SHORTEST PATHS

David Kauchak
CS 140 – Spring 2024

1

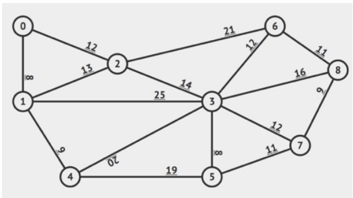
Admin

Assignment 8

No mentor hours Friday

2

Practice



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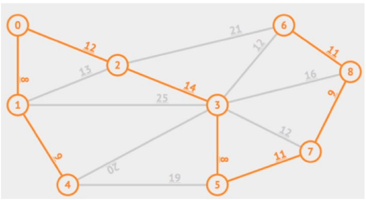
Solution



Sum = 8 + 8 + 9 + 9 + 11 + 11 + 12 + 14 = 82

4

Minimum spanning trees

What is the lowest weight set of edges that connects all vertices of an undirected graph with positive weights

Input: An undirected, positive weight graph, G=(V,E)

Output: A tree T=(V,E') where E' $\subseteq$ E that minimizes

$$weight(T) = \sum_{e \in E'} w_e$$

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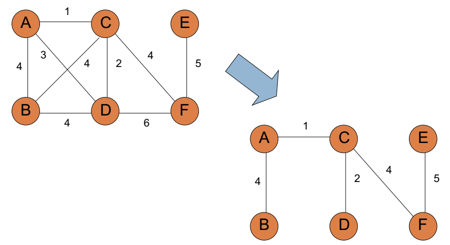
MST example



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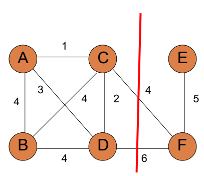
Minimum cut property

Given a partition S, let edge e be the minimum cost edge that **crosses** the partition. *Every* minimum spanning tree contains edge e.



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Prim's algorithm

Start at some root node and build out the MST by adding the lowest weighted edge out of the MST constructed so far

```
Prim(G, r)
for all v ∈ V
        key[v] ← ∞
        prev[v] ← null
key[r] ← 0
H ← MakeHeap(key)
while !Empty(H)
        u ← Extract-Min(H)
        visited[u] ← true
        for each edge (u, v) ∈ E
                if !visited[v] and w(u, v) < key(v)
                        Decrease-Key(v, w(u, v))
                        prev[v] ← u
```

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Correctness of Prim's?

Can we use the min-cut property?

- Given a partion S, let edge e be the minimum cost edge that **crosses** the partition. *Every* minimum spanning tree contains edge e.

Let S be the set of vertices visited so far

The only time we add a new edge is if it's the lowest weight edge from S to V-S

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Prim's

```
while !Empty(H)
    u ← Extract-Min(H)
    visited[u] ← true
    for each edge (u, v) ∈ E
        if !visited[v] and w(u, v) < key(v)
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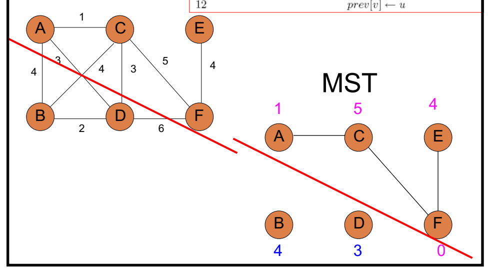

10

Running time of Prim's

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```

11

Running time of Prim's

```
Prim(G, r)
for all v ∈ V                                   Θ(|V|)
    key[v] ← ∞
    prev[v] ← null
key[r] ← 0
H ← MakeHeap(key)                               1 call to MakeHeap
while !Empty(H)
    u ← Extract-Min(H)                          |V| calls to Extract-Min
    visited[u] ← true
    for each edge (u, v) ∈ E
        if !visited[v] and w(u, v) < key(v)
            Decrease-Key(v, w(u, v))            |E| calls to Decrease-Key
            prev[v] ← u
```

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Running time of Prim's

	1 MakeHeap	|V| ExtractMin	|E| DecreaseKey	Total
Array	$\theta(\|V\|)$	$O(\|V\|^2)$	$O(\|E\|)$	$O(\|V\|^2)$
Bin heap	$\theta(\|V\|)$	$O(\|V\| \log \|V\|)$	$O(\|E\| \log \|V\|)$	$O((\|V\|+\|E\|) \log \|V\|)$ $O(\|E\| \log \|V\|)$
Fib heap	$\theta(\|V\|)$	$O(\|V\| \log \|V\|)$	$O(\|E\|)$	**$O(\|V\| \log \|V\| + \|E\|)$**

Kruskal's: $O(|E| \log |E|)$

13

Running time of Prim's

When should we use Kruskal's or Prim's?

	1 MakeHeap	|V| ExtractMin	|E| DecreaseKey	Total
Array	$\theta(\|V\|)$	$O(\|V\|^2)$	$O(\|E\|)$	$O(\|V\|^2)$
Bin heap	$\theta(\|V\|)$	$O(\|V\| \log \|V\|)$	$O(\|E\| \log \|V\|)$	$O((\|V\|+\|E\|) \log \|V\|)$ $O(\|E\| \log \|V\|)$
Fib heap	$\theta(\|V\|)$	$O(\|V\| \log \|V\|)$	$O(\|E\|)$	**$O(\|V\| \log \|V\| + \|E\|)$**

Kruskal's: $O(|E| \log |E|)$

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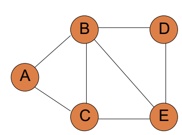
Shortest paths

What is the shortest path from a to d?



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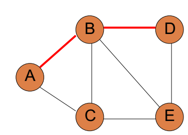
Shortest paths

How can we find this?



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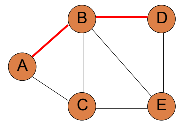
Shortest paths

BFS

```
BFS(G, s)
for each v ∈ V
        dist[v] = ∞
dist[s] = 0
ENQUEUE(Q, s)
while !EMPTY(Q)
        u ← DEQUEUE(Q)
        VISIT(u)
        for each edge (u, v) ∈ E
                if dist[v] = ∞
                        ENQUEUE(Q, v)
                        dist[v] ← dist[u] + 1
```



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Shortest paths

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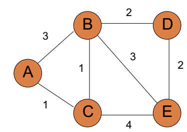

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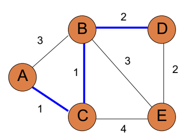
Shortest paths

What is the shortest path from a to d?



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Shortest path algorithms?

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Dijkstra's algorithm

What is dist?

What is prev?

How does it work?

What is the run-time?

How do we get the shortest path?

```
Dijkstra(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
Q ← MakeHeap(V)
while !Empty(Q)
        u ← ExtractMin(Q)
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        DecreaseKey(Q, v, dist[v])
                        prev[v] ← u
```

21

Dijkstra's algorithm

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```

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BFS(G, s)
for each v ∈ V
        dist[v] = ∞
dist[s] = 0
Enqueue(Q, s)
while !Empty(Q)
        u ← Dequeue(Q)
        Visit(u)
        for each edge (u, v) ∈ E
                if dist[v] = ∞
                        Enqueue(Q, v)
                        dist[v] ← dist[u] + 1
```

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Dijkstra's algorithm

prev keeps track of
the shortest path

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23

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24

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25

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26

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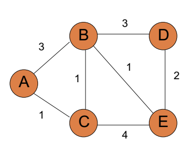

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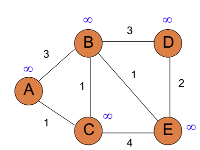

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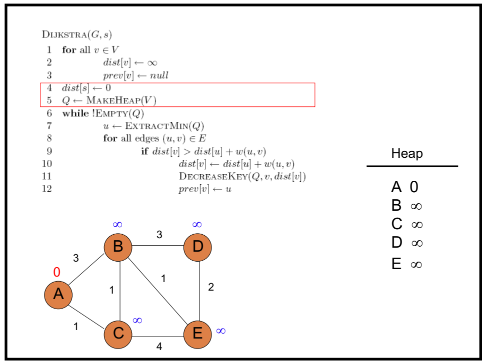
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DecreaseKey(Q, v, dist[v])
prev[v] ← u
Heap
A 0
B ∞
C ∞
D ∞
E ∞

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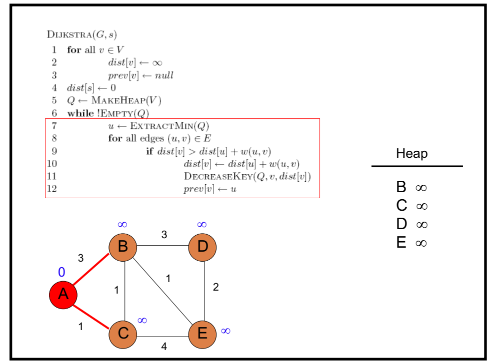
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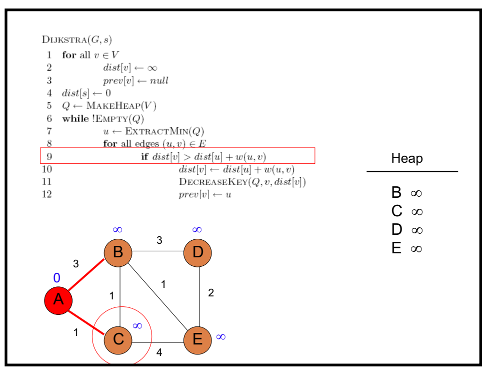
Heap
B ∞
C ∞
D ∞
E ∞

30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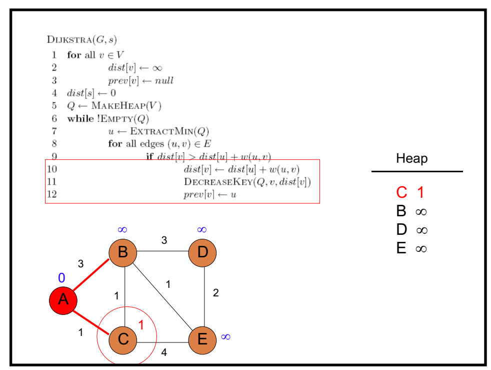
Heap
B ∞
C ∞
D ∞
E ∞

31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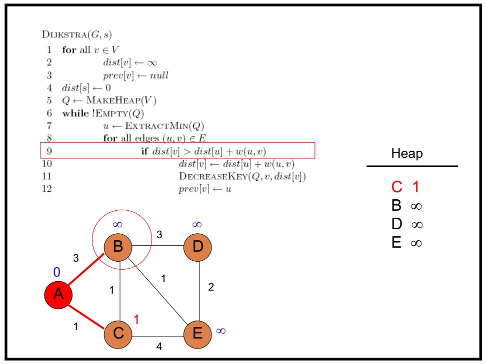
Heap
C 1
B ∞
D ∞
E ∞

32



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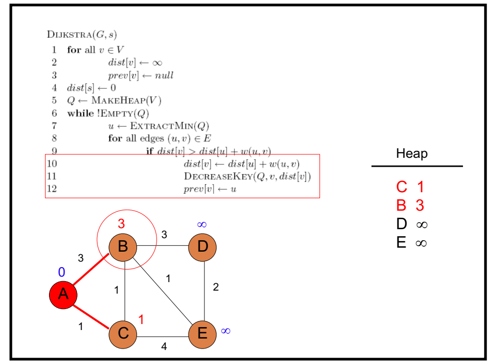
Heap
C 1
B ∞
D ∞
E ∞

33



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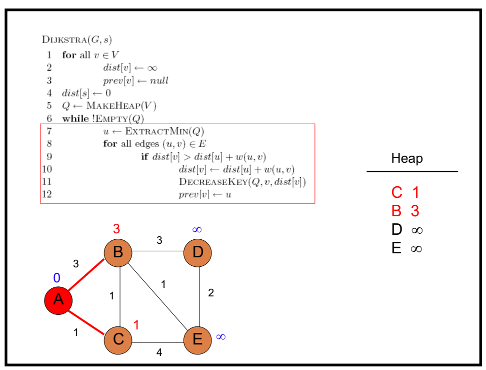
dist[v] ← dist[u] + w(u, v)
DECREASEKEY(Q, v, dist[v])
prev[v] ← u
Heap
C 1
B 3
D ∞
E ∞

34



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Heap
C 1
B 3
D ∞
E ∞

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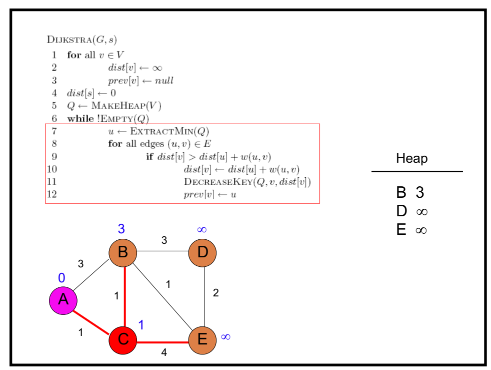

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prev[v] ← u
Heap
B 3
D ∞
E ∞

36

```
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```

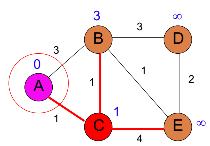

Heap

B 3
D ∞
E ∞

37

```
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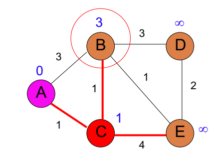

Heap

B 3
D ∞
E ∞

38

```
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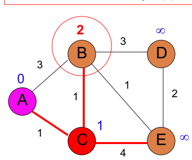

Heap

B 2
D ∞
E ∞

39

```
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while !EMPTY(Q)
        u ← EXTRACTMIN(Q)
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        DECREASEKEY(Q, v, dist[v])
                        prev[v] ← u
```
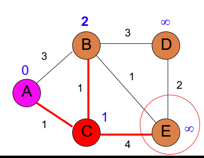



Heap

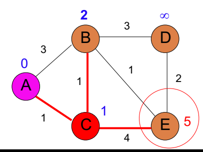
B 2
D ∞
E ∞

40

```
DIJKSTRA(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
Q ← MAKEHEAP(V)
while !EMPTY(Q)
        u ← EXTRACTMIN(Q)
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        DECREASEKEY(Q, v, dist[v])
                        prev[v] ← u
```

Heap

B 2
E 5
D ∞

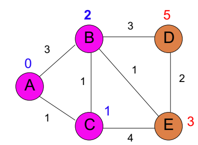


41

```
DIJKSTRA(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
Q ← MAKEHEAP(V)
while !EMPTY(Q)
        u ← EXTRACTMIN(Q)
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        DECREASEKEY(Q, v, dist[v])
                        prev[v] ← u
```

Heap

E 3
D 5

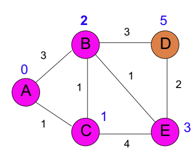


42

```
DIJKSTRA(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
Q ← MAKEHEAP(V)
while !EMPTY(Q)
        u ← EXTRACTMIN(Q)
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        DECREASEKEY(Q, v, dist[v])
                        prev[v] ← u
```

Heap

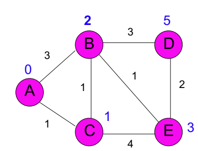
D 5



43

```
DIJKSTRA(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
Q ← MAKEHEAP(V)
while !EMPTY(Q)
        u ← EXTRACTMIN(Q)
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        DECREASEKEY(Q, v, dist[v])
                        prev[v] ← u
```

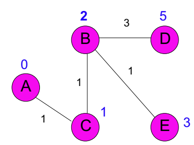
Heap



44

```
DIJKSTRA(G, s)
for all v ∈ V
    dist[v] ← ∞
    prev[v] ← null
dist[s] ← 0
Q ← MAKEHEAP(V)
while !EMPTY(Q)
    u ← EXTRACTMIN(Q)
    for all edges (u, v) ∈ E
        if dist[v] > dist[u] + w(u, v)
            dist[v] ← dist[u] + w(u, v)
            DECREASEKEY(Q, v, dist[v])
            prev[v] ← u
```

Heap

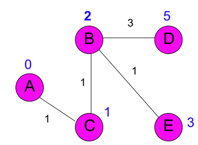


Prev

45

```
DIJKSTRA(G, s)
for all v ∈ V
    dist[v] ← ∞
    prev[v] ← null
dist[s] ← 0
Q ← MAKEHEAP(V)
while !EMPTY(Q)
    u ← EXTRACTMIN(Q)
    for all edges (u, v) ∈ E
        if dist[v] > dist[u] + w(u, v)
            dist[v] ← dist[u] + w(u, v)
            DECREASEKEY(Q, v, dist[v])
            prev[v] ← u
```

Heap



Prev

How do we get the actual paths?

46

Is Dijkstra's algorithm correct?

Invariant: For every vertex removed from the heap, dist[v] is the actual shortest distance from s to v

```
DIJKSTRA(G, s)
for all v ∈ V
    dist[v] ← ∞
    prev[v] ← null
dist[s] ← 0
Q ← MAKEHEAP(V)
while !EMPTY(Q)
    u ← EXTRACTMIN(Q)
    for all edges (u, v) ∈ E
        if dist[v] > dist[u] + w(u, v)
            dist[v] ← dist[u] + w(u, v)
            DECREASEKEY(Q, v, dist[v])
            prev[v] ← u
```

proof?

47

Is Dijkstra's algorithm correct?

Invariant: For every vertex removed from the heap, dist[v] is the actual shortest distance from s to v

- The only time a vertex gets visited is when the distance from s to that vertex is smaller than the distance to any remaining vertex
- Therefore, there cannot be any other path that hasn't been visited already that would result in a shorter path

48

Running time?

```
Dijkstra(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
Q ← MakeHeap(V)
while !Empty(Q)
        u ← ExtractMin(Q)
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        DecreaseKey(Q, v, dist[v])
                        prev[v] ← u
```

49

Running time?

```
Dijkstra(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
Q ← MakeHeap(V)
while !Empty(Q)
        u ← ExtractMin(Q)
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        DecreaseKey(Q, v, dist[v])
                        prev[v] ← u
```

Θ(|V|)

1 call to MakeHeap

|V| calls to Extract-M

|E| calls to Decrease-K

50

Running time?

Depends on the heap implementation

	1 MakeHeap	|V| ExtractMin	|E| DecreaseKey	Total
Array	O(|V|)	O(|V|^2)	O(|E|)	O(|V|^2)
Bin heap	O(|V|)	O(|V| log |V|)	O(|E| log |V|)	O((|V|+|E|) log |V|) O(|E| log |V|)

51

Running time?

Depends on the heap implementation

	1 MakeHeap	|V| ExtractMin	|E| DecreaseKey	Total
Array	O(|V|)	O(|V|^2)	O(|E|)	O(|V|^2)
Bin heap	O(|V|)	O(|V| log |V|)	O(|E| log |V|)	O((|V|+|E|) log |V|) O(|E| log |V|)

Is this an improvement? If \|E\| < \|V\|^2 / log \|V\|

52

Running time?

Depends on the heap implementation

	1 MakeHeap	|V| ExtractMin	|E| DecreaseKey	Total
Array	O(|V|)	O(|V|²)	O(|E|)	O(|V|²)
Bin heap	O(|V|)	O(|V| log |V|)	O(|E| log |V|)	O((|V|+|E|) log |V|) O(|E| log |V|)
Fib heap	O(|V|)	O(|V| log |V|)	O(|E|)	O(|V| log |V| + |E|)

53

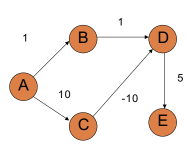
Dijkstra's vs Prim's

```
Dijkstra(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
Q ← MakeHeap(V)
while !Empty(Q)
        u ← ExtractMin(Q)
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        DecreaseKey(Q, v, dist[v])
                        prev[v] ← u
```

```
Prim(G, r)
for all v ∈ V
        key[v] ← ∞
        prev[v] ← null
key[r] ← 0
H ← MakeHeap(key)
while !Empty(H)
        u ← Extract-Min(H)
        visited[u] ← true
        for each edge (u, v) ∈ E
                if !visited[v] and w(u, v) < key(v)
                        Decrease-Key(v, w(u, v))
                        prev[v] ← u
```

54

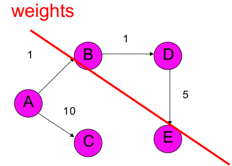
What about Dijkstra's on…?



55

What about Dijkstra's on…?

Dijkstra's algorithm only works for positive edge weights



56

Is Dijkstra's algorithm correct?

Invariant: For every vertex removed from the heap, dist[v] is the actual shortest distance from s to v

- The only time a vertex gets visited is when the distance from s to that vertex is smaller than the distance to any remaining vertex
- Therefore, there cannot be any other path that hasn't been visited already that would result in a shorter path

We relied on having positive edge weights for correctness!

57

Bounding the distance

Another invariant: For each vertex v, dist[v] is an upper bound on the actual shortest distance

```
Dijkstra(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
Q ← MakeHeap(V)
while !Empty(Q)
        u ← ExtractMin(Q)
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        DecreaseKey(Q, v, dist[v])
                        prev[v] ← u
```

Is this a valid invariant?

58

Bounding the distance

Another invariant: For each vertex v, dist[v] is an upper bound on the actual shortest distance

- start off at ∞
- only update the value if we find a shorter distance

An update procedure: for an edge (u,v)

$$dist[v] = \min\{dist[v], dist[u] + w(u,v)\}$$

59

$dist[v] = \min\{dist[v], dist[u] + w(u,v)\}$

Can we ever go wrong applying this update rule?

- We can apply this rule as many times as we want and will never underestimate dist[v]

When will dist[v] be right?

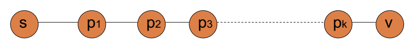
- If u is along the shortest path to v and dist[u] is correct

60

$dist[v] = \min\{dist[v], dist[u] + w(u,v)\}$

dist[v] will be right if u is along the shortest path to v and dist[u] is correct

Consider the shortest path from s to v

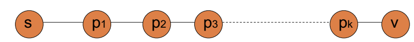


61

$dist[v] = \min\{dist[v], dist[u] + w(u,v)\}$

dist[v] will be right if u is along the shortest path to v and dist[u] is correct

What happens if we update all of the vertices with the above update?

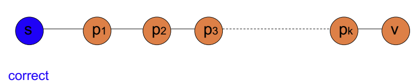


62

$dist[v] = \min\{dist[v], dist[u] + w(u,v)\}$

dist[v] will be right if u is along the shortest path to v and dist[u] is correct

What happens if we update all of the vertices with the above update?

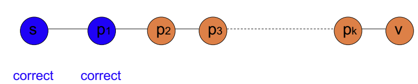


63

$dist[v] = \min\{dist[v], dist[u] + w(u,v)\}$

dist[v] will be right if u is along the shortest path to v and dist[u] is correct

What happens if we update all of the vertices with the above update?

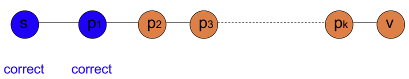


64

$$dist[v] = \min\{dist[v], dist[u] + w(u,v)\}$$

dist[v] will be right if u is along the shortest path to v and dist[u] is correct

Does the order that we update the vertices matter?



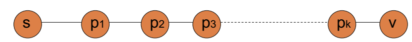
65

$$dist[v] = \min\{dist[v], dist[u] + w(u,v)\}$$

dist[v] will be right if u is along the shortest path to v and dist[u] is correct

How many times do we have to do this for vertex p_i to have the correct shortest path from s?

- i times



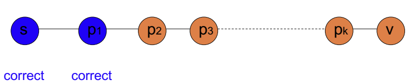
66

$$dist[v] = \min\{dist[v], dist[u] + w(u,v)\}$$

dist[v] will be right if u is along the shortest path to v and dist[u] is correct

How many times do we have to do this for vertex p_i to have the correct shortest path from s?

- i times



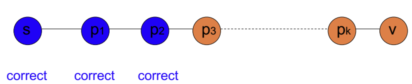
67

$$dist[v] = \min\{dist[v], dist[u] + w(u,v)\}$$

dist[v] will be right if u is along the shortest path to v and dist[u] is correct

How many times do we have to do this for vertex p_i to have the correct shortest path from s?

- i times



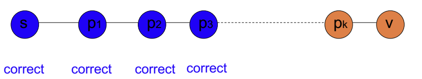
68

$dist[v] = \min\{dist[v], dist[u] + w(u,v)\}$

dist[v] will be right if u is along the shortest path to v and dist[u] is correct

How many times do we have to do this for vertex p_i to have the correct shortest path from s?

- i times



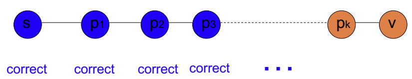
69

$dist[v] = \min\{dist[v], dist[u] + w(u,v)\}$

dist[v] will be right if u is along the shortest path to v and dist[u] is correct

How many times do we have to do this for vertex p_i to have the correct shortest path from s?

- i times



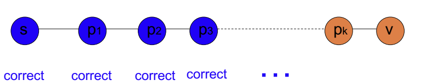
70

$dist[v] = \min\{dist[v], dist[u] + w(u,v)\}$

dist[v] will be right if u is along the shortest path to v and dist[u] is correct

What is the longest (vertex-wise) the path from s to any node v can be?

- |V| - 1 edges/vertices



71

Bellman-Ford algorithm

```
BELLMAN-FORD(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
for i ← 1 to |V| − 1
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        prev[v] ← u
for all edges (u, v) ∈ E
        if dist[v] > dist[u] + w(u, v)
                return false
```

72

Bellman-Ford algorithm

```
BELLMAN-FORD(G, s)
for all v ∈ V
    dist[v] ← ∞
    prev[v] ← null
dist[s] ← 0
for i ← 1 to |V| − 1
    for all edges (u, v) ∈ E
        if dist[v] > dist[u] + w(u, v)
            dist[v] ← dist[u] + w(u, v)
            prev[v] ← u
for all edges (u, v) ∈ E
    if dist[v] > dist[u] + w(u, v)
        return false
```

Initialize all the distances

do it |V| -1 times

iterate over all edges/vertices and apply update rule

73

Bellman-Ford algorithm

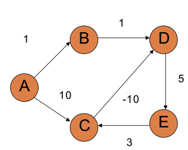
```
BELLMAN-FORD(G, s)
for all v ∈ V
    dist[v] ← ∞
    prev[v] ← null
dist[s] ← 0
for i ← 1 to |V| − 1
    for all edges (u, v) ∈ E
        if dist[v] > dist[u] + w(u, v)
            dist[v] ← dist[u] + w(u, v)
            prev[v] ← u
for all edges (u, v) ∈ E
    if dist[v] > dist[u] + w(u, v)
        return false
```

check for negative cycles

74

Negative cycles

What is the shortest path from a to e?



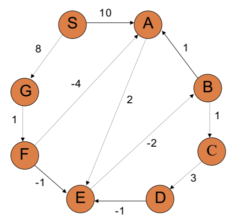
75

Bellman-Ford algorithm

```
BELLMAN-FORD(G, s)
for all v ∈ V
    dist[v] ← ∞
    prev[v] ← null
dist[s] ← 0
for i ← 1 to |V| − 1
    for all edges (u, v) ∈ E
        if dist[v] > dist[u] + w(u, v)
            dist[v] ← dist[u] + w(u, v)
            prev[v] ← u
for all edges (u, v) ∈ E
    if dist[v] > dist[u] + w(u, v)
        return false
```

76

Bellman-Ford algorithm



How many edges is the shortest path from s to:

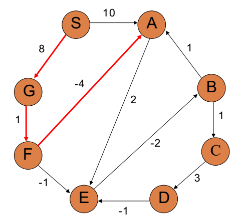
A:

B:

D:

77

Bellman-Ford algorithm

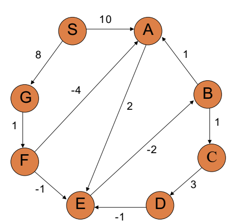


How many edges is the shortest path from s to:

A: 3

78

Bellman-Ford algorithm



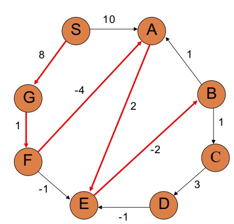
How many edges is the shortest path from s to:

A: 3

B:

79

Bellman-Ford algorithm



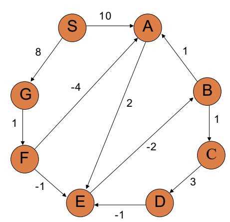
How many edges is the shortest path from s to:

A: 3

B: 5

80

Bellman-Ford algorithm



How many edges is the shortest path from s to:

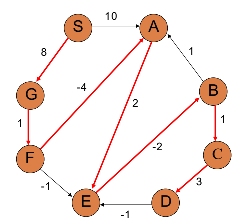
A: 3

B: 5

D:

81

Bellman-Ford algorithm



How many edges is the shortest path from s to:

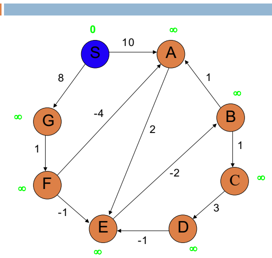
A: 3

B: 5

D: 7

82

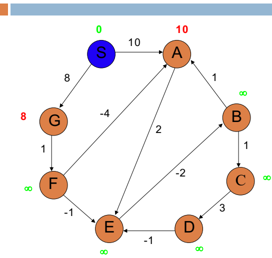
Bellman-Ford algorithm



Iteration: 0

83

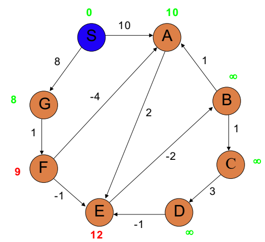
Bellman-Ford algorithm



Iteration: 1

84

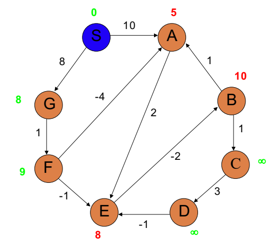
Bellman-Ford algorithm



Iteration: 2

85

Bellman-Ford algorithm

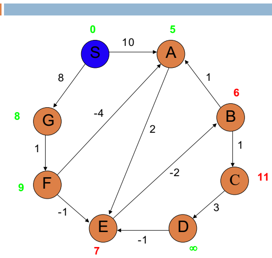


Iteration: 3

A has the correct distance and path

86

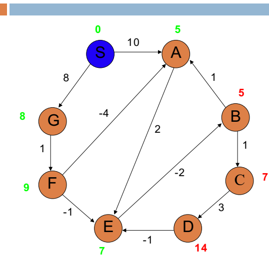
Bellman-Ford algorithm



Iteration: 4

87

Bellman-Ford algorithm

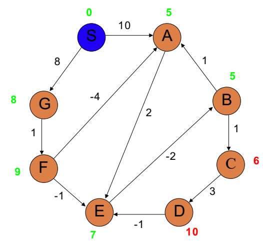


Iteration: 5

B has the correct distance and path

88

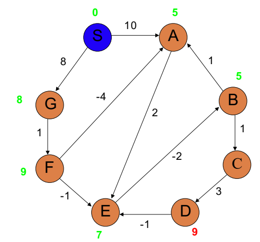
Bellman-Ford algorithm



Iteration: 6

89

Bellman-Ford algorithm



Iteration: 7

D (and all other nodes) have the correct distance and path

90

Correctness of Bellman-Ford

Loop invariant: After iteration i, all vertices with shortest paths from s of length i edges or less have correct distances

```
Bellman-Ford(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
for i ← 1 to |V| − 1
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        prev[v] ← u
for all edges (u, v) ∈ E
        if dist[v] > dist[u] + w(u, v)
                return false
```

91

Runtime of Bellman-Ford

```
Bellman-Ford(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
for i ← 1 to |V| − 1
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        prev[v] ← u
for all edges (u, v) ∈ E
        if dist[v] > dist[u] + w(u, v)
                return false
```

O(|V| |E|)

92

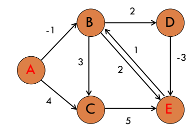
Runtime of Bellman-Ford

```
BELLMAN-FORD(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
for i ← 1 to |V| − 1
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        prev[v] ← u
for all edges (u, v) ∈ E
        if dist[v] > dist[u] + w(u, v)
                return false
```

Can you modify the algorithm to run faster (in some circumstances)?

93

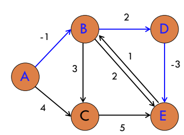
Shortest Paths



What is the shortest path from A to E?

94

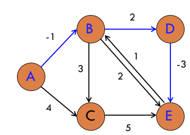
Shortest Paths



-2

95

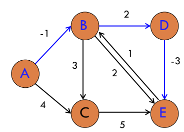
Shortest Paths



What algorithm would we use to calculate this?

96

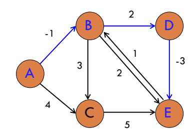
Shortest Paths



- Bellman-Ford (since the graph has negative edges)
- O(VE)

97

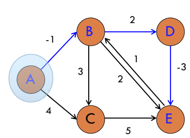
Shortest Paths



- Bellman-Ford (since the graph has negative edges)
- O(VE)
- Called a single-source shortest path algorithm. Why?

98

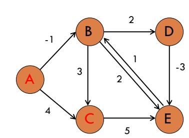
Shortest Paths



- Bellman-Ford (since the graph has negative edges)
- O(VE)
- Calculate all paths from a **single vertex.**

99

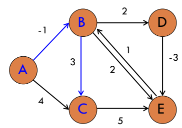
Shortest Paths



What is the shortest path from A to C?
If we already calculated A to E using Bellman-Ford
do we need to do any work?

100

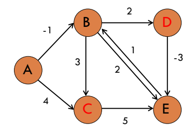
Shortest Paths



No new calculations!
Bellman-Ford calculates all shortest paths starting at A.

101

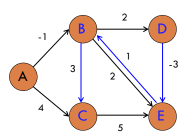
Shortest Paths



What is the shortest path from D to C?
If we already calculated A to E using Bellman-Ford do we need to do any work?

102

Shortest Paths



Different source.
Have to run Bellman-Ford again!

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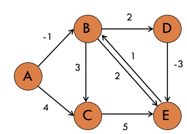
All pairs shortest paths

All pairs shortest paths: calculate the shortest paths between *all* vertices



104

All pairs shortest paths

All pairs shortest paths: calculate the shortest paths between *all* vertices

Easy solution?

105

All pairs shortest paths

All pairs shortest paths: calculate the shortest paths between *all* vertices

Run Bellman-Ford from each vertex!

Running time (in terms of E and V)?

106

All pairs shortest paths

All pairs shortest paths: calculate the shortest paths between *all* vertices

Run Bellman-Ford from each vertex!

$O(V^2E)$
- Bellman-Ford: O(VE)
- V calls, one for each vertex

107

DAGs?

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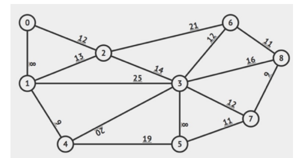
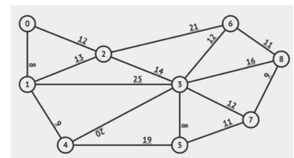
Handout

109



Kruskals

Add smallest edge that connects two sets not already connected

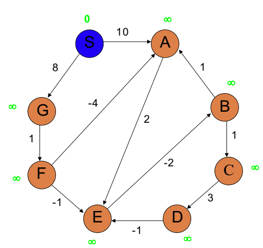


Prims

Starting at 0, grow MST by adding lowest weight edge out of current MST

110

Bellman-Ford algorithm



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