SHORTEST PATHS

David Kauchak
CS 140 – Spring 2023

1

Admin

Assignment 9

Assignment 10 (2 weeks assignment) – don't ignore until next week

Checkpoint 2 next Monday

2

Checkpoint 2

2 pages of notes

2/20 through 4/10 (will not include network flow)

Will make some practice problems available later this week

3

Checkpoint 2 topics

greedy algorithms
- proving correctness
- developing algorithms
- comparing vs. dynamic programming

hashtables
- collision resolution by chaining
- open addressing
- hash functions

Dynamic programming

4

Checkpoint 2 topics

graphs
- different types of graphs
- terminology
- representing graphs (adjacency list/matrix)

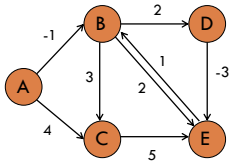
graph algorithms
- Traversal: BFS, DFS
- MST: Prim's, Kruskal's
- Topological sort
- Connectedness
- Detecting cycles
- Single-source shortest paths: Dijkstra's, Bellman-Ford
- All-pairs shortest paths: Floyd-Warshal, Johnson's
- Run-time, why the work, when you can apply them

graph misc
- min-cut property (proving correctness of MST algorithms)

5

All pairs shortest paths

All pairs shortest paths: calculate the shortest paths between *all* vertices



6

All pairs shortest paths

All pairs shortest paths: calculate the shortest paths between *all* vertices

Easy solution?

7

All pairs shortest paths

All pairs shortest paths: calculate the shortest paths between *all* vertices

Run Bellman-Ford from each vertex!

$O(V^2E)$
- Bellman-Ford: $O(VE)$
- V calls, one for each vertex

8

Floyd-Warshall: key idea

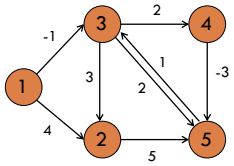
Label all vertices with a number from 1 to V

$d_{ij}{}^k$ = shortest path from vertex i to vertex j using only vertices $\{1, 2, \ldots, k\}$

9

Floyd-Warshall: key idea

$d_{ij}{}^k$ = shortest path from vertex i to vertex j using only vertices $\{1, 2, \ldots, k\}$

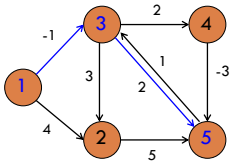


What is $d_{15}{}^3$?

10

Floyd-Warshall: key idea

$d_{ij}{}^k$ = shortest path from vertex i to vertex j using only vertices $\{1, 2, \ldots, k\}$



$d_{15}{}^3 = 1$. Can't use vertex 4.

11

Floyd-Warshall: key idea

Label all vertices with a number from 1 to V

$d_{ij}{}^k$ = shortest path from vertex i to vertex j using only vertices $\{1, 2, \ldots, k\}$

If we want all possibilities, how many values are there (i.e. what is the size of $d_{ij}{}^k$)?

12

Floyd-Warshall: key idea

Label all vertices with a number from 1 to V

$d_{ij}{}^{k}$ = shortest path from vertex i to vertex j using only vertices $\{1, 2, \ldots, k\}$

V^3
- i: all vertices
- j: all vertices
- k: all vertices

13

Floyd-Warshall: key idea

Label all vertices with a number from 1 to V

$d_{ij}{}^{k}$ = shortest path from vertex i to vertex j using only vertices $\{1, 2, \ldots, k\}$

What is $d_{ij}{}^{V}$?
- Distance of the shortest path from i to j
- If we can calculate this, for all (i, j), we're done!

14

Recursive relationship

$d_{ij}{}^{k}$ = shortest path from vertex i to vertex j using only vertices $\{1, 2, \ldots, k\}$

Assume we know $d_{ij}{}^{k}$

How can we calculate $d_{ij}{}^{k+1}$, i.e. shortest path now including vertex k+1? (Hint: in terms of $d_{ij}{}^{k}$)

Two options:
1) Vertex k+1 doesn't give us a shorter path
2) Vertex k+1 does give us a shorter path

15

Recursive relationship

$d_{ij}{}^{k}$ = shortest path from vertex i to vertex j using only vertices $\{1, 2, \ldots, k\}$

Two options:
1) Vertex k+1 doesn't give us a shorter path
2) Vertex k+1 does give us a shorter path

$d_{ij}{}^{k+1}$ = ?

16

Recursive relationship

d_{ij}^{k} = shortest path from vertex i to vertex j using only vertices $\{1, 2, \ldots, k\}$

Two options:
1) Vertex k+1 doesn't give us a shorter path
2) Vertex k+1 does give us a shorter path

$d_{ij}^{k+1} = d_{ij}^{k}$

17

Recursive relationship

d_{ij}^{k} = shortest path from vertex i to vertex j using only vertices $\{1, 2, \ldots, k\}$

Two options:
1) Vertex k+1 doesn't give us a shorter path
2) Vertex k+1 does give us a shorter path

$d_{ij}^{k+1} = ?$

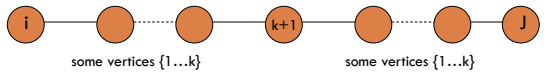
18

Recursive relationship

d_{ij}^{k} = shortest path from vertex i to vertex j using only vertices $\{1, 2, \ldots, k\}$

Two options:
1) Vertex k+1 doesn't give us a shorter path
2) Vertex k+1 does give us a shorter path

$d_{ij}^{k+1} = ?$



What is the cost of this path?

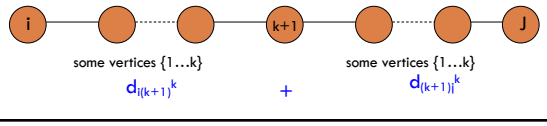
19

Recursive relationship

d_{ij}^{k} = shortest path from vertex i to vertex j using only vertices $\{1, 2, \ldots, k\}$

Two options:
1) Vertex k+1 doesn't give us a shorter path
2) Vertex k+1 does give us a shorter path

$d_{ij}^{k+1} = d_{i(k+1)}^{k} + d_{(k+1)j}^{k}$



20

Recursive relationship

$d_{ij}{}^{k}$ = shortest path from vertex i to vertex j using only vertices $\{1, 2, \ldots, k\}$

Two options:
1) Vertex k+1 doesn't give us a shorter path
2) Vertex k+1 does give us a shorter path

$d_{ij}{}^{k+1}$ = ?

How do we combine these two options?

21

Recursive relationship

$d_{ij}{}^{k}$ = shortest path from vertex i to vertex j using only vertices $\{1, 2, \ldots, k\}$

Two options:
1) Vertex k+1 doesn't give us a shorter path
2) Vertex k+1 does give us a shorter path

$d_{ij}{}^{k+1} = \min(dijk, d_{i(k+1)}{}^{k} + d_{(k+1)j}{}^{k})$

Pick whichever is shorter

22

Floyd-Warshall

Calculate $d_{ij}{}^{k}$ for increasing k, i.e. k = 1 to V

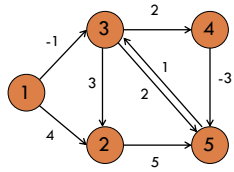
Floyd-Warshall(G = (V,E,W)):
d^0 = W // initialize with edge weights
for k = 1 to V
  for i = 1 to V
    for j = 1 to V
      $\text{dijk} = \min(d_{ij}{}^{k-1}, d_{ik}{}^{k-1} + d_{kj}{}^{k-1})$

return d^V

23

Floyd-Warshall(G = (V,E,W)):
d^0 = W // initialize with edge weights
for k = 1 to V
  for i = 1 to V
    for j = 1 to V
      $\text{dijk} = \min(d_{ij}{}^{k-1}, d_{ik}{}^{k-1} + d_{kj}{}^{k-1})$

return d^V



k = 0

	1	2	3	4	5
1	0	4	−1	∞	∞
2	∞	0	∞	∞	5
3	∞	3	0	2	2
4	∞	∞	∞	0	−3
5	∞	∞	1	∞	0

adjacency matrix

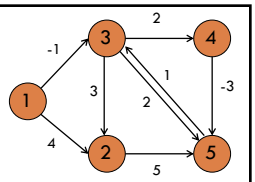
k = 1

	1	2	3	4	5
1	0	4	−1	∞	∞
2	∞	0	∞	∞	5
3	∞	3	0	2	2
4	∞	∞	∞	0	−3
5	∞	∞	1	∞	0

no change

24

Floyd-Warshall(G = (V,E,W)):
$d^0 = W$ // initialize with edge weights
for k = 1 to V
  for i = 1 to V
    for j = 1 to V
      dijk = $\min(d_{ij}{}^{k-1}, d_{ik}{}^{k-1} + d_{kj}{}^{k-1})$
return d^V



k = 1

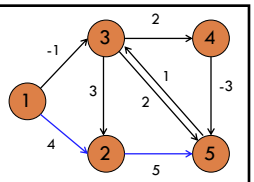
	1	2	3	4	5
1	0	4	−1	∞	∞
2	∞	0	∞	∞	5
3	∞	3	0	2	2
4	∞	∞	∞	0	−3
5	∞	∞	1	∞	0

k = 2

	1	2	3	4	5
1	0	4	−1	∞	?
2					
3					
4					
5					

25

Floyd-Warshall(G = (V,E,W)):
$d^0 = W$ // initialize with edge weights
for k = 1 to V
  for i = 1 to V
    for j = 1 to V
      dijk = $\min(d_{ij}{}^{k-1}, d_{ik}{}^{k-1} + d_{kj}{}^{k-1})$
return d^V



k = 1

	1	2	3	4	5
1	0	4	−1	∞	∞
2	∞	0	∞	∞	5
3	∞	3	0	2	2
4	∞	∞	∞	0	−3
5	∞	∞	1	∞	0

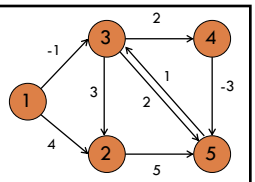
k = 2

	1	2	3	4	5
1	0	4	−1	∞	9
2					
3					
4					
5					

minimum

26

Floyd-Warshall(G = (V,E,W)):
$d^0 = W$ // initialize with edge weights
for k = 1 to V
  for i = 1 to V
    for j = 1 to V
      dijk = $\min(d_{ij}{}^{k-1}, d_{ik}{}^{k-1} + d_{kj}{}^{k-1})$
return d^V



k = 2

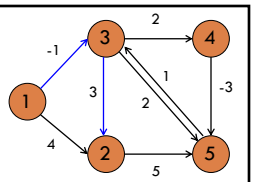
	1	2	3	4	5
1	0	4	−1	∞	9
2	∞	0	∞	∞	5
3	∞	3	0	2	2
4	∞	∞	∞	0	−3
5	∞	∞	1	∞	0

k = 3

	1	2	3	4	5
1	0	?			
2					
3					
4					
5					

27

Floyd-Warshall(G = (V,E,W)):
$d^0 = W$ // initialize with edge weights
for k = 1 to V
  for i = 1 to V
    for j = 1 to V
      dijk = $\min(d_{ij}{}^{k-1}, d_{ik}{}^{k-1} + d_{kj}{}^{k-1})$
return d^V



k = 2

	1	2	3	4	5
1	0	4	−1	∞	9
2	∞	0	∞	∞	5
3	∞	3	0	2	2
4	∞	∞	∞	0	−3
5	∞	∞	1	∞	0

k = 3

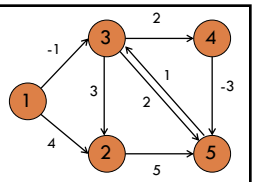
	1	2	3	4	5
1	0	2			
2					
3					
4					
5					

minimum

Found a shorter path!

28

Floyd-Warshall(G = (V,E,W)):
$d^0 = W$ // initialize with edge weights
for k = 1 to V
  for i = 1 to V
    for j = 1 to V
      $dijk = \min(d_{ij}^{k-1}, d_{ik}^{k-1} + d_{kj}^{k-1})$
return d^V



k = 2

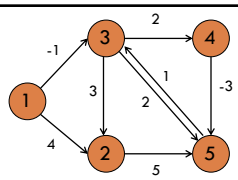
	1	2	3	4	5
1	0	4	−1	∞	9
2	∞	0	∞	∞	5
3	∞	3	0	2	2
4	∞	∞	∞	0	−3
5	∞	∞	1	∞	0

k = 3

	1	2	3	4	5
1	0	2			
2					
3					
4					
5					

29

Floyd-Warshall(G = (V,E,W)):
$d^0 = W$ // initialize with edge weights
for k = 1 to V
  for i = 1 to V
    for j = 1 to V
      $dijk = \min(d_{ij}^{k-1}, d_{ik}^{k-1} + d_{kj}^{k-1})$
return d^V



k = 2

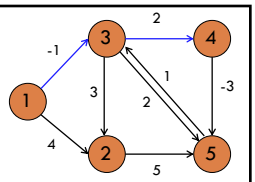
	1	2	3	4	5
1	0	4	−1	∞	9
2	∞	0	∞	∞	5
3	∞	3	0	2	2
4	∞	∞	∞	0	−3
5	∞	∞	1	∞	0

k = 3

	1	2	3	4	5
1	0	2	−1	?	
2					
3					
4					
5					

30

Floyd-Warshall(G = (V,E,W)):
$d^0 = W$ // initialize with edge weights
for k = 1 to V
  for i = 1 to V
    for j = 1 to V
      $dijk = \min(d_{ij}^{k-1}, d_{ik}^{k-1} + d_{kj}^{k-1})$
return d^V



k = 2

	1	2	3	4	5
1	0	4	−1	∞	9
2	∞	0	∞	∞	5
3	∞	3	0	2	2
4	∞	∞	∞	0	−3
5	∞	∞	1	∞	0

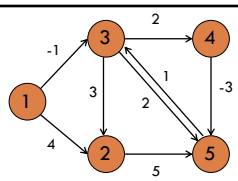
k = 3

	1	2	3	4	5
1	0	2	−1	1	
2					
3					
4					
5					

minimum

31

Floyd-Warshall(G = (V,E,W)):
$d^0 = W$ // initialize with edge weights
for k = 1 to V
  for i = 1 to V
    for j = 1 to V
      $dijk = \min(d_{ij}^{k-1}, d_{ik}^{k-1} + d_{kj}^{k-1})$
return d^V



k = 2

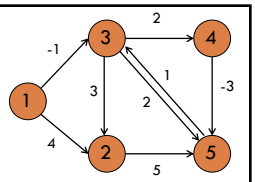
	1	2	3	4	5
1	0	4	−1	∞	9
2	∞	0	∞	∞	5
3	∞	3	0	2	2
4	∞	∞	∞	0	−3
5	∞	∞	1	∞	0

k = 3

	1	2	3	4	5
1	0	2	−1	1	
2					
3					
4					
5					

32

Floyd-Warshall(G = (V,E,W)):
$d^0 = W$ // initialize with edge weights
for $k = 1$ to V
  for $i = 1$ to V
    for $j = 1$ to V
      $\text{dijk} = \min(d_{ij}^{k-1}, d_{ik}^{k-1} + d_{kj}^{k-1})$
return d^V



k = 2

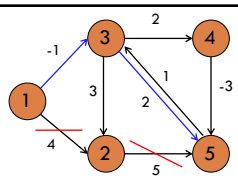
	1	2	3	4	5
1	0	4	−1	∞	9
2	∞	0	∞	∞	5
3	∞	3	0	2	2
4	∞	∞	∞	0	−3
5	∞	∞	1	∞	0

k = 3

	1	2	3	4	5
1	0	2	−1	1	?
2					
3					
4					
5					

33

Floyd-Warshall(G = (V,E,W)):
$d^0 = W$ // initialize with edge weights
for $k = 1$ to V
  for $i = 1$ to V
    for $j = 1$ to V
      $\text{dijk} = \min(d_{ij}^{k-1}, d_{ik}^{k-1} + d_{kj}^{k-1})$
return d^V



k = 2

	1	2	3	4	5
1	0	4	−1	∞	9
2	∞	0	∞	∞	5
3	∞	3	0	2	2
4	∞	∞	∞	0	−3
5	∞	∞	1	∞	0

minimum

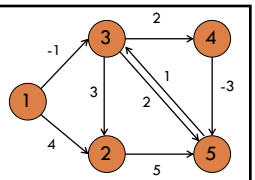
k = 3

	1	2	3	4	5
1	0	2	−1	1	1
2					
3					
4					
5					

Found a shorter path!

34

Floyd-Warshall(G = (V,E,W)):
$d^0 = W$ // initialize with edge weights
for $k = 1$ to V
  for $i = 1$ to V
    for $j = 1$ to V
      $\text{dijk} = \min(d_{ij}^{k-1}, d_{ik}^{k-1} + d_{kj}^{k-1})$
return d^V



k = 2

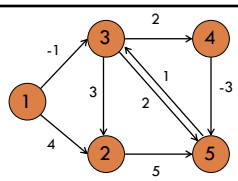
	1	2	3	4	5
1	0	4	−1	∞	9
2	∞	0	∞	∞	5
3	∞	3	0	2	2
4	∞	∞	∞	0	−3
5	∞	∞	1	∞	0

k = 3

	1	2	3	4	5
1	0	2	−1	1	1
2	∞	0	∞	∞	5
3	∞	3	0	2	2
4	∞	∞	∞	0	−3
5	∞	∞	1	∞	0

35

Floyd-Warshall(G = (V,E,W)):
$d^0 = W$ // initialize with edge weights
for $k = 1$ to V
  for $i = 1$ to V
    for $j = 1$ to V
      $\text{dijk} = \min(d_{ij}^{k-1}, d_{ik}^{k-1} + d_{kj}^{k-1})$
return d^V



k = 3

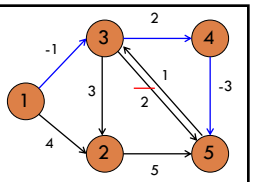
	1	2	3	4	5
1	0	2	−1	1	1
2	∞	0	∞	∞	5
3	∞	3	0	2	2
4	∞	∞	∞	0	−3
5	∞	∞	1	∞	0

k = 4

	1	2	3	4	5
1	0	2	−1	1	?
2					
3					
4					
5					

36

Floyd-Warshall(G = (V,E,W)):
$d^0 = W$ // initialize with edge weights
for k = 1 to V
  for i = 1 to V
    for j = 1 to V
      $\text{dijk} = \min(d_{ij}^{k-1}, d_{ik}^{k-1} + d_{kj}^{k-1})$
return d^V



k = 3

	1	2	3	4	5
1	0	2	−1	1	1
2	∞	0	∞	∞	5
3	∞	3	0	2	2
4	∞	∞	∞	0	−3
5	∞	∞	1	∞	0

minimum

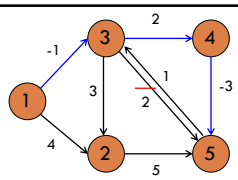
k = 4

	1	2	3	4	5
1	0	2	−1	1	−2
2					
3					
4					
5					

Found a shorter path!

37

Floyd-Warshall(G = (V,E,W)):
$d^0 = W$ // initialize with edge weights
for k = 1 to V
  for i = 1 to V
    for j = 1 to V
      $\text{dijk} = \min(d_{ij}^{k-1}, d_{ik}^{k-1} + d_{kj}^{k-1})$
return d^V



k = 3

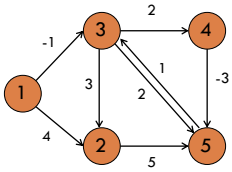
	1	2	3	4	5
1	0	2	−1	1	1
2	∞	0	∞	∞	5
3	∞	3	0	2	2
4	∞	∞	∞	0	−3
5	∞	∞	1	∞	0

k = 4

	1	2	3	4	5
1	0	2	−1	1	−2
2					
3					
4					
5					

38

Floyd-Warshall(G = (V,E,W)):
$d^0 = W$ // initialize with edge weights
for k = 1 to V
  for i = 1 to V
    for j = 1 to V
      $\text{dijk} = \min(d_{ij}^{k-1}, d_{ik}^{k-1} + d_{kj}^{k-1})$
return d^V



k = 3

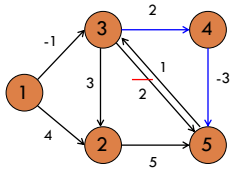
	1	2	3	4	5
1	0	2	−1	1	1
2	∞	0	∞	∞	5
3	∞	3	0	2	2
4	∞	∞	∞	0	−3
5	∞	∞	1	∞	0

k = 4

	1	2	3	4	5
1	0	2	−1	1	−2
2	∞	0	∞	∞	5
3	∞	3	0	2	?
4					
5					

39

Floyd-Warshall(G = (V,E,W)):
$d^0 = W$ // initialize with edge weights
for k = 1 to V
  for i = 1 to V
    for j = 1 to V
      $\text{dijk} = \min(d_{ij}^{k-1}, d_{ik}^{k-1} + d_{kj}^{k-1})$
return d^V



k = 3

	1	2	3	4	5
1	0	2	−1	1	1
2	∞	0	∞	∞	5
3	∞	3	0	2	2
4	∞	∞	∞	0	−3
5	∞	∞	1	∞	0

minimum

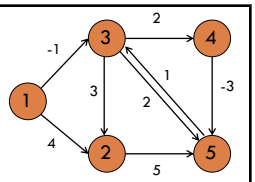
k = 4

	1	2	3	4	5
1	0	2	−1	1	−2
2	∞	0	∞	∞	5
3	∞	3	0	2	−1
4					
5					

Found a shorter path!

40

Floyd-Warshall(G = (V,E,W)):
$d^0 = W$ // initialize with edge weights
for $k = 1$ to V
  for $i = 1$ to V
    for $j = 1$ to V
      $\text{dijk} = \min(d_{ij}^{k-1}, d_{ik}^{k-1} + d_{kj}^{k-1})$
return d^V



k = 3

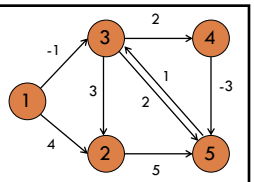
	1	2	3	4	5
1	0	2	−1	1	1
2	∞	0	∞	∞	5
3	∞	3	0	2	2
4	∞	∞	∞	0	−3
5	∞	∞	1	∞	0

k = 4

	1	2	3	4	5
1	0	2	−1	1	−2
2	∞	0	∞	∞	5
3	∞	3	0	2	−1
4	∞	∞	∞	0	−3
5	∞	∞	1	∞	0

41

Floyd-Warshall(G = (V,E,W)):
$d^0 = W$ // initialize with edge weights
for $k = 1$ to V
  for $i = 1$ to V
    for $j = 1$ to V
      $\text{dijk} = \min(d_{ij}^{k-1}, d_{ik}^{k-1} + d_{kj}^{k-1})$
return d^V



k = 4

	1	2	3	4	5
1	0	2	−1	1	−2
2	∞	0	∞	∞	5
3	∞	3	0	2	−1
4	∞	∞	∞	0	−3
5	∞	∞	1	∞	0

k = 5

	1	2	3	4	5
1	0	2	−1	1	−2
2	∞	0	7	9	5
3	∞	3	0	2	−1
4	∞	1	−2	0	−3
5	∞	∞	1	∞	0

Done!

42

Floyd-Warshall analysis

Is it correct?

Floyd-Warshall(G = (V,E,W)):
$d^0 = W$ // initialize with edge weights
for $k = 1$ to V
  for $i = 1$ to V
    for $j = 1$ to V
      $\text{dijk} = \min(d_{ij}^{k-1}, d_{ik}^{k-1} + d_{kj}^{k-1})$
return d^V

43

Floyd-Warshall analysis

Is it correct?

Any assumptions?

Floyd-Warshall(G = (V,E,W)):
$d^0 = W$ // initialize with edge weights
for $k = 1$ to V
  for $i = 1$ to V
    for $j = 1$ to V
      $\text{dijk} = \min(d_{ij}^{k-1}, d_{ik}^{k-1} + d_{kj}^{k-1})$
return d^V

44

Floyd-Warshall analysis

Is it correct?

Assuming the graph has no negative cycles!

What happens if there is a negative cycle?

```
Floyd-Warshall(G = (V,E,W)):
d^0 = W        // initialize with edge weights
for k = 1 to V
  for i = 1 to V
    for j = 1 to V
      dijk = min(d_ij^{k-1}, d_ik^{k-1} + d_kj^{k-1})
return d^V
```

45

Floyd-Warshall analysis

If the graph has a negative weight cycle, at the end, at least one of the diagonal entries will be a negative number, i.e., we there's a way to get back to a vertex using all of the vertices that results in a negative weight

	1	2	3	4	5
1	0	2	−1	1	−2
2	∞	0	7	9	5
3	∞	3	0	2	−1
4	∞	1	−2	0	−3
5	∞	∞	1	∞	0

46

Floyd-Warshall analysis

Run-time?

```
Floyd-Warshall(G = (V,E,W)):
d^0 = W        // initialize with edge weights
for k = 1 to V
  for i = 1 to V
    for j = 1 to V
      dijk = min(d_ij^{k-1}, d_ik^{k-1} + d_kj^{k-1})
return d^V
```

47

Floyd-Warshall analysis

Run-time: $\theta(V^3)$

```
Floyd-Warshall(G = (V,E,W)):
d^0 = W        // initialize with edge weights
for k = 1 to V
  for i = 1 to V
    for j = 1 to V
      dijk = min(d_ij^{k-1}, d_ik^{k-1} + d_kj^{k-1})
return d^V
```

48

Floyd-Warshall analysis

What type of algorithm is Floyd-Warshall?

```
Floyd-Warshall(G = (V,E,W)):
d^0 = W        // initialize with edge weights
for k = 1 to V
  for i = 1 to V
    for j = 1 to V
      dijk = min(d_ij^(k-1), d_ik^(k-1) + d_kj^(k-1))

return d^V
```

49

Floyd-Warshall analysis

Dynamic programming!!
Build up solutions to larger problems using solutions to smaller problems. Use a table to store the values.

```
Floyd-Warshall(G = (V,E,W)):
d^0 = W        // initialize with edge weights
for k = 1 to V
  for i = 1 to V
    for j = 1 to V
      dijk = min(d_ij^(k-1), d_ik^(k-1) + d_kj^(k-1))

return d^V
```

50

Floyd-Warshall analysis

Space usage?

```
Floyd-Warshall(G = (V,E,W)):
d^0 = W        // initialize with edge weights
for k = 1 to V
  for i = 1 to V
    for j = 1 to V
      dijk = min(d_ij^(k-1), d_ik^(k-1) + d_kj^(k-1))

return d^V
```

51

Floyd-Warshall: key idea

Label all vertices with a number from 1 to V

$d_{ij}{}^k$ = shortest path from vertex i to vertex j using only vertices $\{1, 2, \ldots, k\}$

If we want all possibilities, how many values are there (i.e. what is the size of $d_{ij}{}^k$)?

52

Floyd-Warshall: key idea

Label all vertices with a number from 1 to V

d_{ij}^{k} = shortest path from vertex i to vertex j using only vertices $\{1, 2, \ldots, k\}$

V^3

- i: all vertices
- j: all vertices
- k: all vertices

Can we do better?

53

Floyd-Warshall analysis

Space usage: $\theta(V^2)$

Only need the current value and the previous

```
Floyd-Warshall(G = (V,E,W)):
d^0 = W        // initialize with edge weights
for k = 1 to V
  for i = 1 to V
    for j = 1 to V
      dijk = min(d_ij^{k-1}, d_ik^{k-1} + d_kj^{k-1})
return d^V
```

54

All pairs shortest paths

V * Bellman-Ford: $O(V^2E)$

Floyd-Warshall: $\theta(V^3)$

55

All pairs shortest paths

All pairs shortest paths for positive weight graphs:
calculate the shortest paths between *all* points

Easy solution?

56

All pairs shortest paths

All pairs shortest paths for positive weight graphs:
calculate the shortest paths between *all* points

Run Dijsktras from each vertex!

Running time (in terms of E and V)?

57

All pairs shortest paths

All pairs shortest paths for positive weight graphs:
calculate the shortest paths between *all* points

Run Dijsktras from each vertex!

$O(V^2 \log V + V E)$
- V calls to Dijkstras
- Dijkstras: $O(V \log V + E)$

58

All pairs shortest paths

V * Bellman-Ford: $O(V^2E)$

Floyd-Warshall: $\theta(V^3)$

V * Dijkstras: $O(V^2 \log V + V E)$

Is this any better?

59

All pairs shortest paths

V * Bellman-Ford: $O(V^2E)$

Floyd-Warshall: $\theta(V^3)$

V * Dijkstras: $O(V^2 \log V + V E)$

If the graph is sparse!

60

All pairs shortest paths

All pairs shortest paths for positive weight graphs:
calculate the shortest paths between *all* points

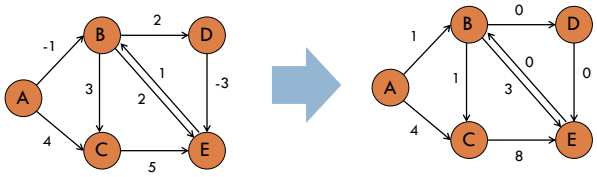
Run Dijsktras from each vertex!

Challenge: Dijkstras assumes positive weights

61

Johnson's: key idea

Reweight the graph to make all edges positive *such that shortest paths are preserved*



What's the shortest path from A to D?

62

Lemma

let *h* be *any* function mapping a vertex to a real value

If we change the graph weights as:

$$\hat{w}(u,v) = w(u,v) + h(u) - h(v)$$

The shortest paths are preserved

63

Lemma: proof $\hat{w}(u,v) = w(u,v) + h(u) - h(v)$

Let $s, v_1, v_2, \ldots, v_k, t$ be a path from s to t

The weight in the reweighted graph is:

$$\hat{w}(s,v_1,\ldots,v_k,t) = w(s,v_1) + h(s) - h(v_1) + \hat{w}(v_1,\ldots,v_k,t)$$

$$= w(s,v_1) + h(s) - h(v_1) + w(v_1,v_2) + h(v_1) - h(v_2) + \hat{w}(v_2,\ldots,v_k,t)$$

$$= w(s,v_1) + h(s) + w(v_1,v_2) - h(v_2) + \hat{w}(v_2,\ldots,v_k,t)$$

$$= w(s,v_1) + h(s) + w(v_1,v_2) - h(v_2) + w(v_2,v_3) + h(v_2) - h(v_3) + \hat{w}(v_3,\ldots,v_k,t)$$

$$= w(s,v_1) + h(s) + w(v_1,v_2) + w(v_2,v_3) - h(v_3) + \hat{w}(v_3,\ldots,v_k,t)$$

$$\ldots$$

$$= w(s,v_1,\ldots,v_k,t) + h(s) - h(t)$$

64

Lemma: proof

$$\hat{w}(s, v_1, \ldots, v_k, t) = w(s, v_1, \ldots, v_k, t) + h(s) - h(t)$$

Claim: the weight change preserves shortest paths, i.e. if a path was the shortest from s to t in the original graph it will still be the shortest path from s to t in the new graph.

Justification?

65

Lemma: proof

$$\hat{w}(s, v_1, \ldots, v_k, t) = w(s, v_1, \ldots, v_k, t) + h(s) - h(t)$$

Claim: the weight change preserves shortest paths, i.e. if a path was the shortest from s to t in the original graph it will still be the shortest path from s to t in the new graph.

h(s) – h(t) is a constant and will be the same for all paths from s to t, so the absolute ordering of all paths from s to t will not change.

66

Lemma

let *h* be *any* function mapping a vertex to a real value

If we change the graph weights as:

$$\hat{w}(u, v) = w(u, v) + h(u) - h(v)$$

The shortest paths are preserved

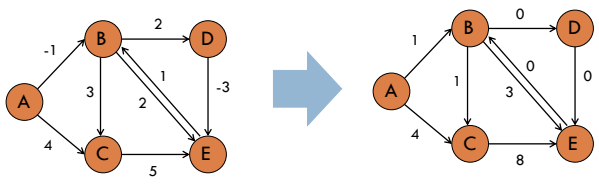
Big question: how do we pick h?

67

Selecting h

Need to pick h such that the resulting graph has all weights as positive

$$\hat{w}(u, v) = w(u, v) + h(u) - h(v)$$



68

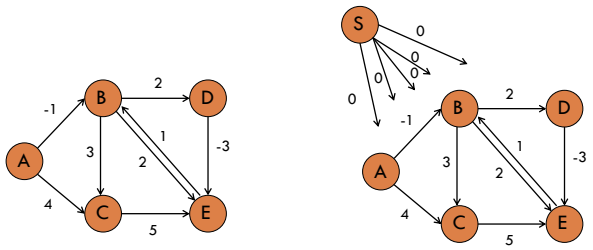
Johnson's algorithm

Create G' with one extra node s with 0 weight edges to all nodes
run Bellman-Ford(G',s)

if no negative-weight cycle
reweight edges in G with h(v)=shortest path from s to v
run Dijkstra's from every vertex
reweight shortest paths based on G

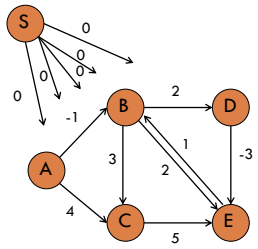
69

Create G'
run Bellman-Ford(G',s)
if no negative-weight cycle
reweight edges in G with h(v)=shortest path from s to v
run Dijkstra's from every vertex
reweight shortest paths based on G



70

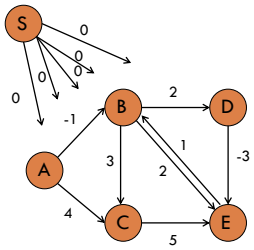
Create G'
run Bellman-Ford(G',s)
if no negative-weight cycle
reweight edges in G with h(v)=shortest path from s to v
run Dijkstra's from every vertex
reweight shortest paths based on G



S➔A: ?
S➔B:
S➔C:
S➔D:
S➔E:

71

Create G'
run Bellman-Ford(G',s)
if no negative-weight cycle
reweight edges in G with h(v)=shortest path from s to v
run Dijkstra's from every vertex
reweight shortest paths based on G



S➔A: 0
S➔B:
S➔C:
S➔D:
S➔E:

72

Create G'

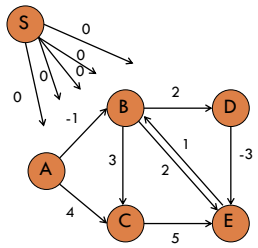
run Bellman-Ford(G',s)

if no negative-weight cycle

reweight edges in G with h(v)=shortest path from s to v

run Dijkstra's from every vertex

reweight shortest paths based on G



S➔A: 0
S➔B: ?
S➔C:
S➔D:
S➔E:

73

Create G'

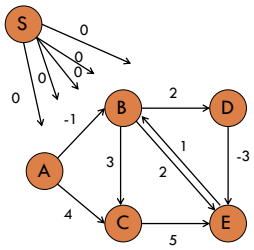
run Bellman-Ford(G',s)

if no negative-weight cycle

reweight edges in G with h(v)=shortest path from s to v

run Dijkstra's from every vertex

reweight shortest paths based on G



S➔A: 0
S➔B: -2
S➔C:
S➔D:
S➔E:

74

Create G'

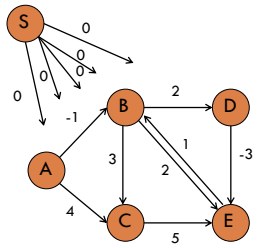
run Bellman-Ford(G',s)

if no negative-weight cycle

reweight edges in G with h(v)=shortest path from s to v

run Dijkstra's from every vertex

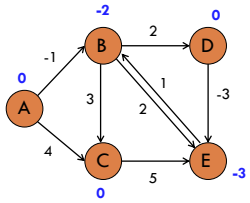
reweight shortest paths based on G



S➔A: 0
S➔B: -2
S➔C: 0
S➔D: 0
S➔E: -3

75

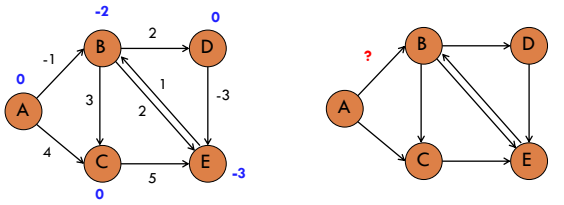
S➔A: 0
S➔B: -2
S➔C: 0
S➔D: 0
S➔E: -3



76

Create G'
run Bellman-Ford(G',s)
if no negative-weight cycle
reweight edges in G with h(v)=shortest path from s to v
run Dijkstra's from every vertex
reweight shortest paths based on G

$$\hat{w}(u,v) = w(u,v) + h(u) - h(v)$$



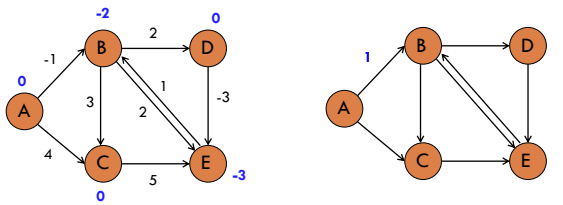
h(v) in blue

77

Create G'
run Bellman-Ford(G',s)
if no negative-weight cycle
reweight edges in G with h(v)=shortest path from s to v
run Dijkstra's from every vertex
reweight shortest paths based on G

$$\hat{w}(u,v) = w(u,v) + h(u) - h(v)$$

-1 + 0 - -2

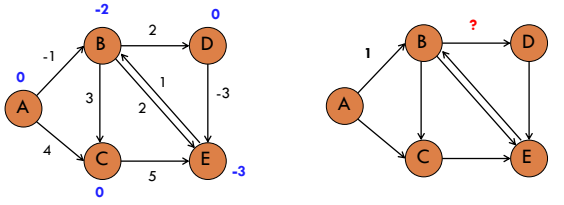


h(v) in blue

78

Create G'
run Bellman-Ford(G',s)
if no negative-weight cycle
reweight edges in G with h(v)=shortest path from s to v
run Dijkstra's from every vertex
reweight shortest paths based on G

$$\hat{w}(u,v) = w(u,v) + h(u) - h(v)$$



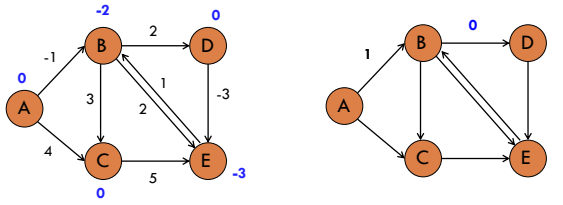
h(v) in blue

79

Create G'
run Bellman-Ford(G',s)
if no negative-weight cycle
reweight edges in G with h(v)=shortest path from s to v
run Dijkstra's from every vertex
reweight shortest paths based on G

$$\hat{w}(u,v) = w(u,v) + h(u) - h(v)$$

2 + -2 - 0

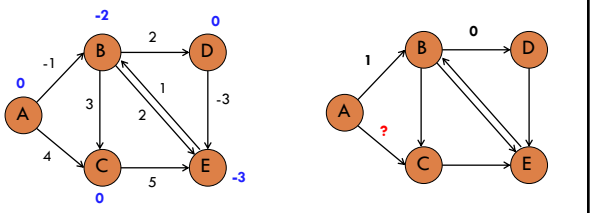


h(v) in blue

80

Create G'
run Bellman-Ford(G',s)
if no negative-weight cycle
reweight edges in G with h(v)=shortest path from s to v
run Dijkstra's from every vertex
reweight shortest paths based on G

$$\hat{w}(u,v) = w(u,v) + h(u) - h(v)$$



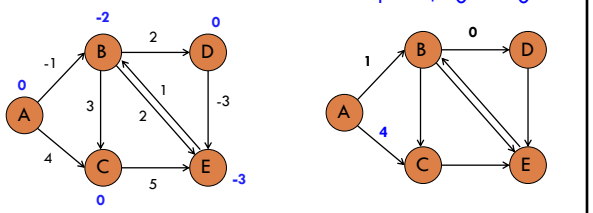
h(v) in blue

81

Create G'
run Bellman-Ford(G',s)
if no negative-weight cycle
reweight edges in G with h(v)=shortest path from s to v
run Dijkstra's from every vertex
reweight shortest paths based on G

$$\hat{w}(u,v) = w(u,v) + h(u) - h(v)$$

4 + 0 - 0

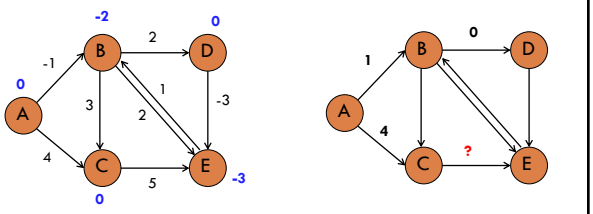


h(v) in blue

82

Create G'
run Bellman-Ford(G',s)
if no negative-weight cycle
reweight edges in G with h(v)=shortest path from s to v
run Dijkstra's from every vertex
reweight shortest paths based on G

$$\hat{w}(u,v) = w(u,v) + h(u) - h(v)$$



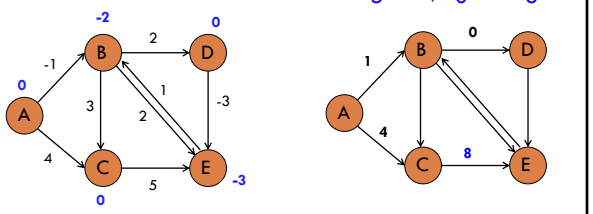
h(v) in blue

83

Create G'
run Bellman-Ford(G',s)
if no negative-weight cycle
reweight edges in G with h(v)=shortest path from s to v
run Dijkstra's from every vertex
reweight shortest paths based on G

$$\hat{w}(u,v) = w(u,v) + h(u) - h(v)$$

5 + 0 - -3

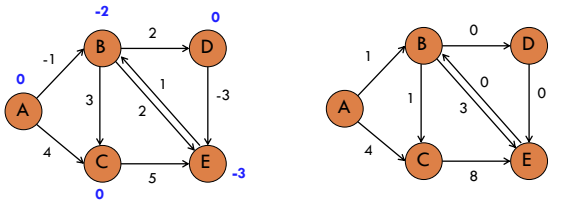


h(v) in blue

84

Create G'
run Bellman-Ford(G',s)
if no negative-weight cycle
reweight edges in G with h(v)=shortest path from s to v
run Dijkstra's from every vertex
reweight shortest paths based on G

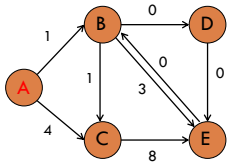
$$\hat{w}(u,v) = w(u,v) + h(u) - h(v)$$



h(v) in blue

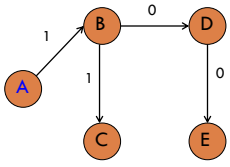
85

Create G'
run Bellman-Ford(G',s)
if no negative-weight cycle
reweight edges in G with h(v)=shortest path from s to v
run Dijkstra's from every vertex
reweight shortest paths based on G



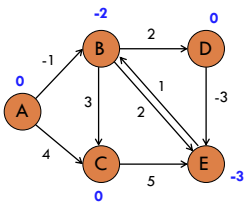
86

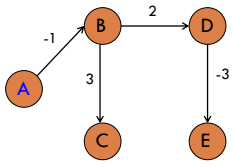
Create G'
run Bellman-Ford(G',s)
if no negative-weight cycle
reweight edges in G with h(v)=shortest path from s to v
run Dijkstra's from every vertex
reweight shortest paths based on G



87

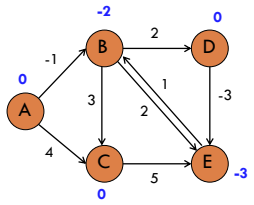
Create G'
run Bellman-Ford(G',s)
if no negative-weight cycle
reweight edges in G with h(v)=shortest path from s to v
run Dijkstra's from every vertex
reweight shortest paths based on G

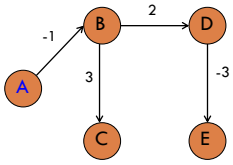




88

A→B: -1
A→C: 2
A→D: 1
A→E: -2





89

Selecting h

Need to pick h such that the resulting graph has all weights as positive

Create G' with one extra node s with 0 weight edges to all nodes
run Bellman-Ford(G',s)
if no negative-weight cycle
    reweight edges in G with h(v)=shortest path from s to v
    run Dijkstra's from every vertex
    reweight shortest paths based on G

Why does this work (i.e. how do we guarantee that reweighted graph has only positive edges)?

90

Reweighted graph is positive

Take two nodes u and v

h(u) shortest distance from s to u
h(v) shortest distance from s to v

Claim: $h(v) \le h(u) + w(u,v)$

Why?

91

Reweighted graph is positive

Take two nodes u and v

h(u) shortest distance from s to u
h(v) shortest distance from s to v

Claim: $h(v) \le h(u) + w(u,v)$

If this weren't true, we could have made a shorter path s to v using u

… but this is in contradiction with how we defined h(v)

92

Reweighted graph is positive

Take two nodes u and v

h(u) shortest distance from s to u
h(v) shortest distance from s to v

$$h(v) \le h(u) + w(u,v)$$

$$\underbrace{w(u,v) + h(u) - h(v)}_{} \ge 0$$

What is this?

93

Reweighted graph is positive

Take two nodes u and v

h(u) shortest distance from s to u
h(v) shortest distance from s to v

$$h(v) \le h(u) + w(u,v)$$

$$\underbrace{w(u,v) + h(u) - h(v)}_{} \ge 0$$

$$\hat{w}(u,v) = w(u,v) + h(u) - h(v)$$

$$\hat{w}(u,v) = w(u,v) + h(u) - h(v) \ge 0$$

All edge weights in reweighted graph are non-negative

94

Johnson's algorithm

Create G'
run Bellman-Ford(G',s)
if no negative-weight cycle
    reweight edges in G
    run Dijkstra's from every vertex
    reweight shortest paths based on G

Run-time?

95

Johnson's algorithm

Create G' — θ(V)
run Bellman-Ford(G',s) — $O(V^2)$
if no negative-weight cycle
    reweight edges in G — θ(E)
    run Dijkstra's from every vertex — $O(V^2 \log V + VE)$
    reweight shortest paths based on G — θ(E)

Run-time?

96

All pairs shortest paths

V * Bellman-Ford: $O(V^2E)$

Floyd-Warshall: $\theta(V^3)$

Johnson's: $O(V^2 \log V + V E)$

97