Mergeable Heaps

David Kauchak
CS140
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Admin

Assignment 2 graded

Assignment 4 – start working!

2

Binary heap

A binary tree where the value of a parent is greater than or equal to the value of its children

Additional restriction: all levels of the tree are **complete** except the last

Max heap vs. min heap

3

Binary heap - operations

Max - return the largest element in the set

ExtractMax – Return and remove the largest element in the set

Insert(val) – insert val into the set

IncreaseElement(x, val) – increase the value of element x to val

BuildHeap(A) – build a heap from an array of elements

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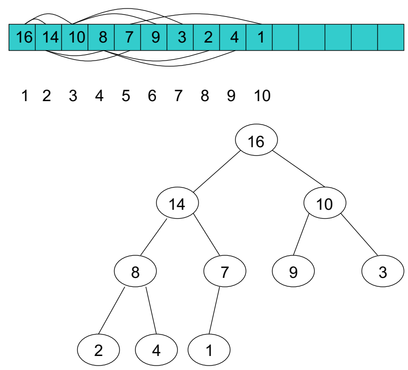
Binary heap representations



5

Heapify

Assume left and right children are heaps, turn current set into a valid heap

HEAPIFY(A, i)
$l \leftarrow$ LEFT(i)
$r \leftarrow$ RIGHT(i)
$largest \leftarrow i$
if $l \leq heap\text{-}size[A]$ and $A[l] > A[i]$
    $largest \leftarrow l$
if $r \leq heap\text{-}size[A]$ and $A[r] > A[largest]$
    $largest \leftarrow r$
if $largest \neq i$
    swap $A[i]$ and $A[largest]$
    HEAPIFY($A, largest$)

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find out which is largest: current, left of right

7

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if $largest \neq i$
    swap $A[i]$ and $A[largest]$
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if a child is larger, swap and recurse

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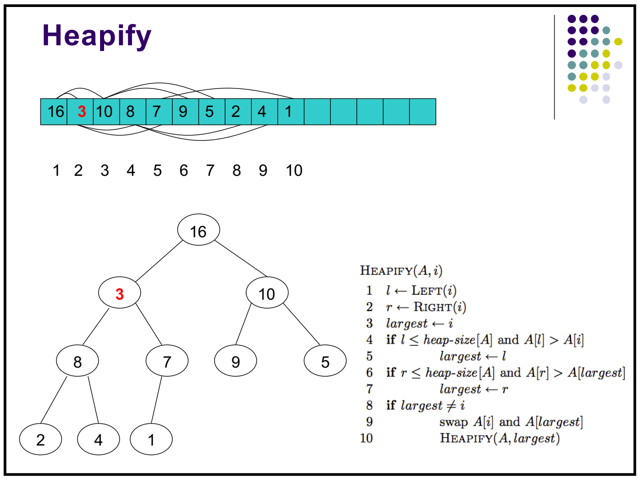
Heapify
16 3 10 8 7 9 5 2 4 1
1 2 3 4 5 6 7 8 9 10
16
3 10
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if largest ≠ i
swap A[i] and A[largest]
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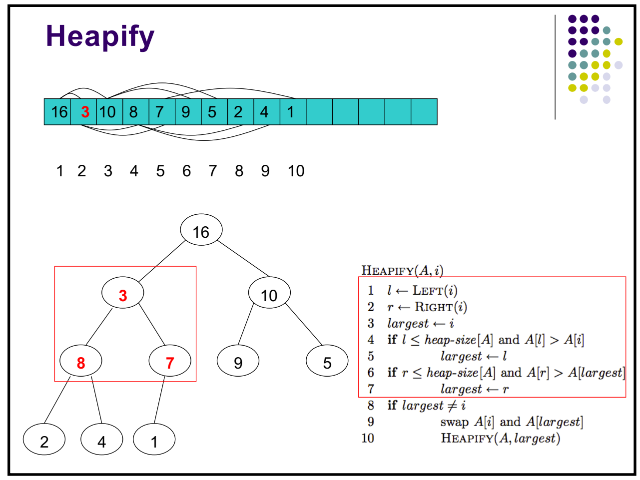
9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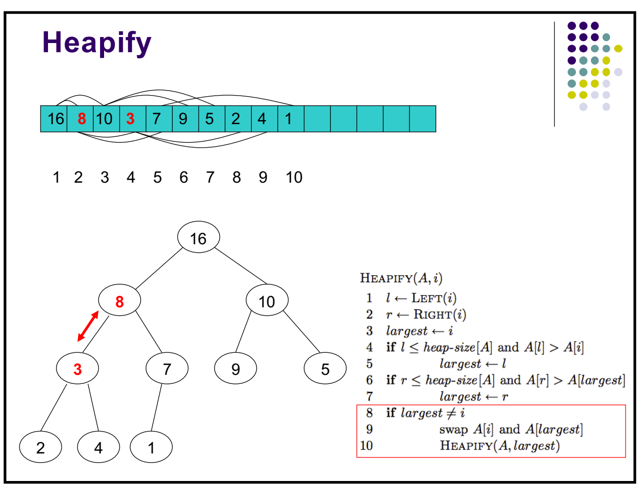
10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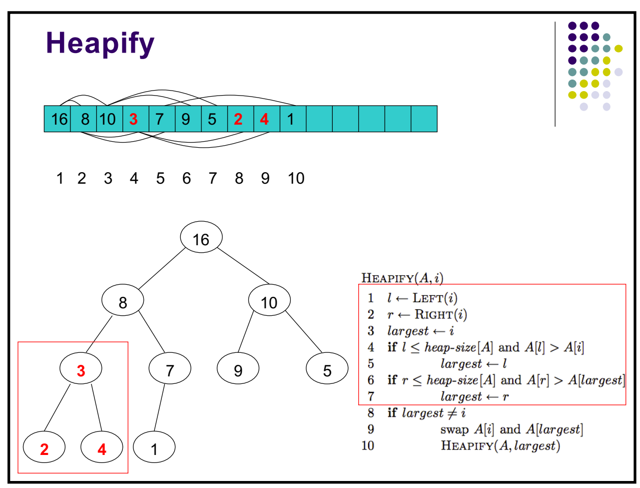
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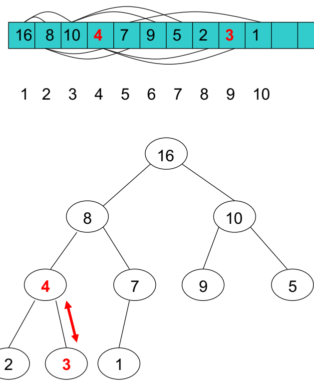
Heapify

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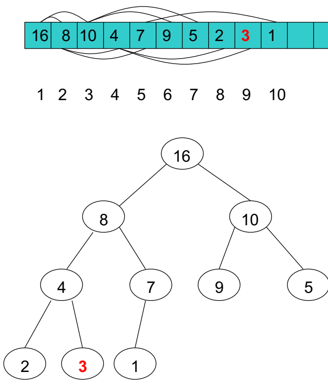
Heapify

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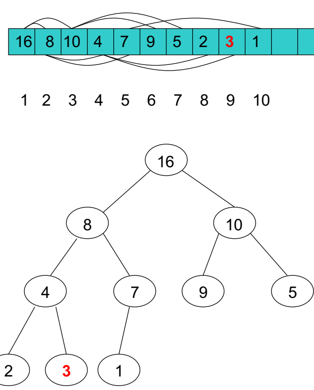
Heapify

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  if largest ≠ i
        swap A[i] and A[largest]
        HEAPIFY(A, largest)
```

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Running time of Heapify

O(height of the tree)

What is the height of the tree?

- Complete binary tree, except for the last level

$$2^h \leq n$$

$$h \leq \log_2 n$$

O(log n)

```
HEAPIFY(A, i)
  l ← LEFT(i)
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Binary heap - operations

Max - return the largest element in the set

ExtractMax – Return and remove the largest element in the set

Insert(val) – insert val into the set

IncreaseElement(x, val) – increase the value of element x to val

BuildHeap(A) – build a heap from an array of elements

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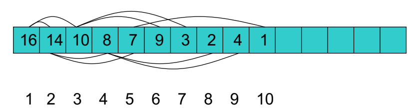
Max

What is the largest element in the set?

Return A[1]



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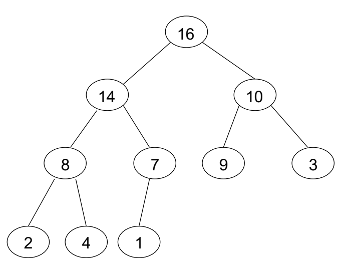
ExtractMax

Return and remove the largest element in the set



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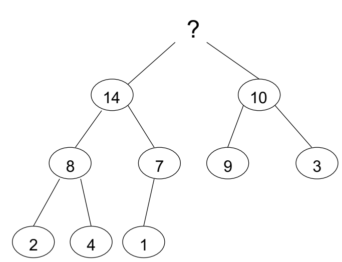

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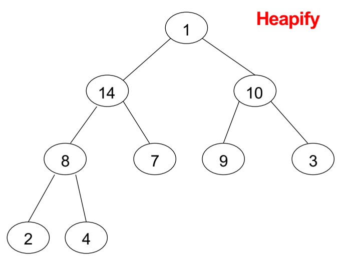
ExtractMax

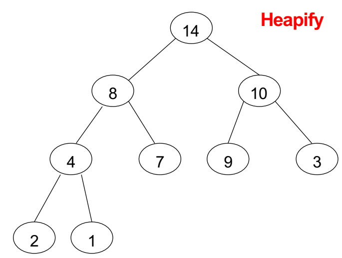
Return and remove the largest element in the set



29

ExtractMax

Return and remove the largest element in the set



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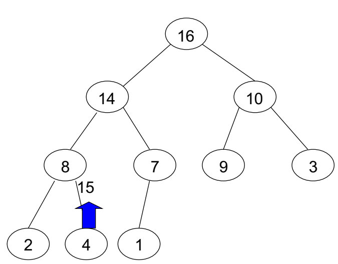
IncreaseElement

Increase the value of element *x* to *val*



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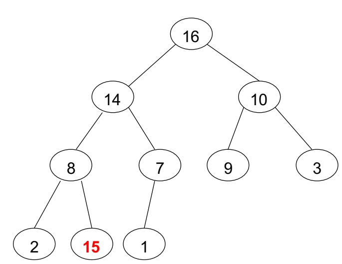

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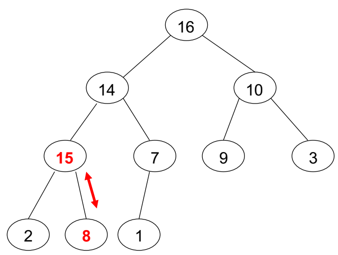

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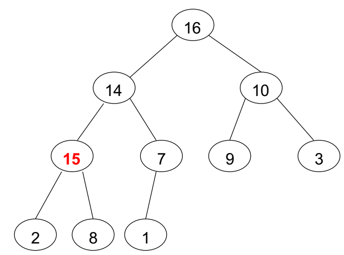

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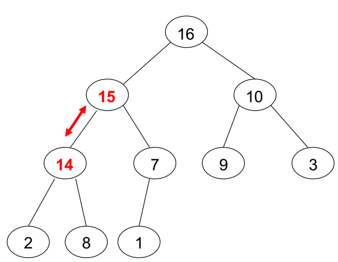

37

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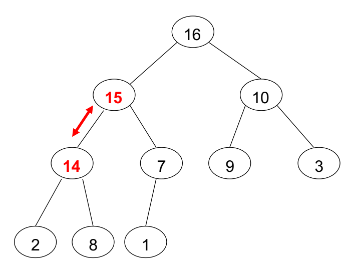

Runtime?

O(height of tree) = O(log n)

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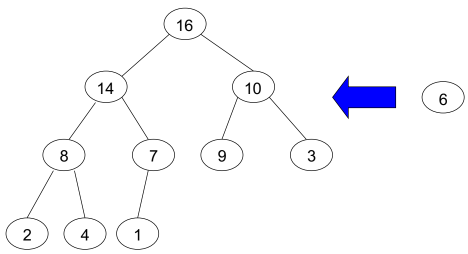
Insert
Insert *val* into the set



43

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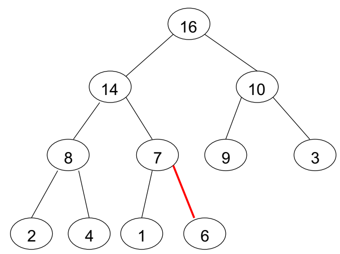

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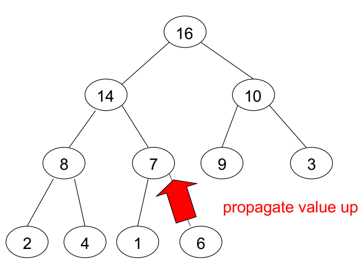

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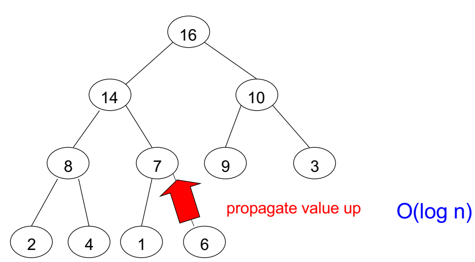
Insert
Insert *val* into the set



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Building a heap

Can we build a heap using the functions we have so far?

- Max
- ExtractMax
- Insert(val)|
- IncreaseElement(x, val)

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Building a heap

For each element x in array:
  insert(x)

```
BUILD-HEAP1(A)
copy A to B
heap-size[A] ← 0
for i ← 1 to length[B]
    INSERT(A, B[i])
```

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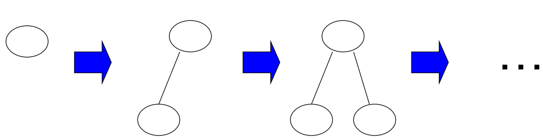
Running time of BuildHeap1

n calls to Insert – O(n log n)

Can we do better?



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Building a heap: take 2

```
BUILD-HEAP2(A)
heap-size[A] ← (length)[A]
for i ← ⌊(length)[A]/2⌋to1
    HEAPIFY(A, i)
```

Start with n/2 “one-node” heaps

call Heapify on element n/2-1, n/2-2, n/2-3 …

all children have smaller indices

building from the bottom up, makes sure that all the children are heaps

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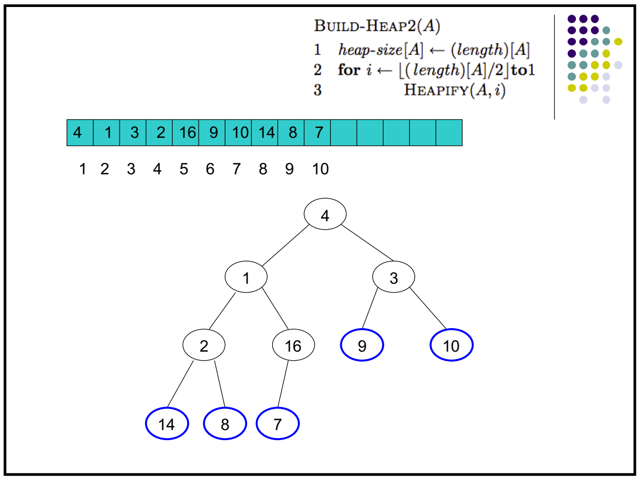
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4 1 3 2 16 9 10 14 8 7
1 2 3 4 5 6 7 8 9 10
4
1 3
2 16 9 10
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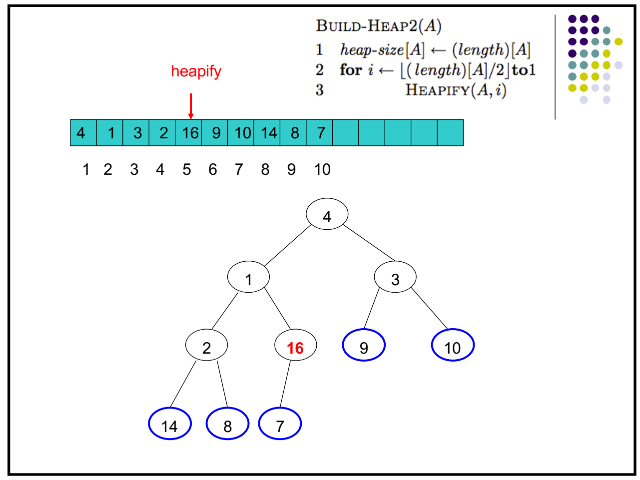
53


BUILD-HEAP2(A)
1 heap-size[A] ← (length)[A]
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heapify
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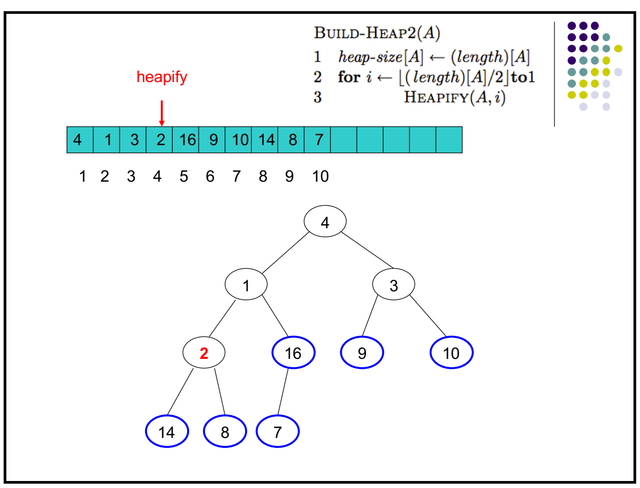
54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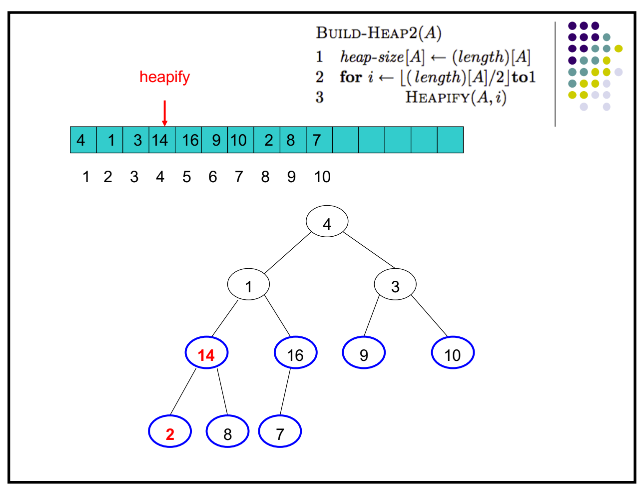
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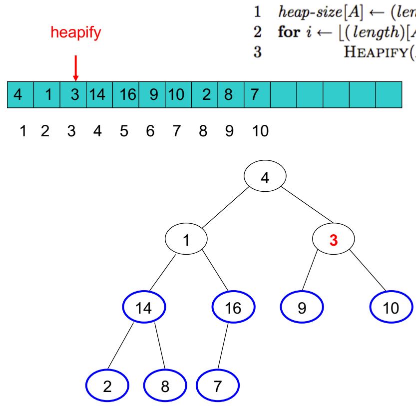
BUILD-HEAP2(A)

$heap\text{-}size[A] \leftarrow (length)[A]$

for $i \leftarrow \lfloor (length)[A]/2 \rfloor$ **to** 1

HEAPIFY(A, i)



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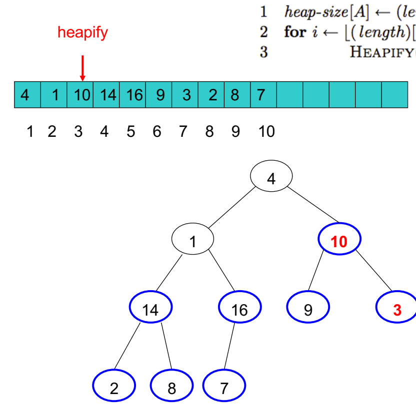

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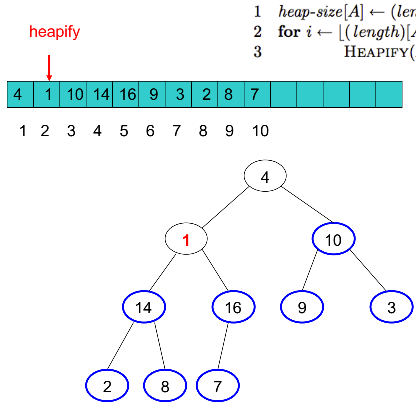

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for $i \leftarrow \lfloor (length)[A]/2 \rfloor$ **to** 1

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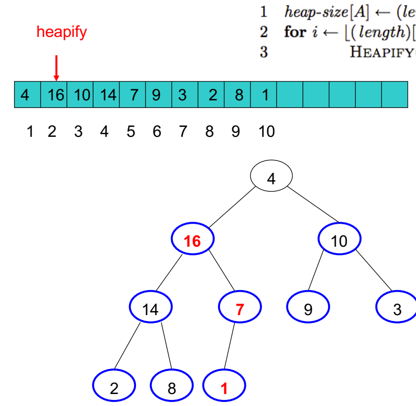

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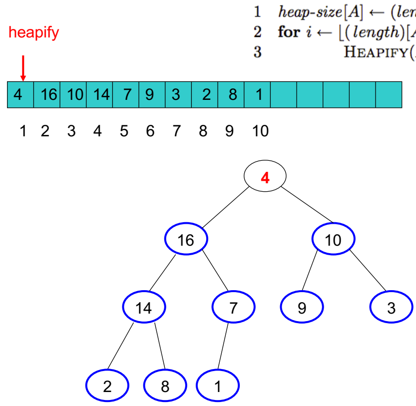

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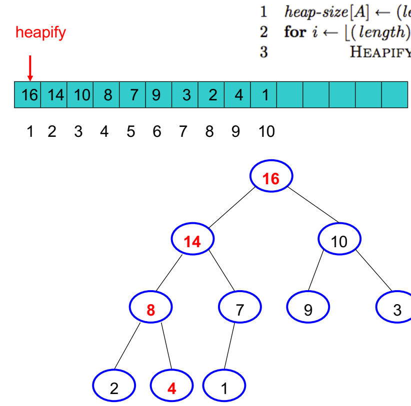

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Running time of BuildHeap2

n/2 calls to Heapify – O(n log n)

Can we get a tighter bound?

BUILD-HEAP2(A)
1 $heap\text{-}size[A] \leftarrow (length)[A]$
2 **for** $i \leftarrow \lfloor (length)[A]/2 \rfloor$ **to** 1
3 HEAPIFY(A, i)

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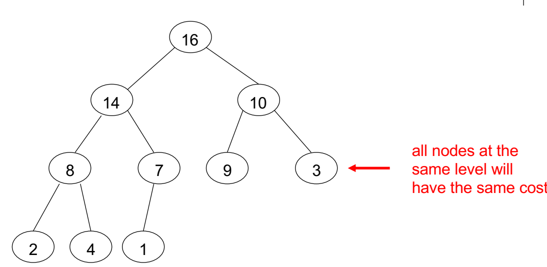
Running time of BuildHeap2



How many nodes are at level *d*? 2^d

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Running time of BuildHeap2

$$T(n) = \sum_{d=0}^{\log n} 2^d O(d)$$

?

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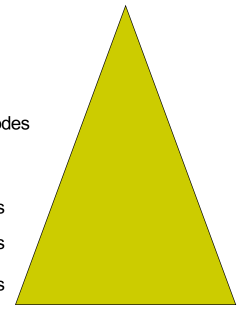
Nodes at height *h*

h < ceil(n/2^{h+1}) nodes

h=2 < ceil(n/8) nodes

h=1 < ceil(n/4) nodes

h=0 < ceil(n/2) nodes



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Running time of BuildHeap2

$$T(n) = \sum_{h=0}^{\log n} \left\lceil \frac{n}{2^{h+1}} \right\rceil O(h)$$

$$= O\left(n \sum_{h=0}^{\log n} \left\lceil \frac{1}{2^{h+1}} \right\rceil h \right)$$

$$= O\left(n \sum_{h=0}^{\log n} \frac{h}{2^h} \right)$$

$$= O\left(n \sum_{h=0}^{\infty} \frac{h}{2^h} \right)$$

$$= O(n)$$

$$\sum_{h=0}^{\infty} \frac{h}{2^h} = \frac{1/2}{(1-1/2)^2} = 2$$

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Binary heaps

Procedure	Binary heap (worst-case)
BUILD-HEAP	$\Theta(n)$
INSERT	$\Theta(\log n)$
MAXIMUM	$\Theta(1)$
EXTRAC-MAX	$\Theta(\log n)$
UNION	
INCREASE-ELEMENT	$\Theta(\log n)$
DELETE	$\Theta(\log n)$

(adapted from Figure 19.1, pg. 456 [1])

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Mergeable heaps

Procedure	Binary heap (worst-case)
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INSERT	$\Theta(\log n)$
MAXIMUM	$\Theta(1)$
EXTRAC-MAX	$\Theta(\log n)$
UNION	
INCREASE-ELEMENT	$\Theta(\log n)$
DELETE	$\Theta(\log n)$

(adapted from Figure 19.1, pg. 456 [1])

- Mergeable heaps support the union operation
- Allows us to combine two heaps to get a single heap
- Union runtime for binary heaps?

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Union for binary heaps

Procedure	Binary heap (worst-case)
BUILD-HEAP	$\Theta(n)$
INSERT	$\Theta(\log n)$
MAXIMUM	$\Theta(1)$
EXTRAC-MAX	$\Theta(\log n)$
UNION	$\Theta(n)$
INCREASE-ELEMENT	$\Theta(\log n)$
DELETE	$\Theta(\log n)$

(adapted from Figure 19.1, pg. 456 [1])

concatenate the arrays and then call Build-Heap

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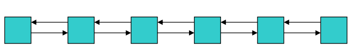
Linked-list heap



- Store the elements in a doubly linked list
- Insert:
- Max:
- Extract-Max:
- Increase:
- Union:

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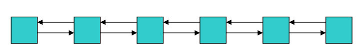
Linked-list heap



- Store the elements in a doubly linked list
- Insert: add to the end/beginning
- Max: search through the linked list
- Extract-Max: search and delete
- Increase: increase value
- Union: concatenate linked lists

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Linked-list heap

Procedure	Binary heap (worst-case)	Linked-list
BUILD-HEAP	$\Theta(n)$	$\Theta(n)$
INSERT	$\Theta(\log n)$	$\Theta(1)$
MAXIMUM	$\Theta(1)$	$\Theta(n)$
EXTRAC-MAX	$\Theta(\log n)$	$\Theta(n)$
UNION	$\Theta(n)$	$\Theta(1)$
INCREASE-ELEMENT	$\Theta(\log n)$	$\Theta(1)$
DELETE	$\Theta(\log n)$	$\Theta(1)$

(adapted from Figure 19.1, pg. 456 [1])

Faster Union, Increase, Insert and Delete… but slower Max operations

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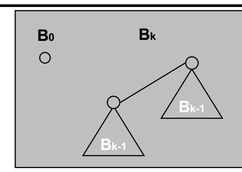
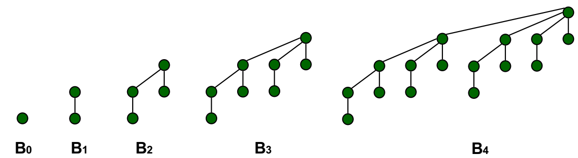
Adapted from: Kevin Wayne

Binomial Tree



B_k: a binomial tree B_{k-1} with the addition of a left child with another binomial tree B_{k-1}



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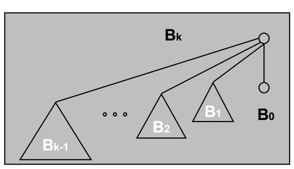
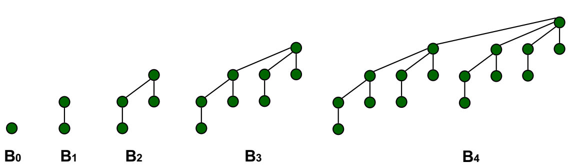
Binomial Tree



Number of nodes with respect to k?

$N(B_o) = 1$
$N(B_k) = 2\ N(B_{k-1}) = 2^k$



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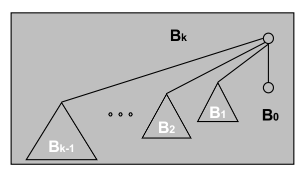
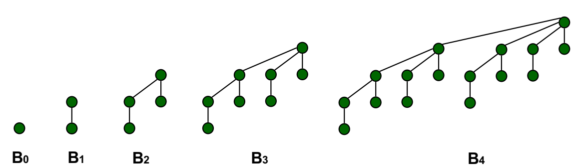
Binomial Tree



Height?

$H(B_o) = 1$
$H(B_k) = 1 + H(B_{k-1}) = k$



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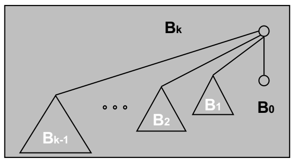
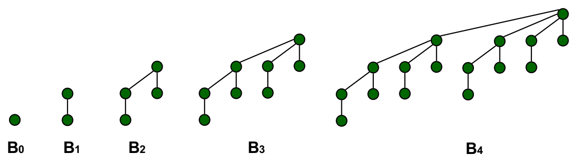
Binomial Tree



Degree of root node?

k, each time we add another binomial tree



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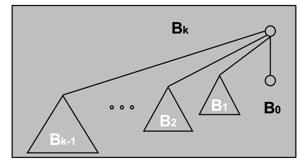
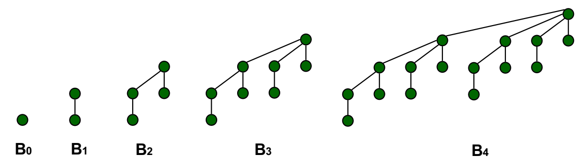
Binomial Tree



What are the children of the root?

k binomial trees:
B_{k-1}, B_{k-2}, …, B_0



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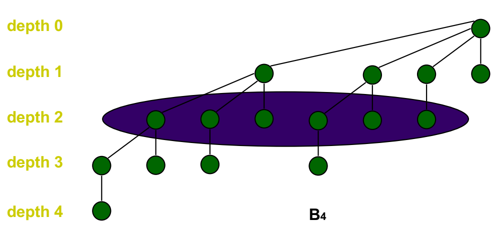
Binomial Tree

Why is it called a binomial tree?



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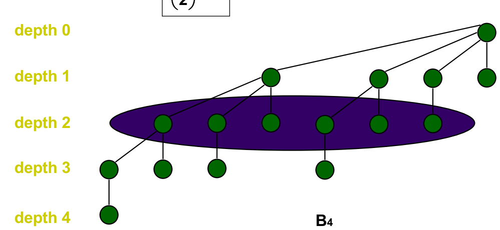
Binomial Tree

B_k has $\binom{k}{i}$ nodes at depth i.

$\binom{4}{2} = 6$



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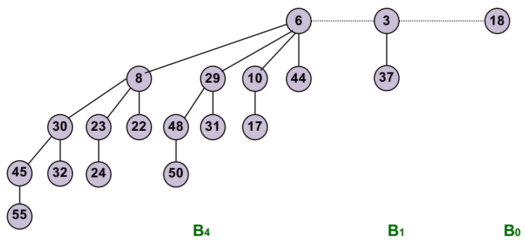
Binomial Heap

Binomial heap Vuillemin, 1978.

Sequence of binomial trees that satisfy binomial heap property:

- each tree is min-heap ordered
- top level: full or empty binomial tree of order k
- which are empty or full is based on the number of elements



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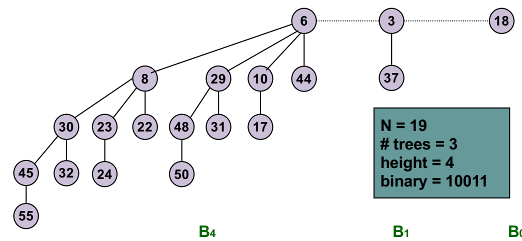
Binomial Heap

A_0: [18]
A_1: [3, 7]
A_2: empty
A_3: empty
A_4: [6, 8, 29, 10, 44, 30, 23, 22, 48, 31, 17, 45, 32, 24, 55]



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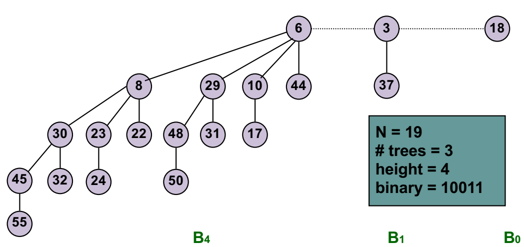
Binomial Heap: Properties

How many heaps?

O(log n) – binary number representation



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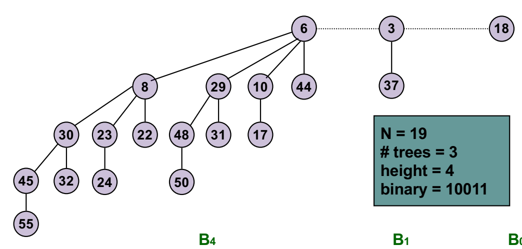
Binomial Heap: Properties

Where is the max/min?

Must be one of the
roots of the heaps



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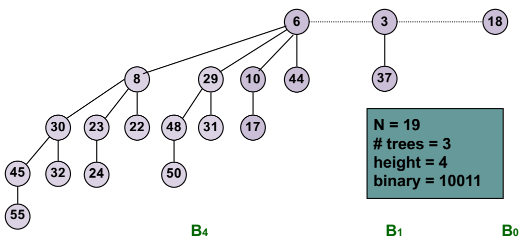
Binomial Heap: Properties

Runtime of max/min?

O(log n)



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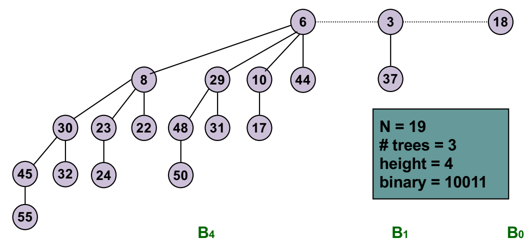
Binomial Heap: Properties

Height?

$\log_2 n$

- largest tree = $B_{\log n}$
- height of that tree is log n



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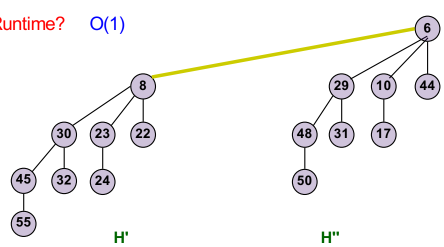
Binomial Heap: Union

How can we merge two binomial tree heaps of the same size (2^k)?

- connect roots of H' and H''
- choose smaller key to be root of H

Runtime? O(1)



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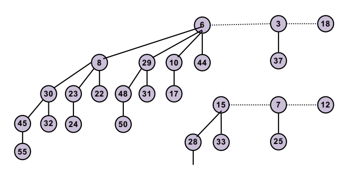
Binomial Heap: Union



How can we combine/merge binomial heaps (i.e. a combination of binomial tree heaps)?

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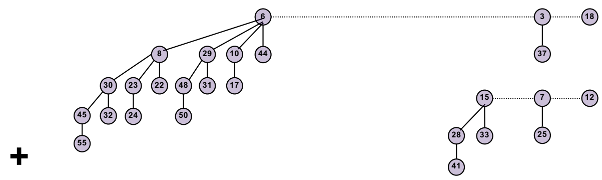
Binomial Heap: Union

Go through each tree size starting at 0 and merge as we go



19 + 7 = 26

		1	1	1	
	1	0	0	1	1
+	0	0	1	1	1
	1	1	0	1	0

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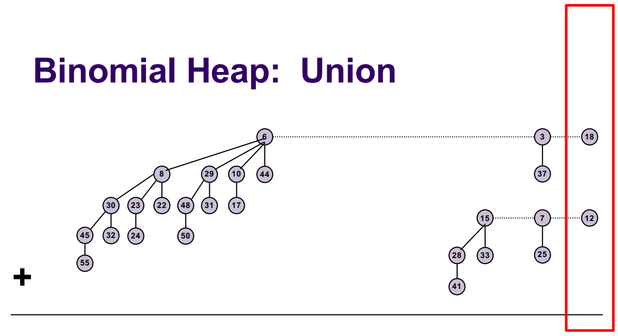
Binomial Heap: Union



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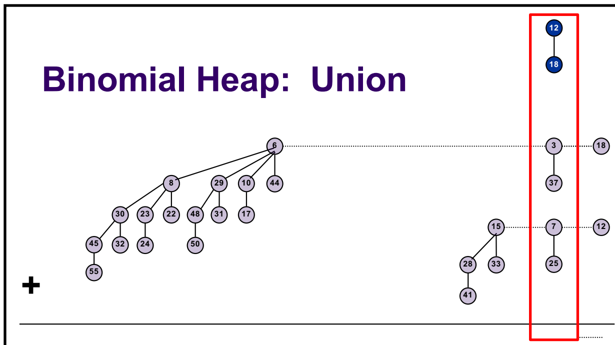
Binomial Heap: Union



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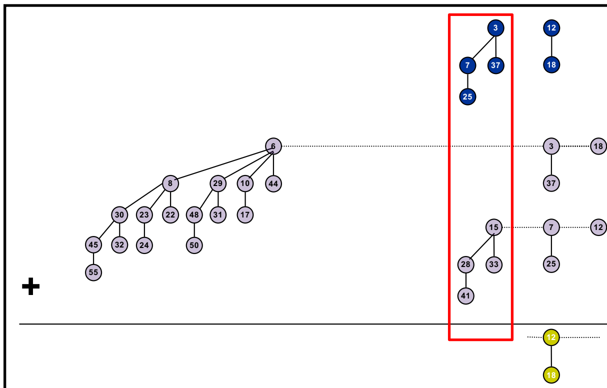

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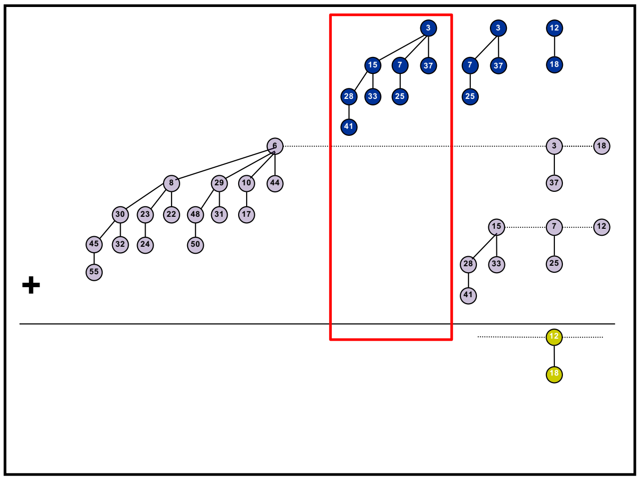

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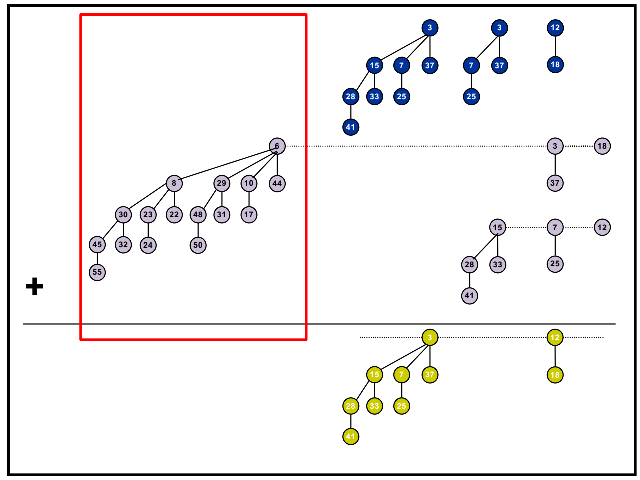
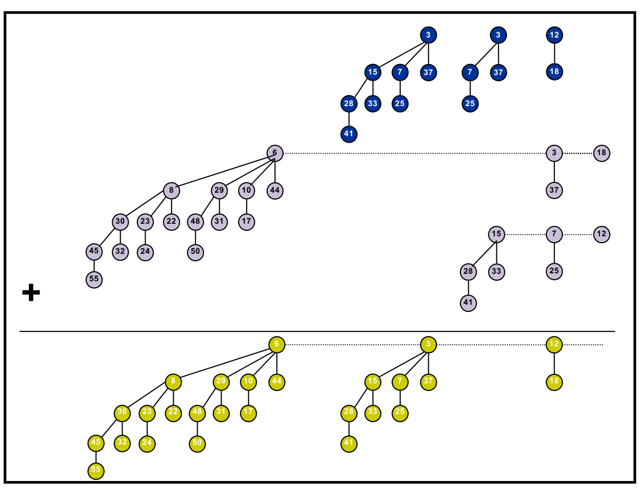


99



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Binomial Heap: Union

Analogous to binary addition

Running time?

- Proportional to number of trees in root lists 2 O($\log_2$ N)
- O(log N)

19 + 7 = 26

		1	1	1	
	1	0	0	1	1
+	0	0	1	1	1
	1	1	0	1	0

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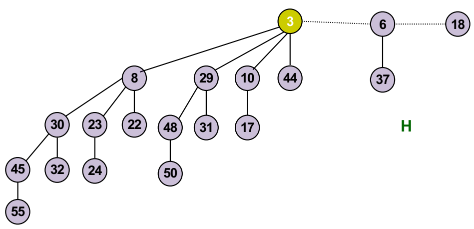
Binomial Heap: Delete Min/Max

We can find the min/max in O(log n).
How can we extract it?

Hint: B_k consists of binomial trees: $B_{k-1}, B_{k-2}, \ldots, B_0$



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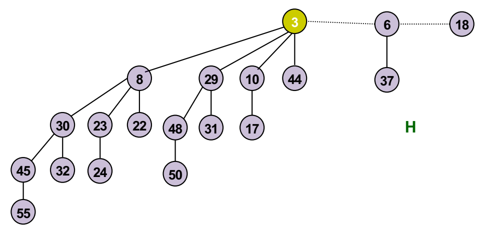
Binomial Heap: Delete Min

Delete node with minimum key in binomial heap H.

- Find root x with min key in root list of H, and delete
- H' ← broken binomial trees
- H ← Union(H', H)



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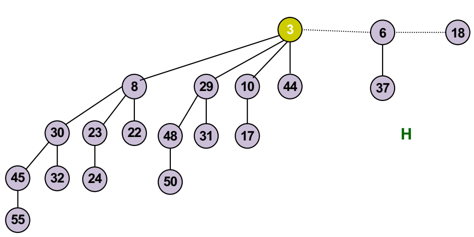
Binomial Heap: Delete Min

Delete node with minimum key in binomial heap H.

- Find root x with min key in root list of H, and delete
- H' ← broken binomial trees
- H ← Union(H', H)

Running time? O(log N)



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Heaps

Procedure	Binary heap (worst-case)	Binomial heap (worst-case)
BUILD-HEAP	$\Theta(n)$	$\Theta(n)$
INSERT	$\Theta(\log n)$	$O(\log n)$
MAXIMUM	$\Theta(1)$	$O(\log n)$
EXTRAC-MAX	$\Theta(\log n)$	$\Theta(\log n)$
UNION	$\Theta(n)$	$\Theta(\log n)$
INCREASE-ELEMENT	$\Theta(\log n)$	$\Theta(\log n)$
DELETE	$\Theta(\log n)$	$\Theta(\log n)$

(adapted from Figure 19.1, pg. 456 [1])

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Fibonacci Heaps

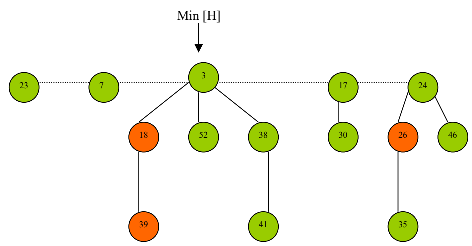
Similar to binomial heap

- A Fibonacci heap consists of a sequence of heaps

More flexible

- Heaps do not have to be binomial trees

More complicated ☺



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Heaps

Procedure	Binary heap (worst-case)	Binomial heap (worst-case)	Fibonacci heap (amortized)
BUILD-HEAP	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
INSERT	$\Theta(\log n)$	$O(\log n)$	$\Theta(1)$
MAXIMUM	$\Theta(1)$	$O(\log n)$	$\Theta(1)$
EXTRAC-MAX	$\Theta(\log n)$	$\Theta(\log n)$	$O(\log n)$
UNION	$\Theta(n)$	$\Theta(\log n)$	$\Theta(1)$
INCREASE-ELEMENT	$\Theta(\log n)$	$\Theta(\log n)$	$\Theta(1)$
DELETE	$\Theta(\log n)$	$\Theta(\log n)$	$O(\log n)$

(adapted from Figure 19.1, pg. 456 [1])

Should you always use a Fibonacci heap?

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Heaps

Procedure	Binary heap (worst-case)	Binomial heap (worst-case)	Fibonacci heap (amortized)
BUILD-HEAP	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
INSERT	$\Theta(\log n)$	$O(\log n)$	$\Theta(1)$
MAXIMUM	$\Theta(1)$	$O(\log n)$	$\Theta(1)$
EXTRAC-MAX	$\Theta(\log n)$	$\Theta(\log n)$	$O(\log n)$
UNION	$\Theta(n)$	$\Theta(\log n)$	$\Theta(1)$
INCREASE-ELEMENT	$\Theta(\log n)$	$\Theta(\log n)$	$\Theta(1)$
DELETE	$\Theta(\log n)$	$\Theta(\log n)$	$O(\log n)$

(adapted from Figure 19.1, pg. 456 [1])

- Extract-Max and Delete are O(n) worst case
- Constants can be large on some of the operations
- Complicated to implement

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Heaps

Procedure	Binary heap (worst-case)	Binomial heap (worst-case)	Fibonacci heap (amortized)
BUILD-HEAP	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
INSERT	$\Theta(\log n)$	$O(\log n)$	$\Theta(1)$
MAXIMUM	$\Theta(1)$	$O(\log n)$	$\Theta(1)$
EXTRAC-MAX	$\Theta(\log n)$	$\Theta(\log n)$	$O(\log n)$
UNION	$\Theta(n)$	$\Theta(\log n)$	$\Theta(1)$
INCREASE-ELEMENT	$\Theta(\log n)$	$\Theta(\log n)$	$\Theta(1)$
DELETE	$\Theta(\log n)$	$\Theta(\log n)$	$O(\log n)$

(adapted from Figure 19.1, pg. 456 [1])

Can we do better?

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