Recurrences

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Administrative

Mentor hours and office hours posted

Assignment 1 out
- Must work in groups of 2-3 within you LC

Learning communities
- still in flux!
- let me know ASAP if you have any problem
- will have first meetings this week

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Divide and Conquer

Divide: Break the problem into smaller sub-problems

Conquer: Solve the sub-problems. Generally, this involves waiting for the problem to be small enough that it is trivial to solve (i.e. 1 or 2 items)

Combine: Given the results of the solved sub-problems, combine them to generate a solution for the complete problem

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Divide and Conquer: some thoughts

Often, the sub-problem is the same as the original problem

Dividing the problem in half frequently does the job

May have to get creative about how the data is split

Splitting tends to generate run times with log n in them

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Divide and conquer

One approach:

- Pretend like you have a working version of your function, but it only works on smaller sub-problems
- If you split up the current problem in some way (e.g. in half) and solved those sub-problems, how could you then get the solution to the larger problem?

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MergeSort

```
MERGE-SORT(A)
  if length[A] == 1
        return A
  else
        q ← ⌊length[A]/2⌋
        create arrays L[1..q] and R[q+1..length[A]]
        copy A[1..q] to L
        copy A[q+1..length[A]] to R
        LS ← MERGE-SORT(L)
        RS ← MERGE-SORT(R)
        return MERGE(LS, RS)
```

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MergeSort: Merge

Assuming L and R are sorted already, merge the two to create a single sorted array

```
MERGE(L, R)
  create array B of length length[L] + length[R]
  i ← 1
  j ← 1
  for k ← 1 to length[B]
        if j > length[R] or (i ≤ length[L] and L[i] ≤ R[j])
              B[k] ← L[i]
              i ← i + 1
        else
              B[k] ← R[j]
              j ← j + 1
  return B
```

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Merge

L: 1 3 5 8 R: 2 4 6 7


```
MERGE(L, R)
  create array B of length length[L] + length[R]
  i ← 1
  j ← 1
  for k ← 1 to length[B]
        if j > length[R] or (i ≤ length[L] and L[i] ≤ R[j])
              B[k] ← L[i]
              i ← i + 1
        else
              B[k] ← R[j]
              j ← j + 1
  return B
```


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Merge

L: 1 3 5 8   R: 2 4 6 7

B:

```
MERGE(L, R)
create array B of length length[L] + length[R]
i ← 1
j ← 1
for k ← 1 to length[B]
        if j > length[R] or (i ≤ length[L] and L[i] ≤ R[j])
                B[k] ← L[i]
                i ← i + 1
        else
                B[k] ← R[j]
                j ← j + 1
return B
```

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Merge

i   j

L: 1 3 5 8   R: 2 4 6 7

B:

```
MERGE(L, R)
create array B of length length[L] + length[R]
i ← 1
j ← 1
for k ← 1 to length[B]
        if j > length[R] or (i ≤ length[L] and L[i] ≤ R[j])
                B[k] ← L[i]
                i ← i + 1
        else
                B[k] ← R[j]
                j ← j + 1
return B
```

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Merge

i   j

L: 1 3 5 8   R: 2 4 6 7

B:

```
MERGE(L, R)
create array B of length length[L] + length[R]
i ← 1
j ← 1
for k ← 1 to length[B]
        if j > length[R] or (i ≤ length[L] and L[i] ≤ R[j])
                B[k] ← L[i]
                i ← i + 1
        else
                B[k] ← R[j]
                j ← j + 1
return B
```

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Merge

i   j

L: 1 3 5 8   R: 2 4 6 7

B: 1

```
MERGE(L, R)
create array B of length length[L] + length[R]
i ← 1
j ← 1
for k ← 1 to length[B]
        if j > length[R] or (i ≤ length[L] and L[i] ≤ R[j])
                B[k] ← L[i]
                i ← i + 1
        else
                B[k] ← R[j]
                j ← j + 1
return B
```

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Merge

i j

L: 1 3 5 8 R: 2 4 6 7

B: 1

```
MERGE(L, R)
create array B of length length[L] + length[R]
i ← 1
j ← 1
for k ← 1 to length[B]
        if j > length[R] or (i ≤ length[L] and L[i] ≤ R[j])
                B[k] ← L[i]
                i ← i + 1
        else
                B[k] ← R[j]
                j ← j + 1
return B
```

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Merge

i j

L: 1 3 5 8 R: 2 4 6 7

B: 1 2

```
MERGE(L, R)
create array B of length length[L] + length[R]
i ← 1
j ← 1
for k ← 1 to length[B]
        if j > length[R] or (i ≤ length[L] and L[i] ≤ R[j])
                B[k] ← L[i]
                i ← i + 1
        else
                B[k] ← R[j]
                j ← j + 1
return B
```

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Merge

i j

L: 1 3 5 8 R: 2 4 6 7

B: 1 2

```
MERGE(L, R)
create array B of length length[L] + length[R]
i ← 1
j ← 1
for k ← 1 to length[B]
        if j > length[R] or (i ≤ length[L] and L[i] ≤ R[j])
                B[k] ← L[i]
                i ← i + 1
        else
                B[k] ← R[j]
                j ← j + 1
return B
```

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Merge

i j

L: 1 3 5 8 R: 2 4 6 7

B: 1 2 3

```
MERGE(L, R)
create array B of length length[L] + length[R]
i ← 1
j ← 1
for k ← 1 to length[B]
        if j > length[R] or (i ≤ length[L] and L[i] ≤ R[j])
                B[k] ← L[i]
                i ← i + 1
        else
                B[k] ← R[j]
                j ← j + 1
return B
```

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Merge

i, j

L: 1 3 5 8 R: 2 4 6 7

B: 1 2 3

```
MERGE(L, R)
create array B of length length[L] + length[R]
i ← 1
j ← 1
for k ← 1 to length[B]
    if j > length[R] or (i ≤ length[L] and L[i] ≤ R[j])
        B[k] ← L[i]
        i ← i + 1
    else
        B[k] ← R[j]
        j ← j + 1
return B
```

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Merge

i, j

L: 1 3 5 8 R: 2 4 6 7

B: 1 2 3 4

```
MERGE(L, R)
create array B of length length[L] + length[R]
i ← 1
j ← 1
for k ← 1 to length[B]
    if j > length[R] or (i ≤ length[L] and L[i] ≤ R[j])
        B[k] ← L[i]
        i ← i + 1
    else
        B[k] ← R[j]
        j ← j + 1
return B
```

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Merge

i, j

L: 1 3 5 8 R: 2 4 6 7

B: 1 2 3 4

```
MERGE(L, R)
create array B of length length[L] + length[R]
i ← 1
j ← 1
for k ← 1 to length[B]
    if j > length[R] or (i ≤ length[L] and L[i] ≤ R[j])
        B[k] ← L[i]
        i ← i + 1
    else
        B[k] ← R[j]
        j ← j + 1
return B
```

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Merge

i, j

L: 1 3 5 8 R: 2 4 6 7

B: 1 2 3 4 5

```
MERGE(L, R)
create array B of length length[L] + length[R]
i ← 1
j ← 1
for k ← 1 to length[B]
    if j > length[R] or (i ≤ length[L] and L[i] ≤ R[j])
        B[k] ← L[i]
        i ← i + 1
    else
        B[k] ← R[j]
        j ← j + 1
return B
```

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Merge

i, j

L: 1 3 5 8 R: 2 4 6 7

B: 1 2 3 4 5

```
MERGE(L, R)
create array B of length length[L] + length[R]
i ← 1
j ← 1
for k ← 1 to length[B]
        if j > length[R] or (i ≤ length[L] and L[i] ≤ R[j])
                B[k] ← L[i]
                i ← i + 1
        else
                B[k] ← R[j]
                j ← j + 1
return B
```

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Merge

i, j

L: 1 3 5 8 R: 2 4 6 7

B: 1 2 3 4 5 6

```
MERGE(L, R)
create array B of length length[L] + length[R]
i ← 1
j ← 1
for k ← 1 to length[B]
        if j > length[R] or (i ≤ length[L] and L[i] ≤ R[j])
                B[k] ← L[i]
                i ← i + 1
        else
                B[k] ← R[j]
                j ← j + 1
return B
```

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Merge

i, j

L: 1 3 5 8 R: 2 4 6 7

B: 1 2 3 4 5 6

```
MERGE(L, R)
create array B of length length[L] + length[R]
i ← 1
j ← 1
for k ← 1 to length[B]
        if j > length[R] or (i ≤ length[L] and L[i] ≤ R[j])
                B[k] ← L[i]
                i ← i + 1
        else
                B[k] ← R[j]
                j ← j + 1
return B
```

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Merge

i, j

L: 1 3 5 8 R: 2 4 6 7

B: 1 2 3 4 5 6 7

```
MERGE(L, R)
create array B of length length[L] + length[R]
i ← 1
j ← 1
for k ← 1 to length[B]
        if j > length[R] or (i ≤ length[L] and L[i] ≤ R[j])
                B[k] ← L[i]
                i ← i + 1
        else
                B[k] ← R[j]
                j ← j + 1
return B
```

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Merge

i j

L: 1 3 5 8   R: 2 4 6 7

B: 1 2 3 4 5 6 7

```
Merge(L, R)
create array B of length length[L] + length[R]
i ← 1
j ← 1
for k ← 1 to length[B]
        if j > length[R] or (i ≤ length[L] and L[i] ≤ R[j])
                B[k] ← L[i]
                i ← i + 1
        else
                B[k] ← R[j]
                j ← j + 1
return B
```

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Merge

i j

L: 1 3 5 8   R: 2 4 6 7

B: 1 2 3 4 5 6 7 8

```
Merge(L, R)
create array B of length length[L] + length[R]
i ← 1
j ← 1
for k ← 1 to length[B]
        if j > length[R] or (i ≤ length[L] and L[i] ≤ R[j])
                B[k] ← L[i]
                i ← i + 1
        else
                B[k] ← R[j]
                j ← j + 1
return B
```

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Merge

Running time?

```
Merge(L, R)
create array B of length length[L] + length[R]
i ← 1
j ← 1
for k ← 1 to length[B]
        if j > length[R] or (i ≤ length[L] and L[i] ≤ R[j])
                B[k] ← L[i]
                i ← i + 1
        else
                B[k] ← R[j]
                j ← j + 1
return B
```

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Merge

Running time? Θ(n) - linear

```
Merge(L, R)
create array B of length length[L] + length[R]
i ← 1
j ← 1
for k ← 1 to length[B]
        if j > length[R] or (i ≤ length[L] and L[i] ≤ R[j])
                B[k] ← L[i]
                i ← i + 1
        else
                B[k] ← R[j]
                j ← j + 1
return B
```

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MergeSort

Running time?

```
MERGE-SORT(A)
if length[A] == 1
        return A
else
        q ← ⌊length[A] /2⌋
        create arrays L[1..q] and R[q + 1.. length[A]]
        copy A[1..q] to L
        copy A[q + 1.. length[A]] to R
        LS ← MERGE-SORT(L)
        RS ← MERGE-SORT(R)
        return MERGE(LS, RS)
```

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Merge-Sort

Running time?

$$T(n) = \begin{cases} c & \text{if } n \text{ is small} \\ 2T(n/2) + D(n) + C(n) & \text{otherwise} \end{cases}$$

D(n): cost of splitting (dividing) the data

C(n): cost of merging/combining the data

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Merge-Sort

Running time?

$$T(n) = \begin{cases} c & \text{if } n \text{ is small} \\ 2T(n/2) + D(n) + C(n) & \text{otherwise} \end{cases}$$

D(n): cost of splitting (dividing) the data - linear Θ(n)

C(n): cost of merging/combining the data – linear Θ(n)

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Merge-Sort

Running time?

$$T(n) = \begin{cases} c & \text{if } n \text{ is small} \\ 2T(n/2) + cn & \text{otherwise} \end{cases}$$

Which is?

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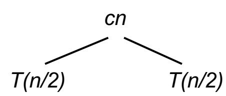
Merge-Sort

$$T(n)=\begin{cases} c & \text{if } n \text{ is small} \\ 2T(n/2)+cn & \text{otherwise} \end{cases}$$



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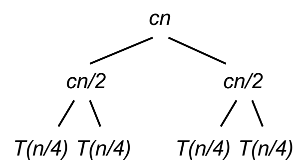
Merge-Sort

$$T(n)=\begin{cases} c & \text{if } n \text{ is small} \\ 2T(n/2)+cn & \text{otherwise} \end{cases}$$



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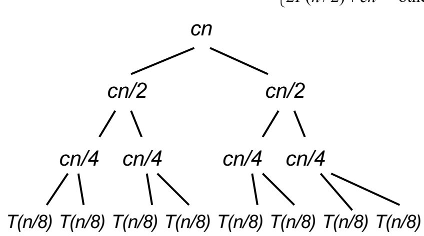
Merge-Sort

$$T(n)=\begin{cases} c & \text{if } n \text{ is small} \\ 2T(n/2)+cn & \text{otherwise} \end{cases}$$



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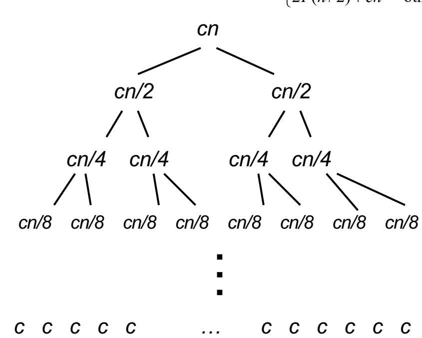
Merge-Sort

$$T(n)=\begin{cases} c & \text{if } n \text{ is small} \\ 2T(n/2)+cn & \text{otherwise} \end{cases}$$



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Merge-Sort

$$T(n) = \begin{cases} c & \text{if } n \text{ is small} \\ 2T(n/2) + cn & \text{otherwise} \end{cases}$$

cn — **cn**

cn/2 *cn/2* — **cn**

cn/4 *cn/4* *cn/4* *cn/4* — **cn**

cn/8 *cn/8* *cn/8* *cn/8* *cn/8* *cn/8* *cn/8* *cn/8* — **cn**

⋮

c c c c c … c c c c c c — **cn**

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Merge-Sort

$$T(n) = \begin{cases} c & \text{if } n \text{ is small} \\ 2T(n/2) + cn & \text{otherwise} \end{cases}$$

cn — **cn**

cn/2 *cn/2* — **cn**

cn/4 *cn/4* *cn/4* *cn/4* — **cn**

cn/8 *cn/8* *cn/8* *cn/8* *cn/8* *cn/8* *cn/8* *cn/8* — **cn**

⋮

c c c c c … c c c c c c — **cn**

Depth?

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Merge-Sort

We can calculate the depth, by determining when the recursion gets to down to a small problem size, e.g. 1

At each level, we divide by 2

$$\frac{n}{2^d} = 1$$

$$2^d = n$$

$$\log 2^d = \log n$$

$$d \log 2 = \log n$$

$$d = \log_2 n$$

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Merge-Sort

$$T(n) = \begin{cases} c & \text{if } n \text{ is small} \\ 2T(n/2) + cn & \text{otherwise} \end{cases}$$

Running time?

- Each level costs *cn*
- log *n* levels

cn log *n* = Θ(*n* log *n*)

Why don't we write it as n $\log_2$ n?

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Log properties

$$\log_a b = \frac{\log b}{\log a}$$

$$n \log_2 n = \frac{n \log n}{\log 2}$$

$$n \log_2 n = \frac{n \log n}{c} = \theta(n \log n)$$

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Recurrence

A function that is defined with respect to itself on smaller inputs

$$T(n) = 2T(n/2) + n$$

$$T(n) = 16T(n/4) + n$$

$$T(n) = 2T(n-1) + n^2$$

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Why are we interested in recurrences?

Computational cost of divide and conquer algorithms

$$T(n) = aT(n/b) + D(n) + C(n)$$

- *a* subproblems of size *n/b*
- *D(n)* the cost of dividing the data
- *C(n)* the cost of recombining the subproblem solutions

In general, the runtimes of most recursive algorithms can be expressed as recurrences

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The challenge

Recurrences are often easy to define because they mimic the structure of the program

But… they do not directly express the computational cost, i.e. n, n^2, …

We want to remove self-recurrence and find a more understandable form for the function

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Three approaches

Substitution method: when you have a good guess of the solution, prove that it's correct

Recursion-tree method: If you don't have a good guess, the recursion tree can help

- Calculate exactly (like we did with MergeSort)
- Use it to get a good quest, then prove with substitution method.

Master method: Provides solutions for recurrences of the form:

$$T(n) = aT(n/b) + f(n)$$

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Substitution method

Guess the form of the solution

Then prove it's correct by induction

$$T(n) = T(n/2) + d$$

Halves the input then a constant amount of work

Guesses?

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Substitution method

Guess the form of the solution

Then prove it's correct by induction

$$T(n) = T(n/2) + d$$

Halves the input then a constant amount of work

Similar to binary search:

Guess: O(log n)

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Proof?

$$T(n) = T(n / 2) + d = O(\log n)$$

Ideas?

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Proof?

$$T(n) = T(n/2) + d = O(\log n)$$

Proof by induction!
-Assume it's true for smaller T(k), i.e. k < n
-prove that it's then true for current T(n)

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$$T(n) = T(n/2) + d$$

Assume *T(k) = O(log k)* for all *k < n*
Show that *T(n) = O(log n)*

From our assumption, *T(n/2) = O(log n/2):*

$$O(g(n)) = \left\{ f(n) : \begin{matrix} \text{there exists positive constants } c \text{ and } n \text{ such that} \\ 0 \le f(n) \le cg(n) \text{ for all } n \ge n_0 \end{matrix} \right\}$$

From the definition of big-O: *T(n/2) ≤ c log(n/2)*

How do we now prove T(n) = O(log n)?

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$$T(n) = T(n/2) + d$$

To prove that *T(n) = O(log n)* identify the appropriate constants:

$$O(g(n)) = \left\{ f(n) : \begin{matrix} \text{there exists positive constants } c \text{ and } n \text{ such that} \\ 0 \le f(n) \le cg(n) \text{ for all } n \ge n_0 \end{matrix} \right\}$$

i.e. some constant *c'* such that *T(n) ≤ c' log n*

$$\begin{aligned} T(n) &= T(n/2) + d \\ &\le c\log\left(\frac{n}{2}\right) + d \quad \text{from our inductive hypothesis} \\ &\le c\log n - c\log 2 + d \\ &\le c\log n - c + d \quad \text{residual} \end{aligned}$$

Key question: does a constant exist such that:
$T(n) \le c' \log n$

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$$T(n) = T(n/2) + d$$

To prove that *T(n) = O(log n)* identify the appropriate constants:

$$O(g(n)) = \left\{ f(n) : \begin{matrix} \text{there exists positive constants } c \text{ and } n \text{ such that} \\ 0 \le f(n) \le cg(n) \text{ for all } n \ge n_0 \end{matrix} \right\}$$

i.e. some constant *c'* such that *T(n) ≤ c' log n*

Key question: does a constant exist such that:
$T(n) \le c' \log n$

$$T(n) \le c\log n - c + d$$

if $c \ge d$, then, just let c' = c

$$T(n) \le c\log n - c + d \le c\log n$$

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$$T(n) = T(n/2) + d$$

To prove that *T(n) = O(log n)* identify the appropriate constants:

$$O(g(n)) = \left\{ f(n): \begin{array}{l} \text{there exists positive constants } c \text{ and } n \text{ such that} \\ 0 \le f(n) \le cg(n) \text{ for all } n \ge n_0 \end{array} \right\}$$

i.e. some constant *c'* such that *T(n) ≤ c' log n*

Key question: does a constant exist such that:
$T(n) \le c' \log n$

$$\mathrm{T(n)} \le c \log n - c + d$$

if $c < d$, let c' = d+1 and

$$\mathrm{T(n)} \le c \log n - c + d \le d \log n + d \log d$$

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Base case?

For an inductive proof we need to show two things:

- *Show that it holds for some base case*
- Assuming it's true for *k* < *n* show it's true for *n*

What is the base case in our situation?

$$T(n) = \begin{cases} \Theta(1) & \text{if } n \text{ is small} \\ T(n/2) + d & \text{otherwise} \end{cases}$$

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$$T(n) = T(n-1) + n$$

Guess the solution?

At each iteration, does a linear amount of work (i.e. iterate over the data) and reduces the size by one at each step

O(n^2)

Assume *T(k) = O(k^2)* for all *k < n*

- again, this implies that *T(n-1) ≤ c(n-1)*^{2}

Show that *T(n) = O(n^2)*, i.e. *T(n) ≤ c'n^2*

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$$\begin{aligned} T(n) &= T(n-1) + n \\ &\le c(n-1)^2 + n \quad \text{from our inductive hypothesis} \\ &= c(n^2 - 2n + 1) + n \\ &= cn^2 - 2cn + c + n \quad \text{residual} \end{aligned}$$

if $-2cn + c + n \le 0$

then let c' = c and there exists a constant such that $T(n) \le c'n^2$

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$$
\begin{aligned}
T(n) &= T(n-1) + n \\
&\le c(n-1)^2 + n \quad \text{from our inductive hypothesis} \\
&= c(n^2 - 2n + 1) + n \\
&= cn^2 - 2cn + c + n \quad \text{residual}
\end{aligned}
$$

$$
\begin{aligned}
-2cn + c + n &\le 0 \\
-2cn + c &\le -n \\
c(-2n+1) &\le -n \\
c &\ge \frac{n}{2n-1} \\
c &\ge \frac{1}{2 - 1/n}
\end{aligned}
$$

which holds for any $c \geq 1$ for $n \geq 1$

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