More Recurrences

David Kauchak
cs140
Spring 2024

1

Administrative

Group sessions

Assignment 1

2

Recurrence

A function that is defined with respect to itself on smaller inputs

$$T(n) = 2T(n/2) + n$$

$$T(n) = 16T(n/4) + n$$

$$T(n) = 2T(n-1) + n^2$$

3

The challenge

Recurrences are often easy to define because they mimic the structure of the program

But… they do not directly express the computational cost, i.e. n, n^2, …

We want to remove self-recurrence and find a more understandable form for the function

4

Three approaches

Substitution method: when you have a good guess of the solution, prove that it's correct

Recursion-tree method: If you don't have a good guess, the recursion tree can help. Then solve with substitution method.

Master method: Provides solutions for recurrences of the form:

$$T(n) = aT(n/b) + f(n)$$

5

Recursion Tree

Guessing the answer can be difficult

$$T(n) = 3T(n/4) + n^2$$
$$T(n) = T(n/3) + 2T(2n/3) + cn$$

The recursion tree approach

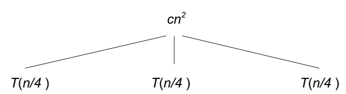
- Draw out the cost of the tree at each level of recursion
- Sum up the cost of the levels of the tree
  - Find the cost of each level with respect to the depth
  - Figure out the depth of the tree
  - Figure out (or bound) the number of leaves
- Verify your answer using the substitution method

12

$$T(n) = 3T(n/4) + n^2$$

cost

cn^2



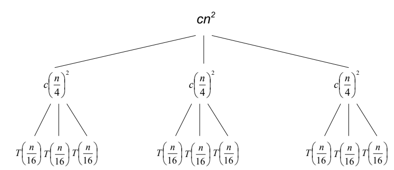
13

$$T(n) = 3T(n/4) + n^2$$

cost

cn^2

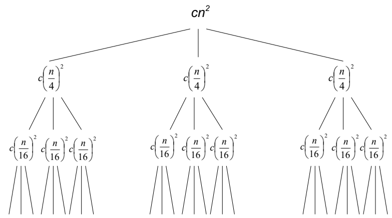
$3/16cn^2$



14

$$T(n) = 3T(n/4) + n^2$$

cost



cn^2

$3/16cn^2$

$(3/16)^2cn^2$

What is the cost at each level? $\left(\frac{3}{16}\right)^d cn^2$

15

What is the depth of the tree?

At each level, the size of the data is divided by 4

$$\frac{n}{4^d} = 1$$

$$\log\left(\frac{n}{4^d}\right) = 0$$

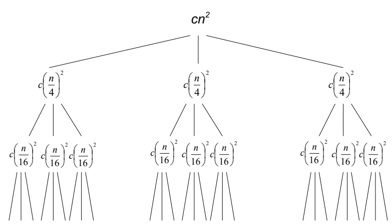
$$\log n - \log 4^d = 0$$

$$d \log 4 = \log n$$

$$d = \log_4 n$$

16

$$T(n) = 3T(n/4) + n^2$$



$$T(1)$$

How many leaves are there?

17

How many leaves?

How many leaves are there in a complete ternary tree of depth *d*?

$$3^d = 3^{\log_4 n}$$

18

Total cost

$$T(n) = cn^2 + \frac{3}{16}cn^2 + \left(\frac{3}{16}\right)^2 cn^2 + \ldots + \left(\frac{3}{16}\right)^{d-1} cn^2 + \Theta(3^{\log_4 n})$$

$$= cn^2 \sum_{i=0}^{\log_4 n - 1} \left(\frac{3}{16}\right)^i + \Theta(3^{\log_4 n})$$

$$< cn^2 \sum_{i=0}^{\infty} \left(\frac{3}{16}\right)^i + \Theta(3^{\log_4 n})$$

$$= \frac{1}{1-(3/16)} cn^2 + \Theta(3^{\log_4 n})$$

$$= \frac{16}{13} cn^2 + \Theta(3^{\log_4 n})$$?

$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$

let x = 3/16

19

Total cost $T(n) = \frac{16}{13}cn^2 + \Theta(3^{\log_4 n})$

$$3^{\log_4 n} = 4^{\log_4 3^{\log_4 n}}$$

$$= 4^{\log_4 n \log_4 3}$$

$$= 4^{\log_4 n^{\log_4 3}}$$

$$= n^{\log_4 3}$$

Assignment 0!

$$T(n) = \frac{16}{13}cn^2 + \Theta(n^{\log_4 3})$$

$$T(n) = O(n^2)$$

20

Recursion tree

If you went through the exact calculation (like we just did), you can be done!

Often, this isn't feasible (or desirable)

Instead, use the recursion tree to get a good guess

21

Verify solution using substitution

$$T(n) = 3T(n/4) + n^2$$

Assume $T(k) = O(k^2)$ for all $k < n$

Show that $T(n) = O(n^2)$

Given that $T(n/4) = O((n/4)^2)$, then

$$O(g(n)) = \left\{ f(n) : \begin{array}{l} \text{there exists positive constants } c \text{ and } n \text{ such that} \\ 0 \le f(n) \le cg(n) \text{ for all } n \ge n_0 \end{array} \right\}$$

$T(n/4) \le c(n/4)^2$

22

$$T(n) = 3T(n/4) + n^2$$

To prove that Show that *T(n) = O(n²)* we need to identify the appropriate constants:

$$O(g(n)) = \left\{ f(n) : \begin{array}{l} \text{there exists positive constants } c \text{ and } n \text{ such that} \\ 0 \le f(n) \le cg(n) \text{ for all } n \ge n_0 \end{array} \right\}$$

i.e. some constant *c* such that $T(n) \le cn^2$

$$\begin{aligned} T(n) &= 3T(n/4) + n^2 \\ &\le 3c(n/4)^2 + n^2 \\ &= cn^2 3/16 + n^2 \\ &= cn^2 - cn^2\frac{13}{16} + n^2 \end{aligned}$$

residual

a constant exists if $-cn^2\frac{13}{16} + n^2 \le 0$

23

$$T(n) = 3T(n/4) + n^2$$

To prove that Show that *T(n) = O(n²)* we need to identify the appropriate constants:

$$O(g(n)) = \left\{ f(n) : \begin{array}{l} \text{there exists positive constants } c \text{ and } n \text{ such that} \\ 0 \le f(n) \le cg(n) \text{ for all } n \ge n_0 \end{array} \right\}$$

i.e. some constant *c* such that $T(n) \le cn^2$

$$-cn^2 * \frac{13}{16} + n^2 \le 0$$

$$cn^2 * \frac{13}{16} \ge n^2$$

$$c \ge \frac{16}{13}$$

24

Master Method

Provides solutions to the recurrences of the form:

$$T(n) = aT(n/b) + f(n)$$

if $f(n) = O(n^{\log_b a - \varepsilon})$ for $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$

if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$

if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for $\varepsilon > 0$ and $af(n/b) \le cf(n)$ for $c < 1$

then $T(n) = \Theta(f(n))$

25

$$T(n) = 16T(n/4) + n$$

if $f(n) = O(n^{\log_b a - \varepsilon})$ for $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$

if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$

if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for $\varepsilon > 0$ and $af(n/b) \le cf(n)$ for $c < 1$

then $T(n) = \Theta(f(n))$

a = 16

b = 4

f(n) = *n*

$$n^{\log_b a} = n^{\log_4 16} = n^2$$

is $n = O(n^{2-\varepsilon})$?

is $n = \Theta(n^2)$?

is $n = \Omega(n^{2+\varepsilon})$?

Case 1: $\Theta(n^2)$

26

$$T(n) = T(n/2) + 2^n$$

if $f(n) = O(n^{\log_b a - \varepsilon})$ for $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$
if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$
if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for $\varepsilon > 0$ and $af(n/b) \le cf(n)$ for $c < 1$
then $T(n) = \Theta(f(n))$

a = 1
b = 2
f(n) = 2^n

$n^{\log_b a} = n^{\log_2 1}$
$= n^0$

is $2^n = O(n^{0-\varepsilon})$?
is $2^n = \Theta(n^0)$?
is $2^n = \Omega(n^{0+\varepsilon})$?

Case 3?
is $2^{n/2} \le c2^n$ for $c < 1$?

27

$$T(n) = T(n/2) + 2^n$$

if $f(n) = O(n^{\log_b a - \varepsilon})$ for $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$
if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$
if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for $\varepsilon > 0$ and $af(n/b) \le cf(n)$ for $c < 1$
then $T(n) = \Theta(f(n))$

is $2^{n/2} \le c2^n$ for $c < 1$?

Let c = 1/2

$2^{n/2} \le (1/2)2^n$
$2^{n/2} \le 2^{-1}2^n$
$2^{n/2} \le 2^{n-1}$

T(*n*) = Θ(2*ⁿ*)

28

$$T(n) = 2T(n/2) + n$$

if $f(n) = O(n^{\log_b a - \varepsilon})$ for $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$
if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$
if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for $\varepsilon > 0$ and $af(n/b) \le cf(n)$ for $c < 1$
then $T(n) = \Theta(f(n))$

a = 2
b = 2
f(n) = *n*

$n^{\log_b a} = n^{\log_2 2}$
$= n^1$

is $n = O(n^{1-\varepsilon})$?
is $n = \Theta(n^1)$?
is $n = \Omega(n^{1+\varepsilon})$?

Case 2: Θ(*n* log *n*)

29

$$T(n) = 16T(n/4) + n!$$

if $f(n) = O(n^{\log_b a - \varepsilon})$ for $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$
if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$
if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for $\varepsilon > 0$ and $af(n/b) \le cf(n)$ for $c < 1$
then $T(n) = \Theta(f(n))$

a = 16
b = 4
f(n) = *n!*

$n^{\log_b a} = n^{\log_4 16}$
$= n^2$

is $n! = O(n^{2-\varepsilon})$?
is $n! = \Theta(n^2)$?
is $n! = \Omega(n^{2+\varepsilon})$?

Case 3?
is $16(n/4)! \le cn!$ for $c < 1$?

30

$$T(n) = 16T(n/4) + n!$$

if $f(n) = O(n^{\log_b a - \varepsilon})$ for $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$

if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$

if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for $\varepsilon > 0$ and $af(n/b) \le cf(n)$ for $c < 1$
then $T(n) = \Theta(f(n))$

is $16(n/4)! \le cn!$ for $c < 1$?

Let c = 1/2

$cn! = 1/2n!$

$> (n/2)!$

therefore,

$16(n/4)! \le (n/2)! < 1/2n!$

T(n) = Θ($n!$)

31

$$T(n) = \sqrt{2}T(n/2) + \log n$$

if $f(n) = O(n^{\log_b a - \varepsilon})$ for $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$

if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$

if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for $\varepsilon > 0$ and $af(n/b) \le cf(n)$ for $c < 1$
then $T(n) = \Theta(f(n))$

a = $\sqrt{2}$
b = 2
$f(n)$ = log$n$

$$n^{\log_b a} = n^{\log_2 \sqrt{2}} = n^{\log_2 2^{1/2}} = \sqrt{n}$$

is $\log n = O(n^{1/2-\varepsilon})$?
is $\log n = \Theta(n^{1/2})$?
is $\log n = \Omega(n^{1/2+\varepsilon})$?

Case 1: $\Theta(\sqrt{n})$

32

$$T(n) = 4T(n/2) + n$$

if $f(n) = O(n^{\log_b a - \varepsilon})$ for $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$

if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$

if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for $\varepsilon > 0$ and $af(n/b) \le cf(n)$ for $c < 1$
then $T(n) = \Theta(f(n))$

a = 4
b = 2
$f(n)$ = n

$$n^{\log_b a} = n^{\log_2 4} = n^2$$

is $n = O(n^{2-\varepsilon})$?
is $n = \Theta(n^2)$?
is $n = \Omega(n^{2+\varepsilon})$?

Case 1: $\Theta(n^2)$

33

Recurrences

$$T(n) = 2T(n/3) + d \qquad T(n) = 7T(n/7) + n$$

if $f(n) = O(n^{\log_b a - \varepsilon})$ for $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$

if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$

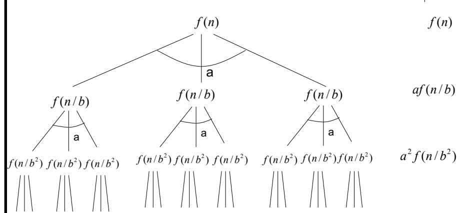
if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for $\varepsilon > 0$ and $af(n/b) \le cf(n)$ for $c < 1$
then $T(n) = \Theta(f(n))$

$$T(n) = T(n-1) + \log n \qquad T(n) = 8T(n/2) + n^3$$

34

Why does the master method work?

$$T(n) = aT(n/b) + f(n)$$



35

What is the depth of the tree?

At each level, the size of the data is divided by b

$$\frac{n}{b^d} = 1$$

$$\log\left(\frac{n}{b^d}\right) = 0$$

$$\log n - \log 4^b = 0$$

$$d \log b = \log n$$

$$d = \log_b n$$

36

How many leaves?

How many leaves are there in a complete *a*-ary tree of depth *d*?

$$a^d = a^{\log_b n}$$

$$= n^{\log_b a}$$

37

Total cost

if $f(n) = O(n^{\log_b a-\varepsilon})$ for $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$

if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$

if $f(n) = \Omega(n^{\log_b a+\varepsilon})$ for $\varepsilon > 0$ and $af(n/b) \le cf(n)$ for $c < 1$ then $T(n) = \Theta(f(n))$

$$T(n) = cf(n) + af(n/b) + a^2 f(n/b^2) + ... + a^{n-1} f(n/b^{n-1}) + \Theta(n^{\log_b a3})$$

$$= \sum_{i=0}^{\log_b n-1} a^i f(n/b^i) + \Theta(n^{\log_b a})$$

Case 1: cost is dominated by the cost of the leaves

$$= \sum_{i=0}^{\log_b n-1} a^i f(n/b^i) < \Theta(n^{\log_b a})$$

38

Total cost

if $f(n) = O(n^{\log_b a-\varepsilon})$ for $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$

if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$

if $f(n) = \Omega(n^{\log_b a+\varepsilon})$ for $\varepsilon > 0$ and $af(n/b) \le cf(n)$ for $c < 1$
then $T(n) = \Theta(f(n))$

$$T(n) = cf(n) + af(n/b) + a^2 f(n/b^2) + ... + a^{n-1} f(n/b^{n-1}) + \Theta(n^{\log_b a3})$$

$$= \sum_{i=0}^{\log_b n-1} a^i f(n/b^i) + \Theta(n^{\log_b a})$$

Case 2: cost is evenly distributed across tree

As we saw with mergesort, log *n* levels to the tree and at each level *f(n)* work

39

Total cost

if $f(n) = O(n^{\log_b a-\varepsilon})$ for $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$

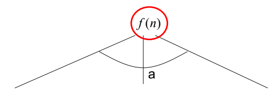
if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$

if $f(n) = \Omega(n^{\log_b a+\varepsilon})$ for $\varepsilon > 0$ and $af(n/b) \le cf(n)$ for $c < 1$
then $T(n) = \Theta(f(n))$

$$T(n) = cf(n) + af(n/b) + a^2 f(n/b^2) + ... + a^{d-1} f(n/b^{d-1}) + \Theta(n^{\log_b a3})$$

$$= \sum_{i=0}^{\log_b n-1} a^i f(n/b^i) + \Theta(n^{\log_b a})$$

Case 3: cost is dominated by the cost of the root



40

Other forms of the master method

$$T(n) = aT(n/b) + O(n^d)$$

$$T(n) = \begin{cases} O(n^d) & \text{if } d > \log_b a \\ O(n^d \log n) & \text{if } d = \log_b a \\ O(n^{\log_b a}) & \text{if } d < \log_b a \end{cases}$$

41

Changing variables

$$T(n) = 2T(\sqrt{n}) + \log n$$

Guesses?

We can do a variable change: let $m = \log_2 n$ (or $n = 2^m$)

$$T(2^m) = 2T(2^{m/2}) + m$$

Now, let $S(m)=T(2^m)$

$$S(m) = 2S(m/2) + m$$

42

Changing variables

$$S(m) = 2S(m/2) + m$$

Guess? $S(m) = O(m \log m)$

$$T(n) = T(2^m) = S(m) = O(m \log m)$$

substituting m=log n

$$T(n) = O(\log n \log \log n)$$

43