 3$

$n_0 = 1$

Search two separate arrays (sequentially) for a given number?

Write an algorithm (in pseudocode): **What is the total running time?**



Search two separate arrays (sequentially) for a given number?

What is the asymptotic running time? T(n) = 4n + 3 = O(n)

$T(n) = O(n)$

$4n + 3 \leq cn \quad \forall\ n \geq n_0$

$4n + 3 \leq 4n + 3n \leq cn$

$\forall\ n \geq n_0$

$4n + 3 \leq 4n + 3n$

$3 \leq 3n \rightarrow n \geq 1$

1

2

$4n + 3n \leq cn$

$7n \leq cn$

$c = 7,\ n_0 = 1$

Naïve

Hash Table → O(n)

Searching two arrays for any common number?

Write an algorithm (in pseudocode): **What is the total running time?**

```
Function FindCommon(array1, array2)
n   For val1 In array1
n       If FindNum(array2, val1)        (2n+1)
            Return True
    Return False
```

$T(n) = n + n(2n+1) + 1$

$= 2n^2 + n + n + 1$

$T(n) \leq c\ f(n)$, where $n \geq n_0$

Searching two arrays for any common number?

What is the asymptotic running time? $T(n) = 2n^2 + 2n + 1$

$T(n) \neq O(n)$

$\frac{2n^2}{n} + \frac{2n+1}{n} \leq \frac{cn}{n}$ $\forall n \geq n_0$

$2n + 2 + \frac{1}{n} \leq c$

$202 + 1 = 203$

$1,000,000$

$n_0 = ?$

$= 200$

$T(n) \leq c\ f(n)$, where $n \geq n_0$

Searching two arrays for any common number?

What is the asymptotic running time? $T(n) = 2n^2 + 2n + 1$

$T(n) = O(n^2)$

$2n^2 + 2n + 1 \leq cn^2$

$2n^2 + 2n + 1 \leq 2n^2 + 2n^2 + 1n^2 \leq cn^2 \quad \forall n \geq n_0$

$2n^2 + 2n + 1 \leq 2n^2 + 2n^2 + n^2$

$2n \leq 2n^2 \qquad 1 \leq n^2$

$1 \leq n \qquad n \geq 1$

$2n^2 + 2n^2 + n^2 \leq cn^2 \quad \forall n \geq n_0$

$5n^2 \leq cn^2$

$c = 5, \ n_0 = 1$

Searching a single array for duplicate numbers?

Write an algorithm (in pseudocode): **What is the total running time?**

```
Function FindDuplicate(array)
    array = MergeSort(array)        ↙ 2lnlgn + 2ln
    For i In [1..<array.length]     ↙ 1n
        If array[i-1] == array[i]   ↙ 3n
            Return True
    Return False + 1
```

2lnlgn + 2ln + 4n + 1

2lnlgn + 2n + 1

Searching a single array for duplicate numbers?

What is the asymptotic running time? T(n) = 21nlgn + 25n + 1

$T(n) = O(n \lg n)$

$\frac{21 n \lg n}{n \lg n} + \frac{25n}{n \lg n} + \frac{1}{n \lg n} \leq \frac{c\, n \lg n}{n \lg n} \quad \forall\ n \geq n_0$

$\rightarrow 21 + \frac{25}{\lg n} + \frac{1}{n \lg n} \leq c \quad \forall\ n \geq n_0$

$n \geq 2^{25}$

$\frac{25}{\lg n} \leq 1 \rightarrow \frac{25}{\lg 2^{25}} \rightarrow \frac{25}{25 \lg 2} \rightarrow \frac{1}{\lg 2} \rightarrow 1 \not< 1$

26

$T(n) \leq c\ f(n)$, where $n \geq n_0$

Searching a single array for duplicate numbers?

What is the asymptotic running time? T(n) = 21nlgn + 25n + 1 $= O(n \lg n)$

$21 + 1 + \frac{1}{n \lg n} \leq c \quad \forall\ n \geq n_0$

$n_0 \geq 2^{25}$

$\frac{1}{n \lg n} \leq 1$

$21 + 1 + 1 \leq c \quad \forall\ n \geq 2^{25}$

$n \lg n \geq 1$

$n \geq 2$ ✓

$c = 23, \quad n_0 = 2^{25}$.

Exercise

Big-O Examples

$T(n) = O(f(n))$

If and only if we can find values for c, $n_0 > 0$, such that

$T(n) \leq c\ f(n)$, where $n \geq n_0$

Note: c, n_0 cannot depend on n

- Claim: $2^{n+10} = O(2^n)$

$2^{n+10} \leq c\,2^n \quad \forall\ n \geq n_0$

$2^n 2^{10} \leq c\,2^n \quad \forall\ n \geq n_0$

$c = 2^{10}, \quad n_0 = 1$

Big-O Examples

$T(n) = O(f(n))$

If and only if we can find values for c, $n_0 > 0$, such that

$T(n) \leq c\ f(n)$, where $n \geq n_0$

Note: c, n_0 cannot depend on n

- Claim: $2^{10n} \neq O(2^n)$

$$\frac{2^{10n}}{2^n} \leq c\,2^n \quad \forall\ n \geq n_0$$

$$2^{10n-n} \leq c$$

$$2^{9n} \leq c \quad \forall\ n \geq n_0$$

$$2^{10n} \neq O(2^n)$$

Big-O Examples

$T(n) = O(f(n))$

If and only if we can find values for c, $n_0 > 0$, such that

$T(n) \leq c\ f(n)$, where $n \geq n_0$

Note: c, n_0 cannot depend on n

- Claim: for every k >= 1, n^k is **not** $O(n^{k-1})$

$\forall k \geq 1 \quad n^k \neq O(n^{k-1})$

$n^k \leq c\, n^{k-1} \quad \forall n \geq n_0$

$n^k \leq c\, n^k n^{-1} \quad \forall n \geq n_0$

$n \leq c \quad \forall n \geq n_0$

$\square$ Claim is true

Θ Examples

$T(n) = \Theta(f(n))$

If and only if we can find values for c, $n_0 > 0$, such that

$c_1\ f(n) \leq T(n) \leq c_2\ f(n)$, where $n \geq n_0$

Note: c_1, c_2, n_0 cannot depend on n

- Claim: $21n\ (\log_2(n) + 1) = \Theta(n\log_2 n)$

Other Notations

- Big-O (≤) : $T(n) = O(f(n))$ if $T(n) \leq c\, f(n)$, where $n \geq n_0$
- Big-Omega (≥) : $T(n) = \Omega(f(n))$ if $T(n) \geq c\, f(n)$, where $n \geq n_0$
- Theta (=) : $T(n) = \Theta(f(n))$ if $T(n) = O(f(n))$ and $T(n) = \Omega(f(n))$

$$c_1\, f(n) \leq T(n) \leq c_2\, f(n), \text{ where } n \geq n_0$$

Other Notations

- Big-O (≤) : $T(n) = O(f(n))$ if $T(n) \leq c\ f(n)$, where $n \geq n_0$
- little-o (<)

- Big-Omega (≥) : $T(n) = \Omega(f(n))$ if $T(n) \geq c\ f(n)$, where $n \geq n_0$
- Little-omega (>)

Θ Examples

$T(n) = \Theta(f(n))$

If and only if we can find values for c, $n_0 > 0$, such that

$c_1\ f(n) \le T(n) \le c_2\ f(n)$, where $n \ge n_0$

Note: c_1, c_2, n_0 cannot depend on n

Big-O upper bound

- Claim: $21n\ (\log_2(n) + 1) = \Theta(n\log_2 n)$

$21n \lg n + 21n \le c_2 n \lg n \quad \forall n \ge n_0$

$21 n \lg n + 21n \le 21 n \lg n + 21 n \lg n \le c_2 n \lg n$

$21 n \lg n \le 21 n \lg n$ ✓

$42 n \lg n \le c_2 n \lg n$

$21n \le 21n \lg n$

$1 \le \lg n \quad n \ge 2$

$c_2 = 42, \quad n_0 = 2$

Θ Examples

$T(n) = \Theta(f(n))$

If and only if we can find values for c, $n_0 > 0$, such that

$c_1\ f(n) \leq T(n) \leq c_2\ f(n)$, where $n \geq n_0$

Note: c_1, c_2, n_0 cannot depend on n

Big-O

- Claim: $21n\ (\log_2(n) + 1) = \Theta(n\log_2 n)$

$c_2 = 22$

$n_0 = 2^{21}$

Θ Examples

$T(n) = \Theta(f(n))$

If and only if we can find values for $c, n_0 > 0$, such that

$c_1 f(n) \leq T(n) \leq c_2 f(n)$, where $n \geq n_0$

Note: c_1, c_2, n_0 cannot depend on n

42 2

- Claim: $21n(\log_2(n) + 1) = \Theta(n\log_2 n)$

$c_1 n \lg n \leq 21 n \lg n + 21n \quad \forall n \geq n_0$

$c_1 n \lg n \leq 21 n \lg n \leq 21 n \lg n + 21n$

$n \geq 1$

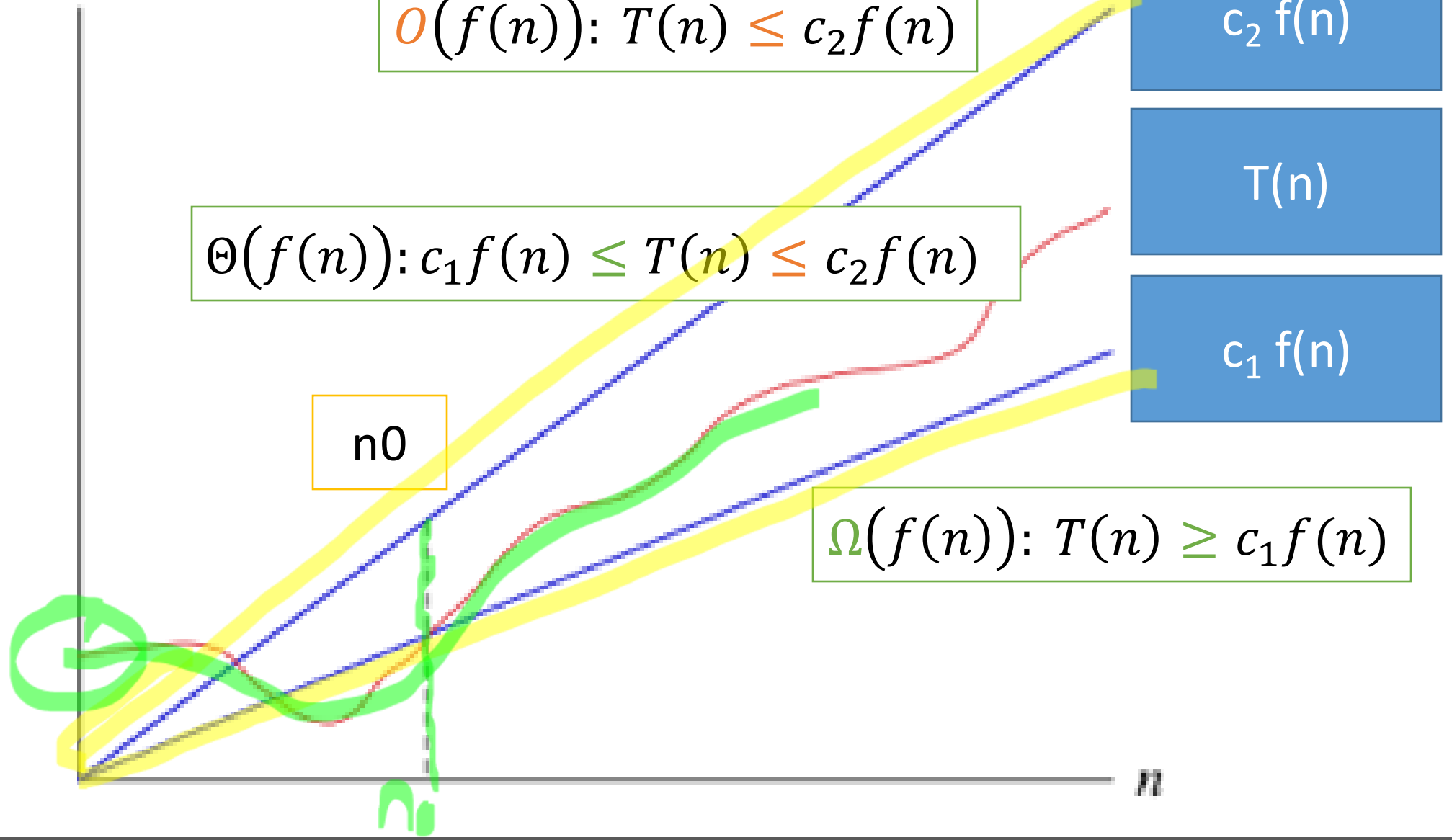
$c_1 n \lg n \leq 21 n \lg n$

$c_1 = 21, \; c_2 = 42, \; n_0 = 2$

$O(f(n))$: $T(n) \leq c_2 f(n)$

$\Theta(f(n))$: $c_1 f(n) \leq T(n) \leq c_2 f(n)$

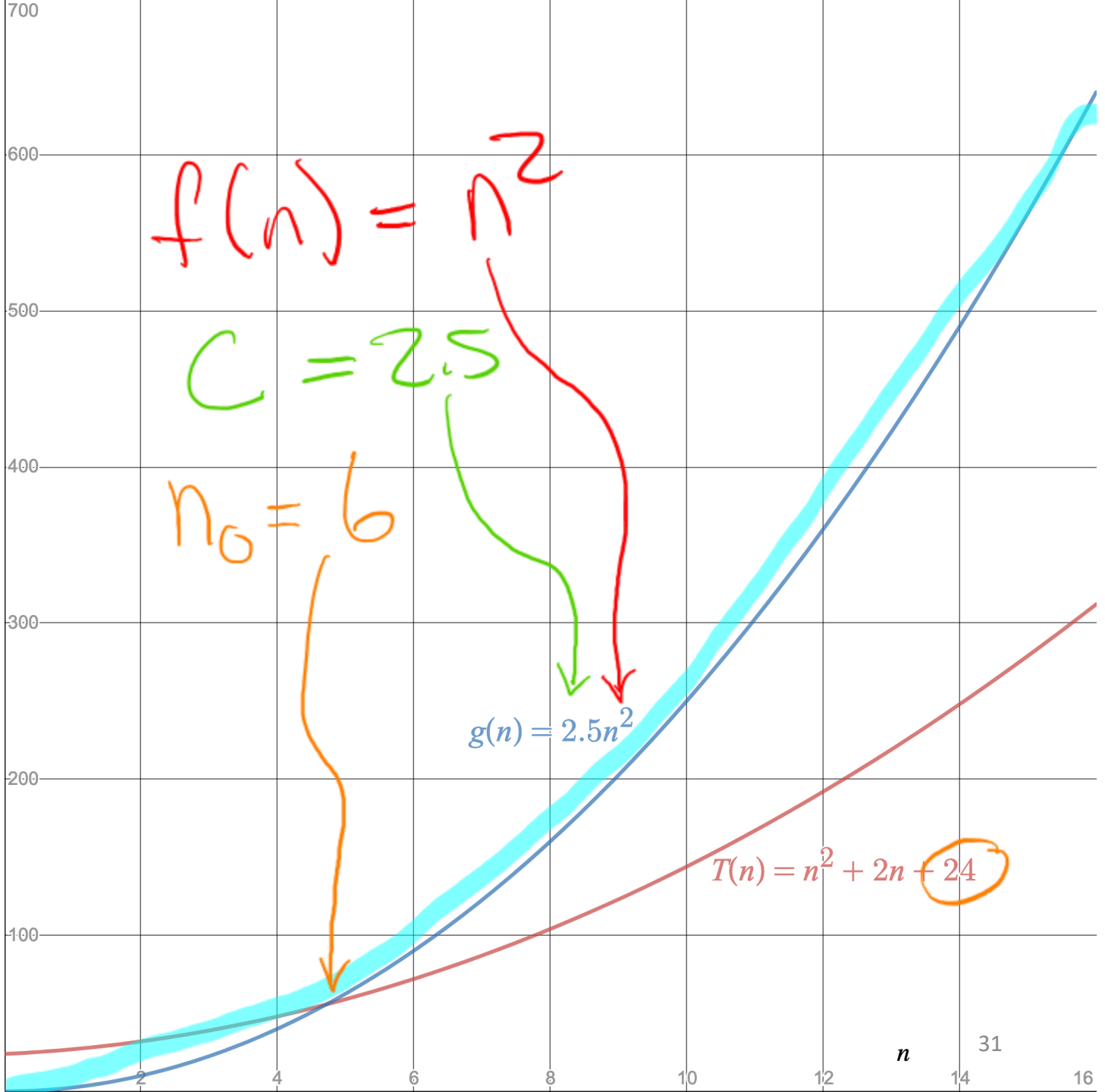
$\Omega(f(n))$: $T(n) \geq c_1 f(n)$



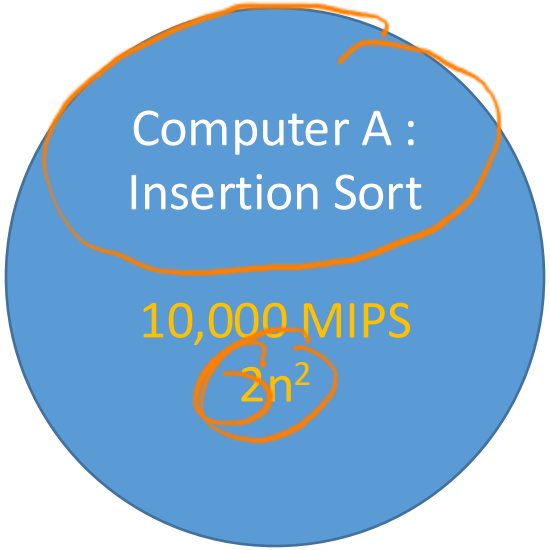
What is f(n)?

What are good values for:

- c
- n_0

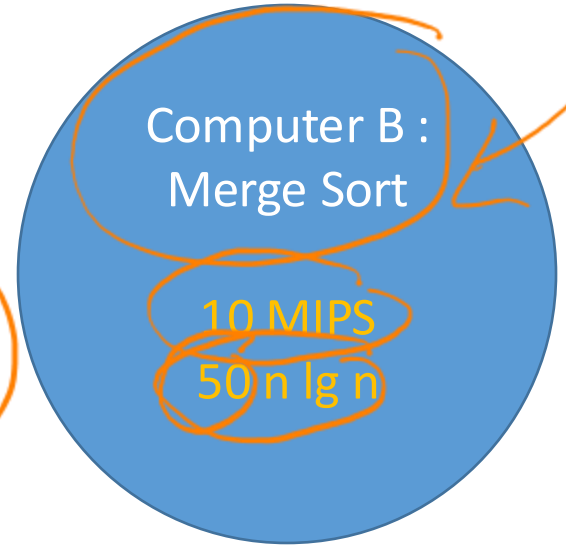


Insertion Sort vs Merge Sort



$O(n^2)$

$O(n \lg n)$



5.5 hours	10 million numbers	20 minutes
23 days	100 million numbers	4 hours

Simplifying the Comparison

- Why can we remove leading coefficients?
- Why can we remove lower order terms?

- They are both insignificant when compared with the growth of the function.
- They both get factored into the constant “c”