MAX FLOW

David Kauchak
CS 140 – Spring 2023

1

Admin

Assignment 10

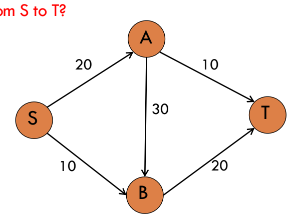
Checkpoint next Monday (sample problems coming soon)

2

Student networking

You decide to create your own computer network:

- You get three of your friends and string some network cables
- Because of capacity (due to cable type, distance, computer, etc) you can only send a certain amount of data to each person
- If edges denote capacity, what is the maximum throughput you can you send from S to T?

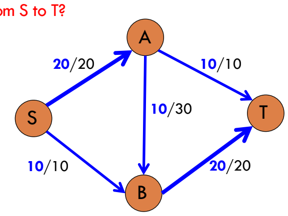


3

Student networking

You decide to create your own campus network:

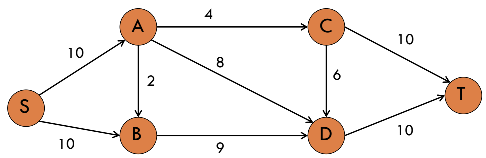
- You get three of your friends and string some network cables
- Because of capacity (due to cable type, distance, computer, etc) you can only send a certain amount of data to each person
- If edges denote capacity, what is the maximum throughput you can you send from S to T?



30 units

4

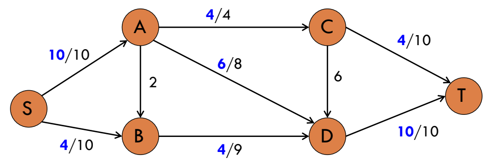
Another flow problem



How much water flow can we continually send from s to t?

5

Another flow problem



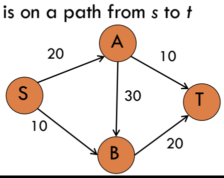
14 units

6

Flow graph/networks

Flow network

- directed, weighted graph (V, E)
- positive edge weights indicating the "capacity" (generally, assume integers)
- contains a single source $s \in V$ with no incoming edges
- contains a single sink/target $t \in V$ with no outgoing edges
- every vertex is on a path from s to t



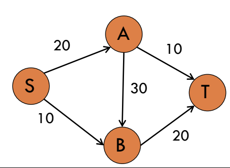
7

Flow constraints

in-flow = out-flow for every vertex (except s, t)

flow along an edge cannot exceed the edge capacity

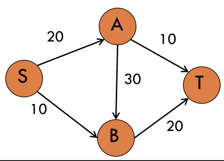
flows are positive



9

Max flow problem

Given a flow network: *what is the maximum flow we can send from s to t that meet the flow constraints?*



10

Applications?

network flow
- water, electricity, sewage, cellular…
- traffic/transportation capacity

bipartite matching

sports elimination

…

11

Max flow origins

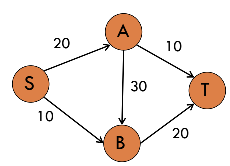
Rail networks of the Soviet Union in the 1950's
The US wanted to know how quickly the Soviet Union could get supplies through its rail network to its satellite states in Eastern Europe.
In addition, the US wanted to know which rails it could destroy most easily to cut off the satellite states from the rest of the Soviet Union.

These two problems are closely related: solving the **max flow problem** also solves the **min cut problem** of figuring out the cheapest way to cut off the Soviet Union from its satellites.

Source: lbackstrom, The Importance of Algorithms, at www.topcoder.com

12

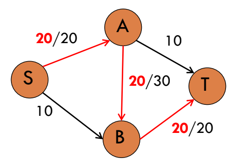
Algorithm idea



14

Algorithm idea

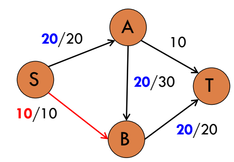
send some flow down a path



15

Algorithm idea

send some flow down a path

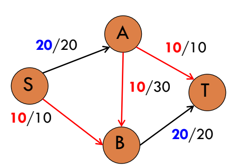


Now what?

16

Algorithm idea

reroute some of the flow

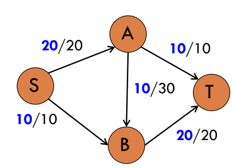


Total flow?

17

Algorithm idea

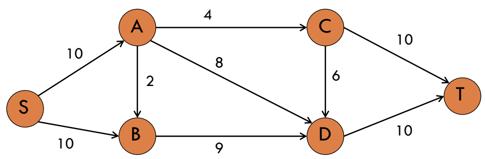
reroute some of the flow



30

18

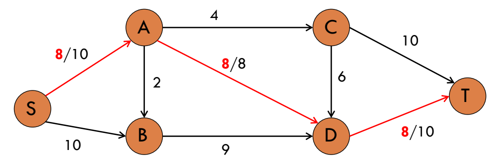
Algorithm idea



19

Algorithm idea

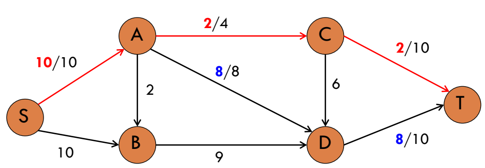
send some flow down a path



20

Algorithm idea

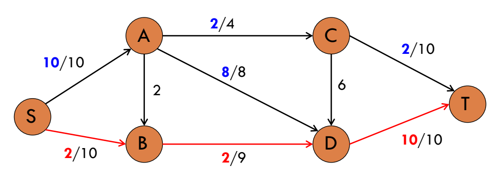
send some flow down a path



21

Algorithm idea

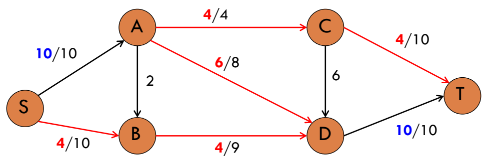
send some flow down a path



22

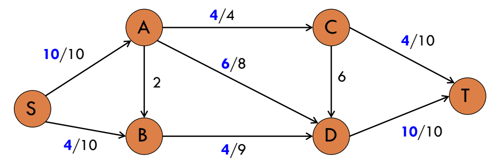
Algorithm idea

reroute some of the flow



23

Algorithm idea

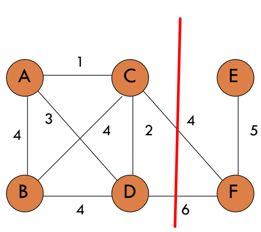


Are we done?
Is this the best we can do?

24

Cuts

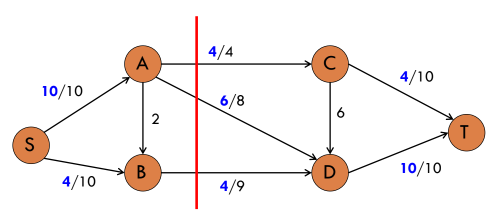
A cut is a partitioning of the vertices into two sets S_s and $S_t = V\text{-}S_s$



25

Flow across cuts

In flow graphs, we're interested in cuts that separate s from t, that is $s \in S_s$ and $t \in S_t$

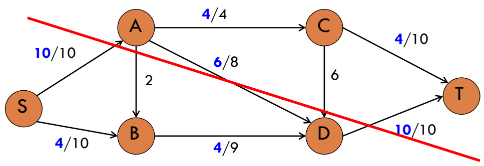


26

Flow across cuts

The flow "across" a cut is the total flow from nodes in S_s to nodes in S_t ***minus*** the total from nodes in S_t to S_s

What is the flow across this cut?

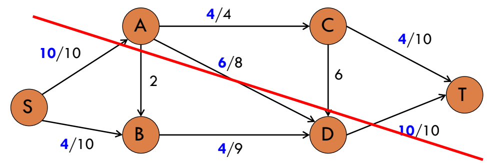


27

Flow across cuts

The flow "across" a cut is the total flow from nodes in S_s to nodes in S_t ***minus*** the total from nodes in S_t to S_s

10+10-6 = 14

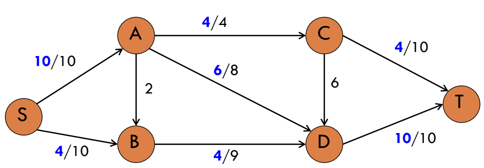


28

Flow across cuts

Consider any cut where $s \in S_s$ and $t \in S_t$, i.e. the cut partitions the source from the sink

What do we know about the flow across the any such cut?

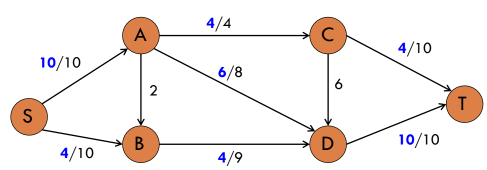


29

Flow across cuts

Consider any cut where $s \in S_s$ and $t \in S_t$, i.e. the cut partitions the source from the sink

The flow across ANY such cut is the same and is the current flow in the network

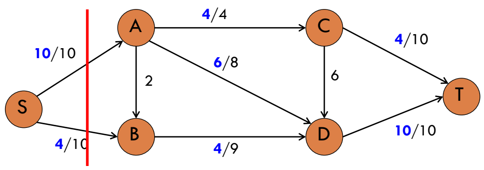


30

Flow across cuts

Consider any cut where s ∈ S_s and t ∈ S_t, i.e. the cut partitions the source from the sink

4+10 = 14

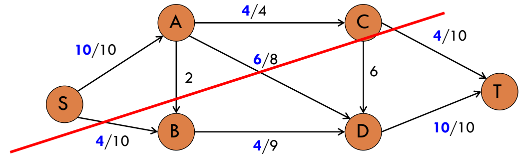


31

Flow across cuts

Consider any cut where s ∈ S_s and t ∈ S_t, i.e. the cut partitions the source from the sink

4+6+4 = 14

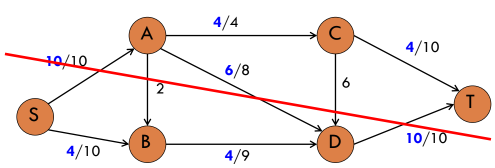


32

Flow across cuts

Consider any cut where s ∈ S_s and t ∈ S_t, i.e. the cut partitions the source from the sink

10+10-6 = 14



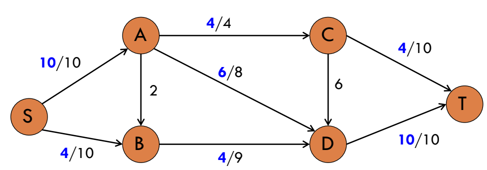
33

Flow across cuts

Consider any cut where s ∈ S_s and t ∈ S_t, i.e. the cut partitions the source from the sink

The flow across ANY such cut is the same and is the current flow in the network

Why? Can you prove it?



34

Flow across cuts

The flow across ANY such cut is the same and is the current flow in the network

Inductively?

- every vertex is on a path from *s* to *t*
- in-flow = out-flow for every vertex (except s, t)
- flow along an edge cannot exceed the edge capacity
- flows are positive

35

Flow across cuts

The flow across ANY such cut is the same and is the current flow in the network

Base case: $S_s = s$

- Flow is total from from s to t: therefore the total flow out of s should be the flow
- All flow from s gets to t
  - every vertex is on a path from s to t
  - in-flow = out-flow

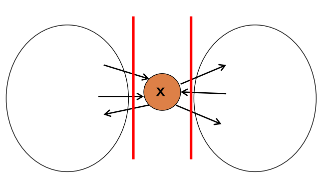
36

Flow across cuts

The flow across ANY such cut is the same and is the current flow in the network

Inductive case: Consider moving a node *x* from S_t to S_s

Is the flow across the different partitions the same?



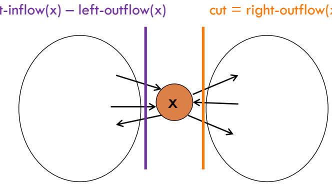
37

Flow across cuts

Inductive case: Consider moving a node *x* from S_t to S_s

cut = left-inflow(x) – left-outflow(x)

cut = right-outflow(x) – right-inflow(x)



left-inflow(x) + right-inflow(x) = left-outflow(x) + right-outflow(x)

in-flow = out-flow

left-inflow(x) - left-outflow(x) = right-outflow(x) – right-inflow(x)

38

Flow across cuts

Consider any cut where $s \in S_s$ and $t \in S_t$, i.e. the cut partitions the source from the sink

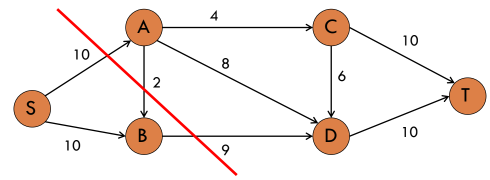
The flow across ANY such cut is the same and is the current flow in the network

39

Capacity of a cut

The "**capacity of a cut**" is the maximum flow that we *could* send from nodes in S_s to nodes in S_t (i.e. across the cut)

How do we calculate the capacity?

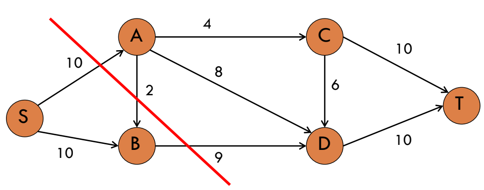


40

Capacity of a cut

The "**capacity of a cut**" is the maximum flow that we *could* send from nodes in S_s to nodes in S_t (i.e. across the cut)

Capacity is the sum of the edges from S_s to S_t



10 + 9 = 19

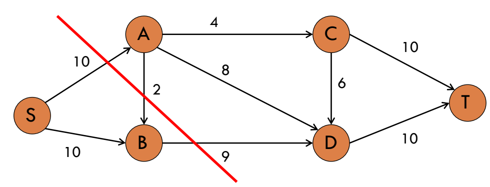
41

Capacity of a cut

The "**capacity of a cut**" is the maximum flow that we *could* send from nodes in S_s to nodes in S_t (i.e. across the cut)

Capacity is the sum of the edges from S_s to S_t

Why?



42

Capacity of a cut

The "**capacity of a cut**" is the maximum flow that we *could* send from nodes in S_s to nodes in S_t (i.e. across the cut)

Capacity is the sum of the edges from S_s to S_t

- Any more and we would violate the edge capacity constraint
- Any less and it would not be maximal, since we could simply increase the flow

43

Max Power

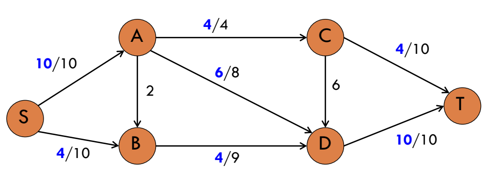
https://www.youtube.com/watch?v=BSVms6cT9nk

44

Maximum flow

For any cut where $s \in S_s$ and $t \in S_t$

- the flow across the cut is the same
- the maximum capacity (i.e. flow) across the cut is the sum of the capacities for edges from S_s to S_t



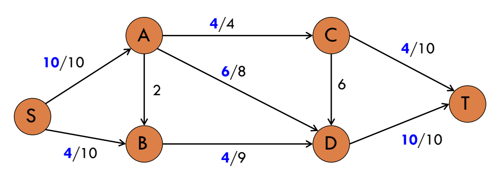
Are we done?
Is this the best we can do?

46

Maximum flow

For any cut where $s \in S_s$ and $t \in S_t$

- the flow across the cut is the same
- the maximum capacity (i.e. flow) across the cut is the sum of the capacities for edges from S_s to S_t



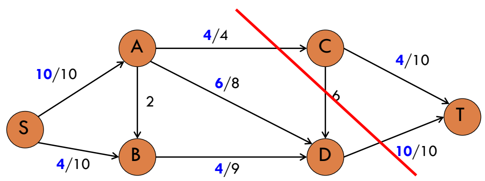
We can do no better than the minimum capacity cut!

47

Maximum flow

What is the minimum capacity cut for this graph?

Capacity = 10 + 4



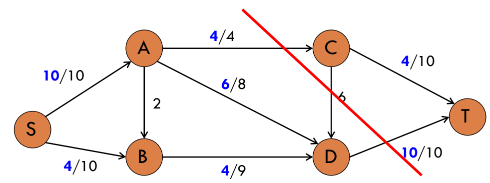
Is this the best we can do?

48

Maximum flow

What is the minimum capacity cut for this graph?

Capacity = 10 + 4

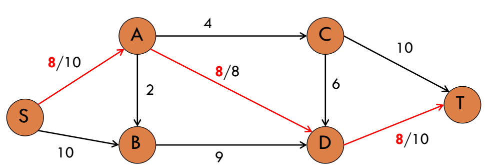


flow = minimum capacity, so we can do no better

49

Algorithm idea

send some flow down a path

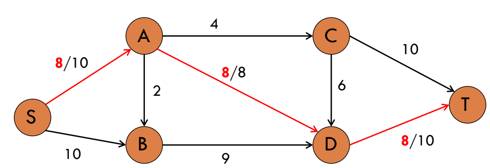


How do we determine the path to send flow down?

50

Algorithm idea

send some flow down a path

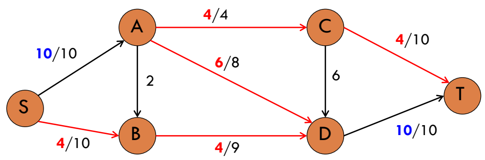


Search for a path with remaining capacity from s to t

51

Algorithm idea

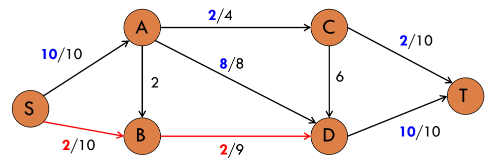
reroute some of the flow



How do we handle "rerouting" flow?

52

Algorithm idea

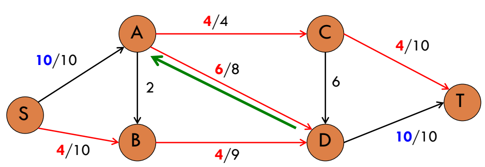


During the search, if an edge has some flow, we consider "reversing" some of that flow

53

Algorithm idea

reroute some of the flow



During the search, if an edge has some flow, we consider "reversing" some of that flow

54

The residual graph

The *residual graph* G_f is constructed from G

For each edge e in the original graph (G):

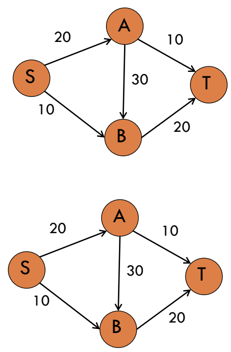
- if *flow(e)* < capacity(e)
  - introduce an edge in G_f with capacity = capacity(e)-flow(e)
  - this represents the remaining flow we can still push
- if flow(e) > 0
  - introduce an edge in G_f in the *opposite direction* with capacity = flow(e)
  - this represents the flow that we can reroute/reverse

55

Algorithm idea

G

G_f



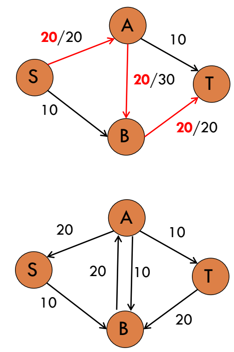
Find a path from s to t in G_f

56

Algorithm idea

G

G_f



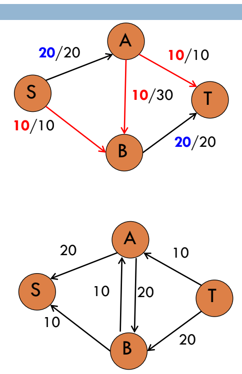
Find a path from s to t in G_f

57

Algorithm idea

G

G_f



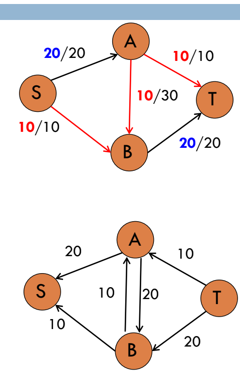
Find a path from s to t in G_f

58

Algorithm idea

G

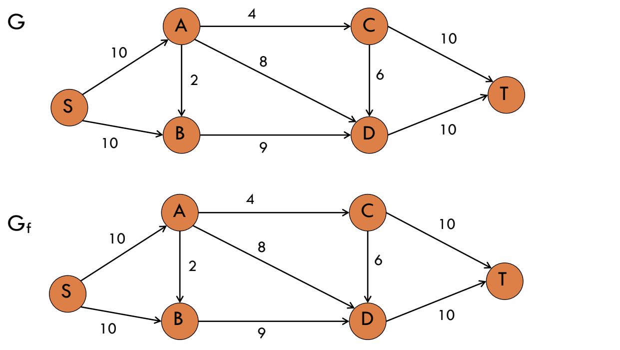
G_f



None exist… done!

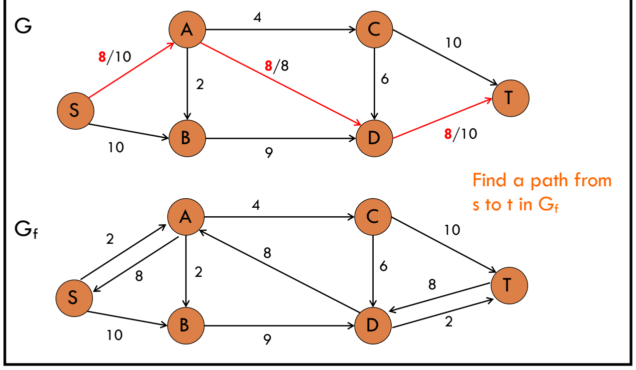
59

Algorithm idea



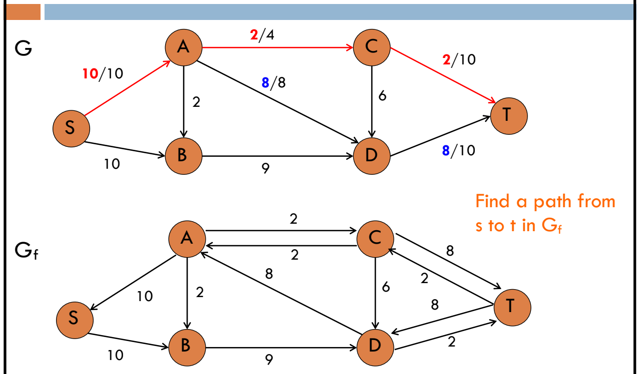
60

Algorithm idea



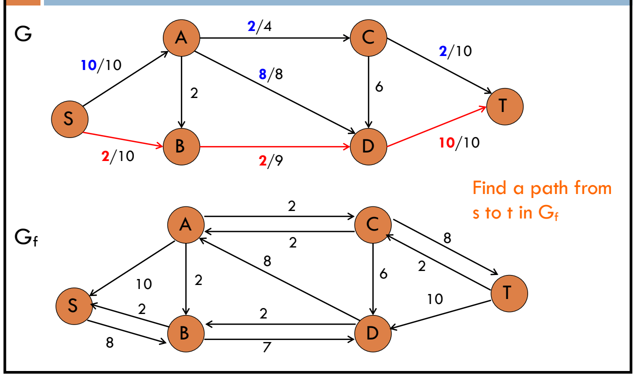
61

Algorithm idea



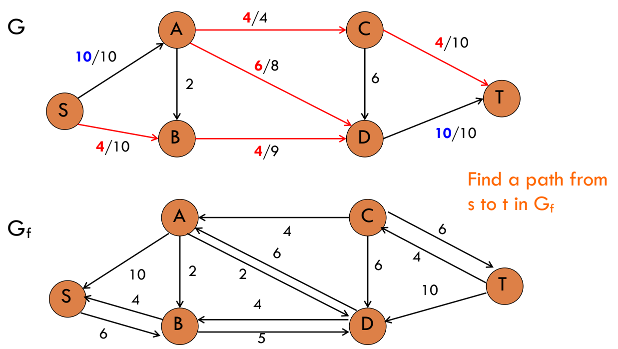
62

Algorithm idea



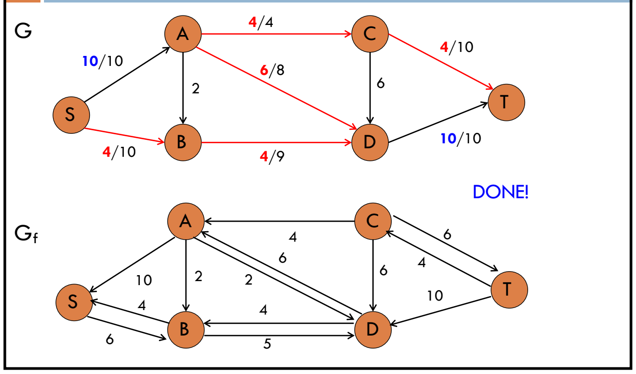
63

Algorithm idea



64

Algorithm idea



65

Ford-Fulkerson

Ford-Fulkerson(G, s, t)
  flow = 0 for all edges
  G_f = residualGraph(G)
  while a simple path exists from s to t in G_f
    send as much flow along the path as possible
    G_f = residualGraph(G)
  return flow

a simple path contains no repeated vertices

66

Ford-Fulkerson: runtime?

Ford-Fulkerson(G, s, t)
  flow = 0 for all edges
  G_f = residualGraph(G)
  while a simple path exists from s to t in G_f
    send as much flow along path as possible
    G_f = residualGraph(G)
  return flow

71

Ford-Fulkerson: runtime?

```
Ford-Fulkerson(G, s, t)
  flow = 0 for all edges
  Gf = residualGraph(G)
  while a simple path exists from s to t in Gf
    send as much flow along path as possible
    Gf = residualGraph(G)
  return flow
```

- traverse the graph
- at most add 2 edges for original edge
- $\theta(V + E)$

Can we simplify this expression?

72

Ford-Fulkerson: runtime?

```
Ford-Fulkerson(G, s, t)
  flow = 0 for all edges
  Gf = residualGraph(G)
  while a simple path exists from s to t in Gf
    send as much flow along path as possible
    Gf = residualGraph(G)
  return flow
```

- traverse the graph
- at most add 2 edges for original edge
- $\theta(V + E) = \theta(E)$
- (all nodes exists on paths from s to t)

73

Ford-Fulkerson: runtime?

```
Ford-Fulkerson(G, s, t)
  flow = 0 for all edges
  Gf = residualGraph(G)
  while a simple path exists from s to t in Gf
    send as much flow along path as possible
    Gf = residualGraph(G)
  return flow
```

- BFS or DFS
- $O(V + E) = O(E)$

74

Ford-Fulkerson: runtime?

```
Ford-Fulkerson(G, s, t)
  flow = 0 for all edges
  Gf = residualGraph(G)
  while a simple path exists from s to t in Gf
    send as much flow along path as possible
    Gf = residualGraph(G)
  return flow
```

- max-flow!
- increases ever iteration
- integer capacities, so integer increases

Can we bound the number of times the loop will execute?

75

Ford-Fulkerson: runtime?

Ford-Fulkerson(G, s, t)
  flow = 0 for all edges
  G_f = residualGraph(G)
  while a simple path exists from s to t in G_f
    send as much flow along path as possible
    G_f = residualGraph(G)
  return flow

- max-flow!
- increases ever iteration
- integer capacities, so integer increases

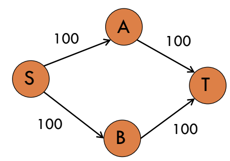
Overall runtime? O(max-flow * E)

76

O(max-flow * E)

Can you construct a graph that *could get* this running time?

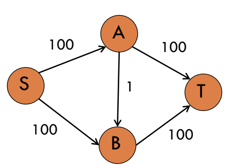
Hint:



77

O(max-flow * E)

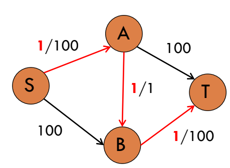
Can you construct a graph that *could get* this running time?



78

O(max-flow * E)

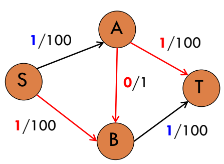
Can you construct a graph that *could get* this running time?



79

O(max-flow * E)

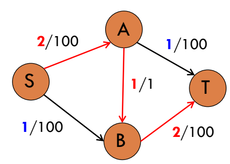
Can you construct a graph that *could get* this running time?



80

O(max-flow * E)

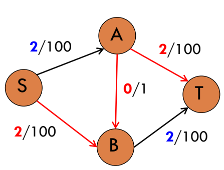
Can you construct a graph that *could get* this running time?



81

O(max-flow * E)

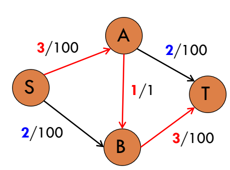
Can you construct a graph that *could get* this running time?



82

O(max-flow * E)

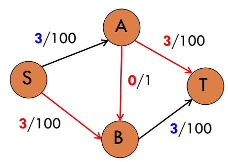
Can you construct a graph that *could get* this running time?



83

O(max-flow * E)

Can you construct a graph that *could get* this running time?



What is the problem here?
Could we do better?

84

Faster variants

Edmunds-Karp

- Select the *shortest path* (in number of edges) from s to t in G_f
  - How can we do this?
    - use BFS for search
- Running time: $O(V E^2)$
  - avoids issues like the one we just saw
  - see the book for the proof
  - or http://www.cs.cornell.edu/courses/CS4820/2011sp/handouts/edmondskarp.pdf

preflow-push (aka push-relabel) algorithms

- $O(V^3)$

85

Other variations…

Method	Complexity
Linear programming	
Ford–Fulkerson algorithm	$O(E \max\|f\|)$
Edmonds–Karp algorithm	$O(VE^2)$
Dinitz blocking flow algorithm	$O(V^2E)$
General push-relabel maximum flow algorithm	$O(V^2E)$
Push-relabel algorithm with *FIFO* vertex selection rule	$O(V^3)$
Dinitz blocking flow algorithm with dynamic trees	$O(VE \log(V))$
Push-relabel algorithm with dynamic trees	$O(VE \log(V^2/E))$
Binary blocking flow algorithm [1]	$O(E \min(V^{2/3}, \sqrt{E}) \log(V^2/E) \log U)$
MPM (Malhotra, Pramodh-Kumar and Maheshwari) algorithm	$O(V^3)$

TABLE I. POLYNOMIAL-TIME ALGORITHMS FOR THE MAXIMUM FLOW PROBLEM[a]

Algorithm no.	Date	Discoverer	Running time	References
1	1969	Edmonds and Karp	$O(nm^2)$	[5]
2	1970	Dinic	$O(n^2m)$	[4]
3	1974	Karzanov	$O(n^3)$	[18]
4	1977	Cherkasky	$O(n^2m^{1/2})$	[3]
5	1978	Malhotra, Pramodh Kumar, and Maheshwari	$O(n^3)$	[21]
6	1978	Galil	$O(n^{5/3}m^{2/3})$	[11]
7	1978	Galil and Naamad; Shiloach	$O(nm(\log n)^2)$	[12, 25]
8	1980	Sleator and Tarjan	$O(nm \log n)$	[27, 28]
9	1982	Shiloach and Vishkin	$O(n^3)$	[26]
10	1983	Gabow	$O(nm \log U)$	[10]
11	1984	Tarjan	$O(n^3)$	[31]
12	1985	Goldberg	$O(n^3)$	[14]
13	1986	Goldberg and Tarjan	$O(nm \log(n^2/m))$	[16, 15]
14	1986	Ahuja and Orlin	$O(nm + n^2 \log U)$	[1]

[a] Algorithm 13 is presented in this paper.

http://akira.ruc.dk/~keld/teaching/algoritmedesign_f03/Artikler/08/Goldberg88.pdf

http://en.wikipedia.org/wiki/Maximum_flow

86

Network flow properties

If one of these is true then all are true (i.e. each implies the the others):

f is a maximum flow

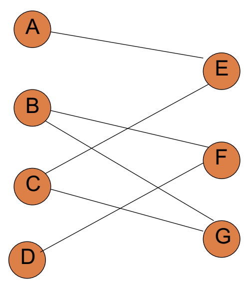
G_f (residual graph) has no paths from s to t

|f| = minimum capacity cut

87

Application: bipartite graph matching

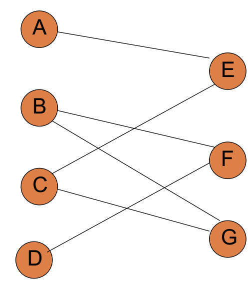
Bipartite graph – a graph where every vertex can be partitioned into two sets X and Y such that all edges connect a vertex $u \in X$ and a vertex $v \in Y$



89

Application: bipartite graph matching

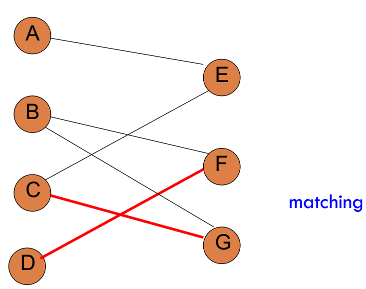
A *matching* M is a subset of edges such that each node occurs **at most once** in M



90

Application: bipartite graph matching

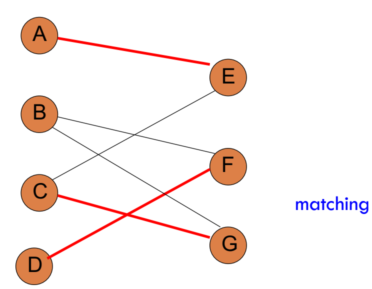
A *matching* M is a subset of edges such that each node occurs **at most once** in M



91

Application: bipartite graph matching

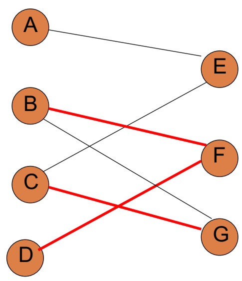
A *matching* M is a subset of edges such that each node occurs **at most once** in M



92

Application: bipartite graph matching

A *matching* M is a subset of edges such that each node occurs **at most once** in M

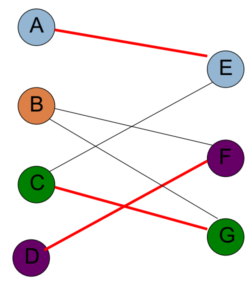


not a matching

93

Application: bipartite graph matching

A *matching* can be thought of as pairing the vertices



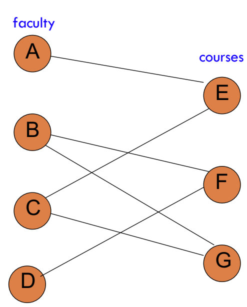
94

Application: bipartite graph matching

Bipartite matching problem: find the *largest* matching in a bipartite graph

Where might this problem come up?

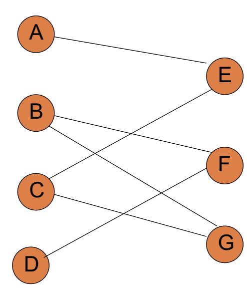
- CS department has n courses and m faculty
- Every instructor can teach *some* of the courses
- What course should each person teach?
- Anytime we want to match n things with m, but not all things can match



95

Application: bipartite graph matching

Setup as a flow problem:

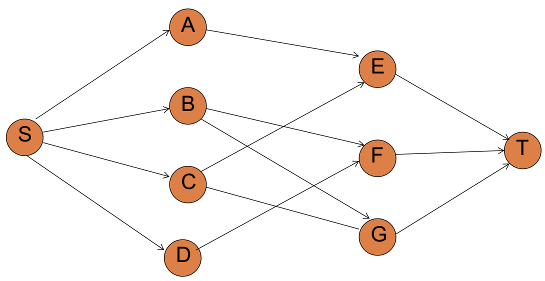


97

Application: bipartite graph matching

Setup as a flow problem:

edge weights?

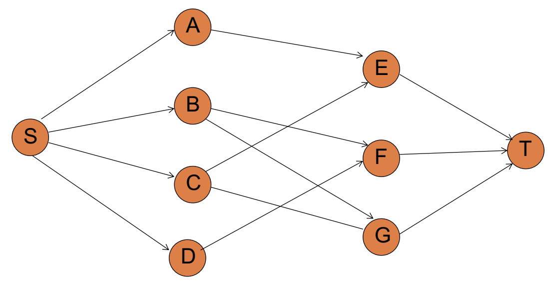


98

Application: bipartite graph matching

Setup as a flow problem:

all edge weights are 1

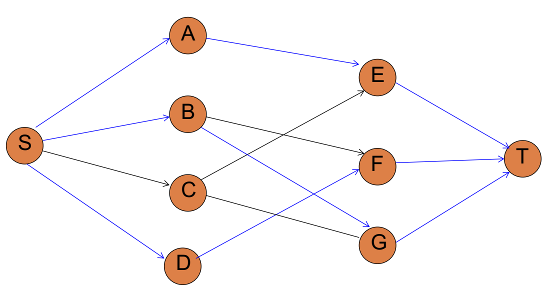


99

Application: bipartite graph matching

Setup as a flow problem:

after we find the flow, how do we find the matching?

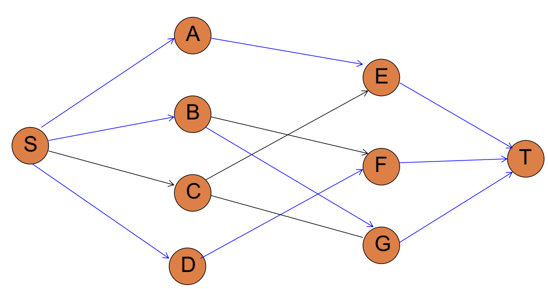


100

Application: bipartite graph matching

Setup as a flow problem:

match those nodes with flow between them



101

Application: bipartite graph matching

Run-time?

Cost to build the flow?

- O(E)
  - each existing edge gets a capacity of 1
  - introduce V new edges (to and from s and t)
  - V is O(E) (for non-degenerate bipartite matching problems)

Max-flow calculation?

- Basic Ford-Fulkerson: O(max-flow * E)
- Edmunds-Karp: $O(V E^2)$
- Preflow-push: $O(V^3)$

103

Application: bipartite graph matching

Run-time?

Cost to build the flow?

- O(E)
  - each existing edge gets a capacity of 1
  - introduce V new edges (to and from s and t)
  - V is O(E) (for non-degenerate bipartite matching problems)

Max-flow calculation?

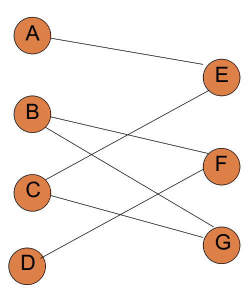
- Basic Ford-Fulkerson: O(max-flow * E)
  - max-flow = O(V)
  - O(V E)

104

Application: bipartite graph matching

Bipartite matching problem: find the *largest* matching in a bipartite graph

- CS department has n courses and m faculty
- Every instructor can teach *some* of the courses
- What course should each person teach?
- Each faculty can teach at most 3 courses a semester?

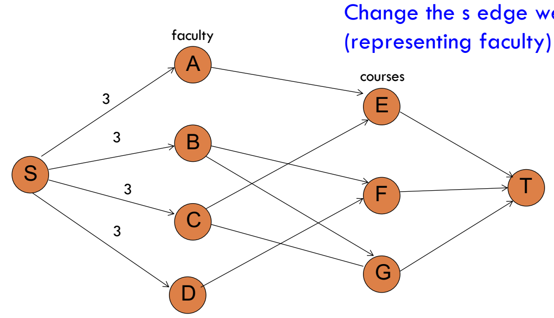


Change the s edge weights (representing faculty) to 3

105

Application: bipartite graph matching

Change the s edge weights (representing faculty) to 3



All others are capacity 1

106

Survey Design

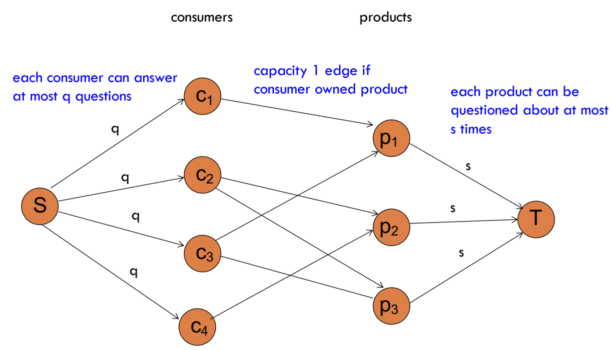
Design a survey with the following requirements:

- Design survey asking *n* consumers about *m* products
- Can only survey consumer about a product if they own it
- Question consumers about at most *q* products
- Each product should be surveyed at most *s* times
- Maximize the number of surveys/questions asked

How can we do this?

107

Survey Design



108

Survey design

Is it correct?

- Each of the comments above the flow graph match the problem constraints
- max-flow finds the maximum matching, given the problem constraints

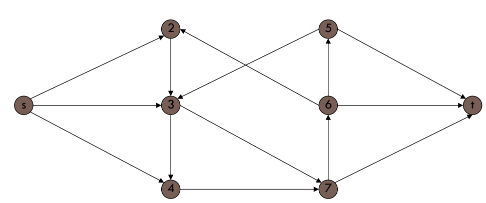
What is the run-time?

- Basic Ford-Fulkerson: O(max-flow * E)
- Edmunds-Karp: $O(V E^2)$
- Preflow-push: $O(V^3)$

109

Edge Disjoint Paths

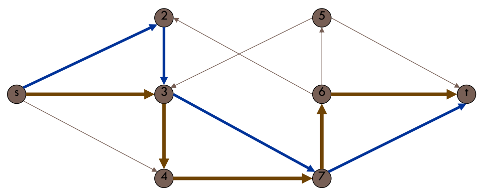
Two paths are edge-disjoint if they have no edge in common



110

Edge Disjoint Paths

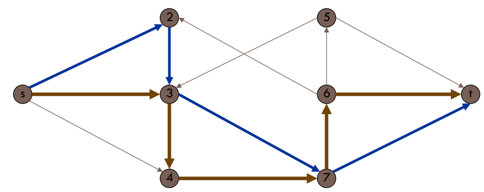
Two paths are edge-disjoint if they have no edge in common



111

Edge Disjoint Paths Problem

Given a directed graph G = (V, E) and two nodes s and t, find the max number of edge-disjoint paths from s to t



Why might this be useful?

112

Edge Disjoint Paths Problem

Given a directed graph G = (V, E) and two nodes s and t, find the max number of edge-disjoint paths from s to t

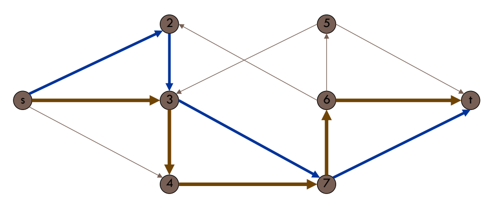
Why might this be useful?

- edges are unique resources (e.g. communications, transportation, etc.)
- how many *concurrent (non-conflicting)* paths do we have from s to t

113

Edge Disjoint Paths

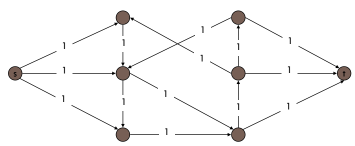
Algorithm ideas?



114

Edge Disjoint Paths

Max flow formulation: assign unit capacity to every edge

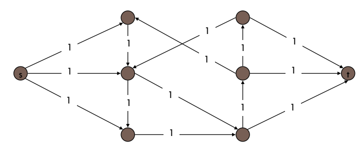


What does the max flow represent?
Why?

115

Edge Disjoint Paths

Max flow formulation: assign unit capacity to every edge

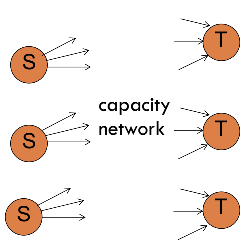


- max-flow = maximum number of disjoint paths
- correctness:
  - each edge can have at most flow = 1, so can only be traversed once
  - therefore, each unit out of s represents a separate path to t

116

Max-flow variations

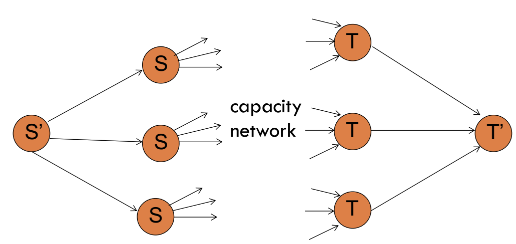
What if we have multiple sources and multiple sinks (e.g. the Russian train problem has multiple sinks)?



117

Max-flow variations

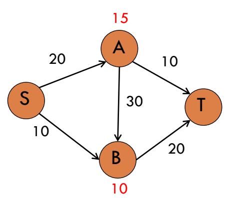
Create a new source and sink and connect up with infinite capacities…



118

Max-flow variations

Vertex capacities: in addition to having edge capacities we can also restrict the amount of flow through each vertex

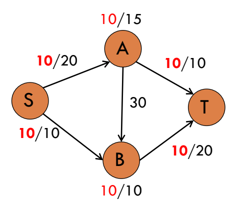


What is the max-flow now?

119

Max-flow variations

Vertex capacities: in addition to having edge capacities we can also restrict the amount of flow through each vertex

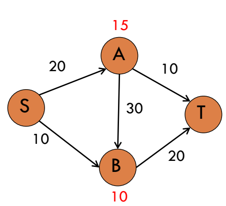


20 units

120

Max-flow variations

Vertex capacities: in addition to having edge capacities we can also restrict the amount of flow through each vertex



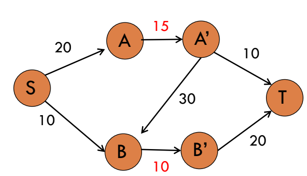
How can we solve this problem?

121

Max-flow variations

For each vertex v

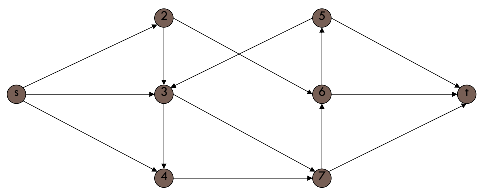
- create a new node v’
- create an edge with the vertex capacity from v to v’
- move all outgoing edges from v to v’



122

More problems: maximum independent path

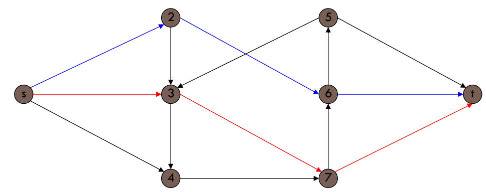
Two paths are independent if they have no *vertices* in common



126

More problems: maximum independent path

Two paths are independent if they have no *vertices* in common

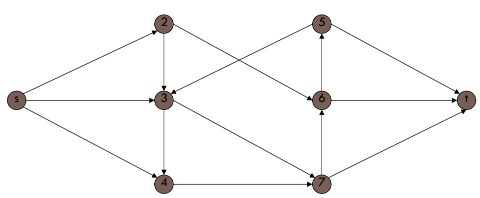


127

More problems: maximum independent path

Find the maximum number of independent paths

Ideas?

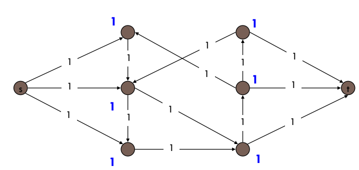


128

maximum independent path

Max flow formulation:

- assign unit capacity to every edge (though any value would work)
- assign unit capacity to every vertex



Same idea as the maximum edge-disjoint paths, but now we also constrain the vertices

129