BINARY SEARCH TREES

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CS 140 – Spring 2022

1

Number guessing game

I'm thinking of a number between 1 and n

You are trying to guess the answer

For each guess, I'll tell you "correct", "higher" or "lower"

Describe an algorithm that minimizes the number of guesses

2

Binary Search Trees

3

Binary Search Trees

BST – A binary tree where a parent's value is greater than all values in the left subtree and less than or equal to all the values in the right subtree

$$leftTree(i) < i \leq rightTree(i)$$

and the left and right children are also binary search trees

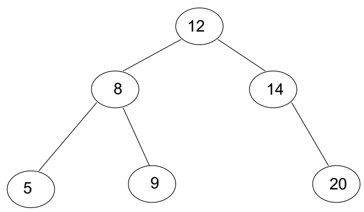
Why not?

$$leftTree(i) \leq i \leq rightTree(i)$$

Ambiguous about where elements that are equal would reside

4

Example

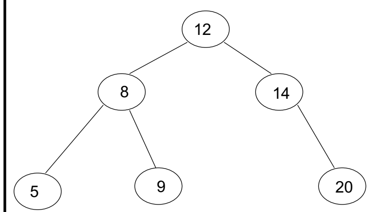


Can be implemented with with references or an array

5

What else can we conclude?

$$leftTree(i) < i \leq rightTree(i)$$



The smallest element is the left-most element

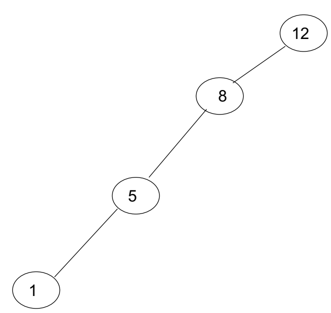
The largest element is the right-most element

6

Another example: the solo tree

7

Another example: the twig



8

Operations

Search(T,k) – Does value k exist in tree T
Insert(T,k) – Insert value k into tree T
Delete(T,x) – Delete node x from tree T
Minimum(T) – What is the smallest value in the tree?
Maximum(T) – What is the largest value in the tree?
Successor(T,x) – What is the next element in sorted order after x
Predecessor(T,x) – What is the previous element in sorted order of x
Median(T) – return the median of the values in tree T

9

Search

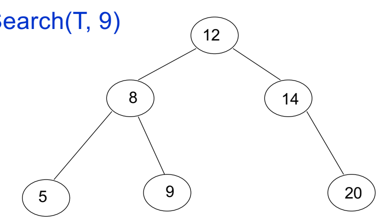
How do we find an element?

```
BSTSearch(x, k)
if x = null or k = x
        return x
elseif k < x
        return BSTSearch(Left(x), k)
else
        return BSTSearch(Right(x), k)
```

10

Finding an element

Search(T, 9)




```
BSTSearch(x, k)
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elseif k < x
        return BSTSearch(Left(x), k)
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        return BSTSearch(Right(x), k)
```
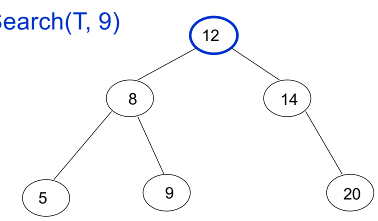

11

Finding an element

Search(T, 9)

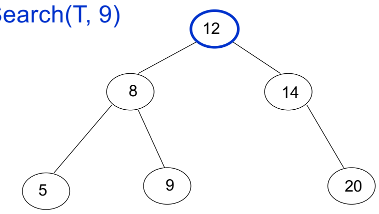



```
BSTSearch(x, k)
if x = null or k = x
        return x
elseif k < x
        return BSTSearch(Left(x), k)
else
        return BSTSearch(Right(x), k)
```

12

Finding an element

Search(T, 9)



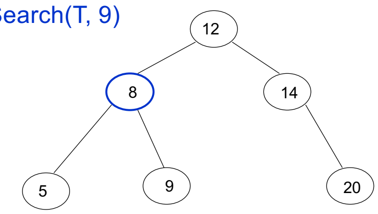
9 > 12?

```
BSTSearch(x, k)
if x = null or k = x
        return x
elseif k < x
        return BSTSearch(Left(x), k)
else
        return BSTSearch(Right(x), k)
```

13

Finding an element

Search(T, 9)




```
BSTSearch(x, k)
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        return x
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else
        return BSTSearch(Right(x), k)
```
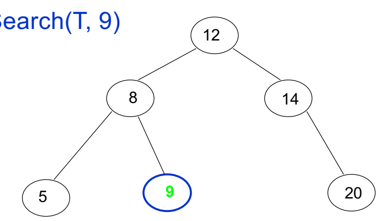

14

Finding an element

Search(T, 9)

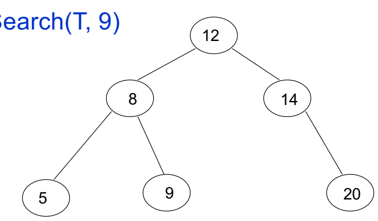



```
BSTSearch(x, k)
if x = null or k = x
        return x
elseif k < x
        return BSTSearch(Left(x), k)
else
        return BSTSearch(Right(x), k)
```

15

Finding an element

Search(T, 9)

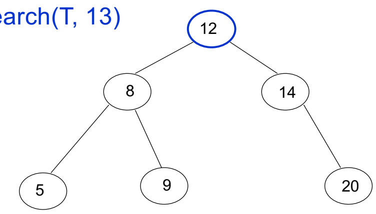



```
BSTSearch(x, k)
if x = null or k = x
        return x
elseif k < x
        return BSTSearch(Left(x), k)
else
        return BSTSearch(Right(x), k)
```

16

Finding an element

Search(T, 13)

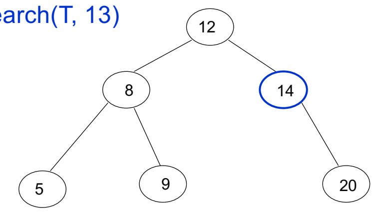



```
BSTSearch(x, k)
if x = null or k = x
        return x
elseif k < x
        return BSTSearch(Left(x), k)
else
        return BSTSearch(Right(x), k)
```

17

Finding an element

Search(T, 13)

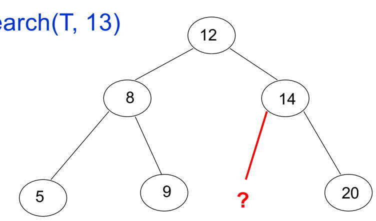



```
BSTSearch(x, k)
if x = null or k = x
        return x
elseif k < x
        return BSTSearch(Left(x), k)
else
        return BSTSearch(Right(x), k)
```

18

Finding an element

Search(T, 13)




```
BSTSearch(x, k)
if x = null or k = x
        return x
elseif k < x
        return BSTSearch(Left(x), k)
else
        return BSTSearch(Right(x), k)
```

19

Iterative search

```
IterativeBSTSearch(x, k)
while x ≠ null and k ≠ x
        if k < x
                x ← Left(x)
        else
                x ← Right(x)
return x
```

```
BSTSearch(x, k)
if x = null or k = x
        return x
elseif k < x
        return BSTSearch(Left(x), k)
else
        return BSTSearch(Right(x), k)
```

20

Is BSTSearch correct?

```
BSTSearch(x, k)
if x = null or k = x
        return x
elseif k < x
        return BSTSearch(Left(x), k)
else
        return BSTSearch(Right(x), k)
```

$$leftTree(i) < i \leq rightTree(i)$$

21

Running time of BSTSearch

Worst case?

- θ(height of the tree)

Average case?

- O(height of the tree)

Best case?

- O(1)

22

Height of the tree

Worst case height?

- n-1
- "the twig"

Best case height?

- $\lfloor \log_2 n \rfloor$
- complete (or near complete) binary tree

Average case height?

- Depends on two things:
  - the data
  - how we build the tree!

23

Insertion

```
BSTInsert(T, x)
if Root(T) = null
        Root(T) ← x
else
        y ← Root(T)
        while y ≠ null
                prev ← y
                if x < y
                        y ← Left(y)
                else
                        y ← Right(y)
        Parent(x) ← prev
        if x < prev
                Left(prev) ← x
        else
                Right(prev) ← x
```

24

Insertion

```
BSTInsert(T, x)
if Root(T) = null
        Root(T) ← x
else
        y ← Root(T)
        while y ≠ null
                prev ← y
                if x < y
                        y ← Left(y)
                else
                        y ← Right(y)
        Parent(x) ← prev
        if x < prev
                Left(prev) ← x
        else
                Right(prev) ← x
```

Similar to search

```
IterativeBSTSearch(x, k)
while x ≠ null and k ≠ x
        if k < x
                x ← Left(x)
        else
                x ← Right(x)
return x
```

25

Insertion

```
BSTInsert(T, x)
if Root(T) = null
        Root(T) ← x
else
        y ← Root(T)
        while y ≠ null
                prev ← y
                if x < y
                        y ← Left(y)
                else
                        y ← Right(y)
        Parent(x) ← prev
        if x < prev
                Left(prev) ← x
        else
                Right(prev) ← x
```

Similar to search

Find the correct location in the tree

26

Insertion

```
BSTInsert(T, x)
if Root(T) = null
        Root(T) ← x
else
        y ← Root(T)
        while y ≠ null
                prev ← y
                if x < y
                        y ← Left(y)
                else
                        y ← Right(y)
        Parent(x) ← prev
        if x < prev
                Left(prev) ← x
        else
                Right(prev) ← x
```

keeps track of the previous node we visited so when we fall off the tree, we know

27

Insertion

```
BSTInsert(T, x)
if Root(T) = null
        Root(T) ← x
else
        y ← Root(T)
        while y ≠ null
                prev ← y
                if x < y
                        y ← Left(y)
                else
                        y ← Right(y)
        Parent(x) ← prev
        if x < prev
                Left(prev) ← x
        else
                Right(prev) ← x
```

add node onto the bottom of the tree

28

Correctness?

```
BSTInsert(T, x)
if Root(T) = null
    Root(T) ← x
else
    y ← Root(T)
    while y ≠ null
        prev ← y
        if x < y
            y ← Left(y)
        else
            y ← Right(y)
    Parent(x) ← prev
    if x < prev
        Left(prev) ← x
    else
        Right(prev) ← x
```

maintain BST property

29

Correctness

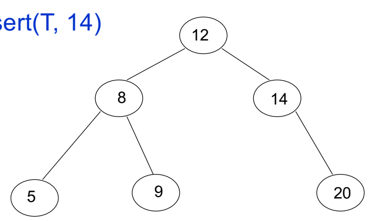
```
BSTInsert(T, x)
if Root(T) = null
    Root(T) ← x
else
    y ← Root(T)
    while y ≠ null
        prev ← y
        if x < y
            y ← Left(y)
        else
            y ← Right(y)
    Parent(x) ← prev
    if x < prev
        Left(prev) ← x
    else
        Right(prev) ← x
```

What happens if it is a duplicate?

30

Inserting duplicate

Insert(T, 14)

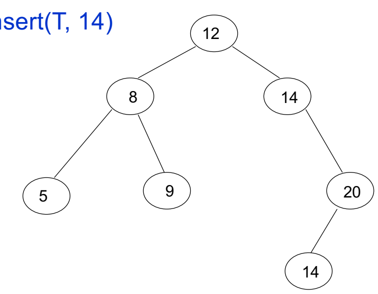


$$leftTree(i) < i \leq rightTree(i)$$

31

Inserting duplicate

Insert(T, 14)



$$leftTree(i) < i \leq rightTree(i)$$

32

Running time

```
BSTInsert(T, x)
if Root(T) = null
    Root(T) ← x
else
    y ← Root(T)
    while y ≠ null
        prev ← y
        if x < y
            y ← Left(y)
        else
            y ← Right(y)
    Parent(x) ← prev
    if x < prev
        Left(prev) ← x
    else
        Right(prev) ← x
```

O(height of the tree)

33

Running time

```
BSTInsert(T, x)
if Root(T) = null
    Root(T) ← x
else
    y ← Root(T)
    while y ≠ null
        prev ← y
        if x < y
            y ← Left(y)
        else
            y ← Right(y)
    Parent(x) ← prev
    if x < prev
        Left(prev) ← x
    else
        Right(prev) ← x
```

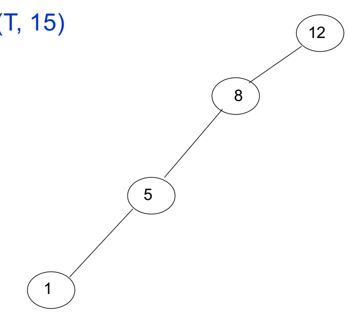
O(height of the tree)

Why not
Θ(height of the tree)?

34

Running time

Insert(T, 15)



35

Height of the tree

Worst case: "the twig" – When will this happen?

```
BSTInsert(T, x)
if Root(T) = null
    Root(T) ← x
else
    y ← Root(T)
    while y ≠ null
        prev ← y
        if x < y
            y ← Left(y)
        else
            y ← Right(y)
    Parent(x) ← prev
    if x < prev
        Left(prev) ← x
    else
        Right(prev) ← x
```

36

Height of the tree

Best case: “complete” – When will this happen?

```
BSTInsert(T, x)
if Root(T) = null
    Root(T) ← x
else
    y ← Root(T)
    while y ≠ null
        prev ← y
        if x < y
            y ← Left(y)
        else
            y ← Right(y)
    Parent(x) ← prev
    if x < prev
        Left(prev) ← x
    else
        Right(prev) ← x
```

37

Height of the tree

Average case for random data?

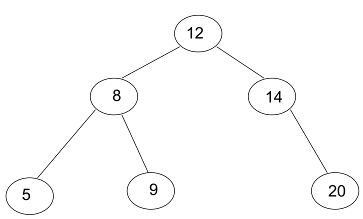
```
BSTInsert(T, x)
if Root(T) = null
    Root(T) ← x
else
    y ← Root(T)
    while y ≠ null
        prev ← y
        if x < y
            y ← Left(y)
        else
            y ← Right(y)
    Parent(x) ← prev
    if x < prev
        Left(prev) ← x
    else
        Right(prev) ← x
```

Randomly inserting data into a BST generates a tree on average that is O(log n)

38

Visiting all nodes

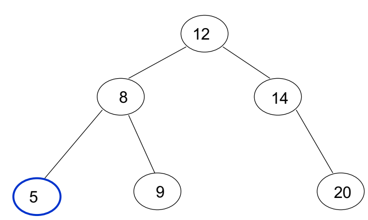
In sorted order



39

Visiting all nodes

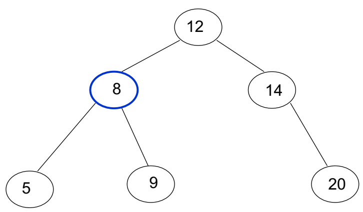
In sorted order 5



40

Visiting all nodes

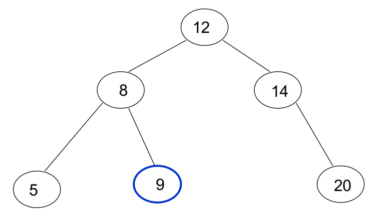
In sorted order 5, 8



41

Visiting all nodes

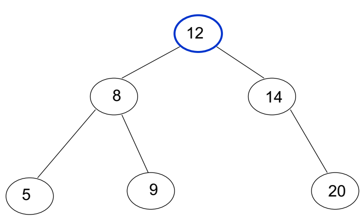
In sorted order 5, 8, 9



42

Visiting all nodes

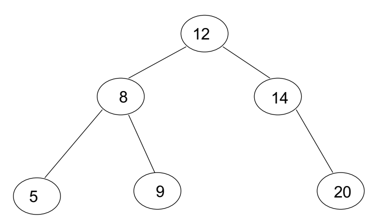
In sorted order 5, 8, 9, 12



43

Visiting all nodes

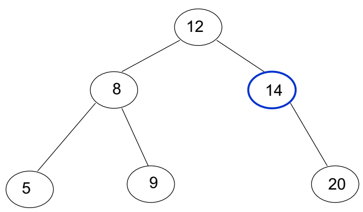
What's happening? 5, 8, 9, 12



44

Visiting all nodes

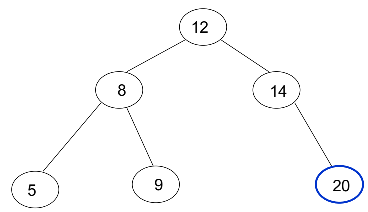
In sorted order 5, 8, 9, 12, 14



45

Visiting all nodes

In sorted order 5, 8, 9, 12, 14, 20



46

Visiting all nodes in order

```
InorderTreeWalk(x)
if x ≠ null
        InorderTreeWalk(Left(x))
        print x
        InorderTreeWalk(Right(x))
```

47

Visiting all nodes in order

```
InorderTreeWalk(x)
if x ≠ null
        InorderTreeWalk(Left(x))
        print x
        InorderTreeWalk(Right(x))
```

any operation

48

Is it correct?

```
INORDERTREEWALK(x)
if x ≠ null
    INORDERTREEWALK(LEFT(x))
    print x
    INORDERTREEWALK(RIGHT(x))
```

Does it print out all of the nodes in sorted order?

$$leftTree(i) < i \le rightTree(i)$$

49

What about?

```
TREEWALK(x)
if x ≠ null
    print x
    TREEWALK(LEFT(x))
    TREEWALK(RIGHT(x))
```

51

Preorder traversal

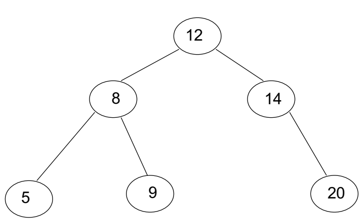
```
TREEWALK(x)
if x ≠ null
    print x
    TREEWALK(LEFT(x))
    TREEWALK(RIGHT(x))
```

12, 8, 5, 9, 14, 20

How is this useful?

Tree copying: insert in to new tree in preorder

prefix notation: (2+3)*4 -> * + 2 3 4



52

What about?

```
TREEWALK(x)
if x ≠ null
    TREEWALK(LEFT(x))
    TREEWALK(RIGHT(x))
    print x
```

53

Postorder traversal

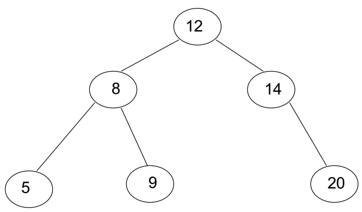
```
TreeWalk(x)
if x ≠ null
    TreeWalk(Left(x))
    TreeWalk(Right(x))
    print x
```

5, 9, 8, 20, 14, 12

How is this useful?

postfix notation: (2+3)*4
-> 4 3 2 + *

?

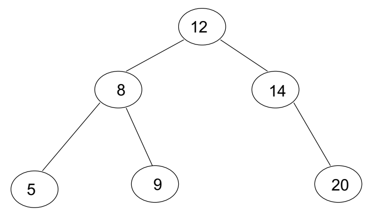


54

Min/Max

```
BSTMin(x)
if Left(x) = null
    return x
else
    return BSTMin(Left(x))
```

```
IterativeBSTMin(x)
while Left(x) ≠ null
    x ← Left(x)
return x
```



55

Running time of min/max?

```
BSTMin(x)
if Left(x) = null
    return x
else
    return BSTMin(Left(x))
```

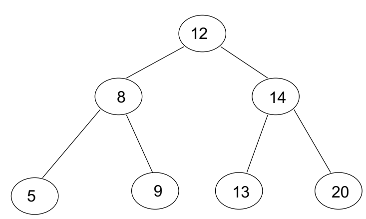
```
IterativeBSTMin(x)
while Left(x) ≠ null
    x ← Left(x)
return x
```

O(height of the tree)

56

Successor and predecessor

Predecessor(12)? 9



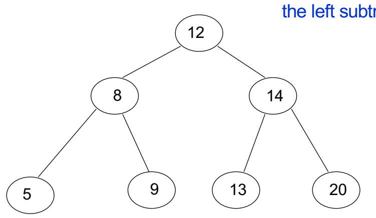
57

Successor and predecessor

Predecessor in general?

largest node of all those smaller than this node

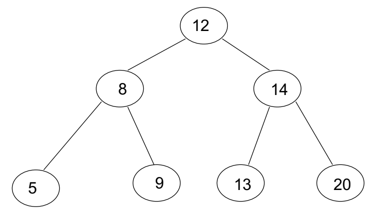
rightmost element of the left subtree



58

Successor

Successor(12)? 13



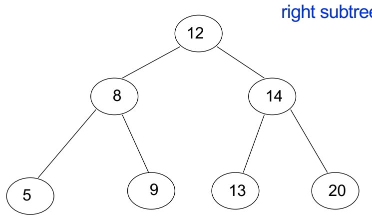
59

Successor

Successor in general?

smallest node of all those larger than this node

leftmost element of the right subtree



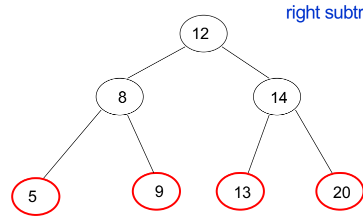
60

Successor

What if the node doesn't have a right subtree?

smallest node of all those larger than this node

leftmost element of the right subtree



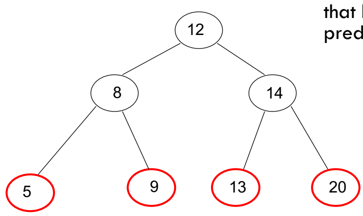
61

Successor

What if the node doesn't have a right subtree?

node is the largest

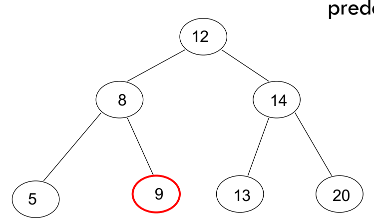
the successor is the node that has x as a predecessor



62

Successor

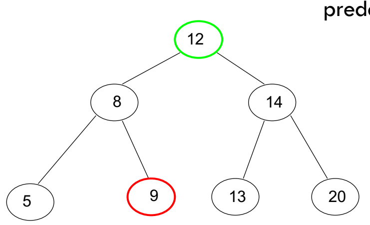
successor is the node that has x as a predecessor



63

Successor

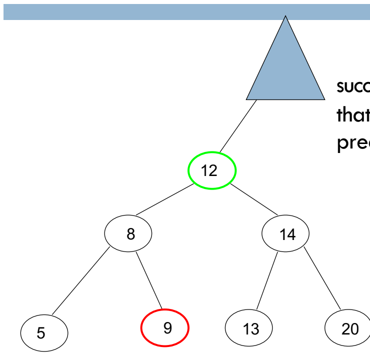
successor is the node that has x as a predecessor



64

Successor

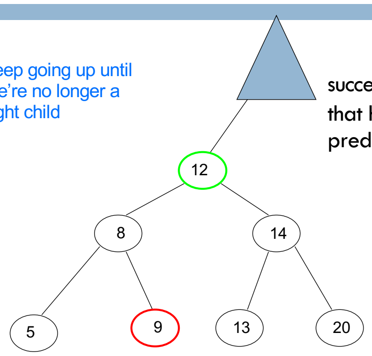
successor is the node that has x as a predecessor



65

Successor

keep going up until we're no longer a right child



successor is the node that has x as a predecessor

66

Successor

```
SUCCESSOR(x)
if RIGHT(x) ≠ null
        return BSTMIN(RIGHT(x))
else
        y ← PARENT(x)
        while y ≠ null and x = RIGHT(y)
                x ← y
                y ← PARENT(y)
return y
```

67

Successor

```
SUCCESSOR(x)
if RIGHT(x) ≠ null
        return BSTMIN(RIGHT(x))
else
        y ← PARENT(x)
        while y ≠ null and x = RIGHT(y)
                x ← y
                y ← PARENT(y)
return y
```

if we have a right subtree, return the smallest of the right subtree

68

Successor

```
SUCCESSOR(x)
if RIGHT(x) ≠ null
        return BSTMIN(RIGHT(x))
else
        y ← PARENT(x)
        while y ≠ null and x = RIGHT(y)
                x ← y
                y ← PARENT(y)
return y
```

find the node that x is the predecessor of

keep going up until we're no longer a right child

69

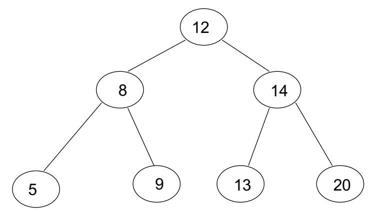
Successor running time

O(height of the tree)

```
SUCCESSOR(x)
if RIGHT(x) ≠ null
        return BSTMIN(RIGHT(x))
else
        y ← PARENT(x)
        while y ≠ null and x = RIGHT(y)
                x ← y
                y ← PARENT(y)
return y
```

70

Deletion



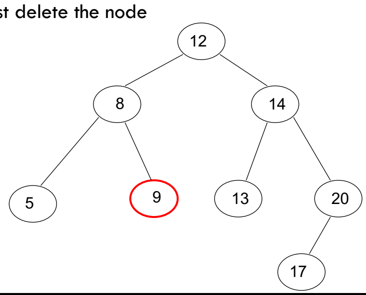
Three cases!

71

Deletion: case 1

No children

Just delete the node

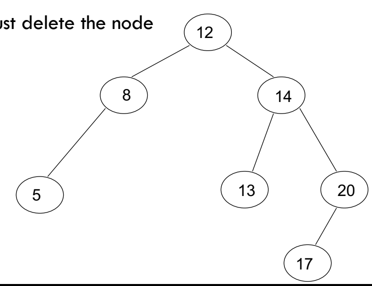


72

Deletion: case 1

No children

Just delete the node

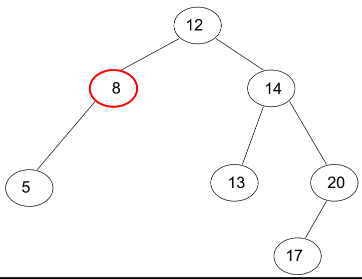


73

Deletion: case 2

One child

Splice out the node

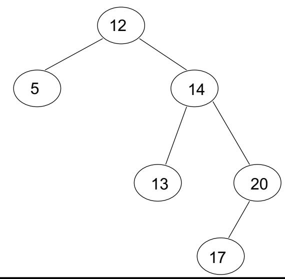


74

Deletion: case 2

One child

Splice out the node

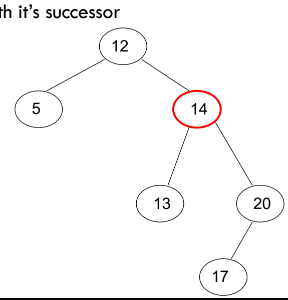


75

Deletion: case 3

Two children

Replace x with it's successor

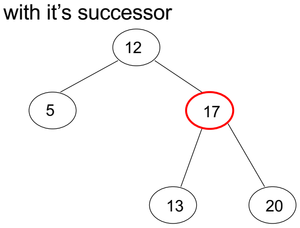


76

Deletion: case 3

Two children

Replace x with it's successor



77

Deletion: case 3

Two children

Will we always have a successor?

Why successor?

- Larger than the left subtree
- Less than or equal to right subtree

78

Height of the tree

Most of the operations take time
O(height of the tree)

We said trees built from random data have height
O(log n), which is asymptotically tight

Two problems:

- We can't always insure random data
- What happens when we delete nodes and insert others after building a tree?

79

Balanced trees

Make sure that the trees remain balanced!

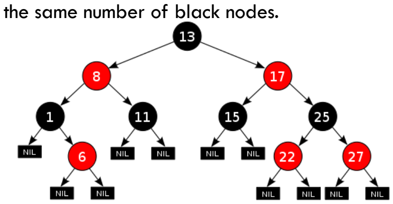
- Red-black trees
- AVL trees
- 2-3-4 trees
- …

B-trees

80

Red-black trees: BST (plus some)

1. every node is either red or black
2. root is black
3. leaves (NIL) are black
4. if a node is red, both children are black
5. for every node, all paths from the node to descendant leaves contain the same number of black nodes.



https://en.wikipedia.org/wiki/Red–black_tree

81

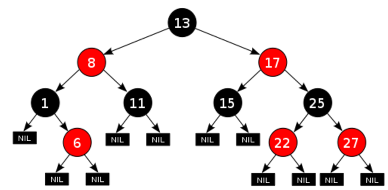
Red-black trees: BST (plus some)

1. every node is either red or black
2. root is black
3. leaves (NIL) are black
4. if a node is red, both children are black
5. for every node, all paths from the node to descendant leaves contain the same number of black nodes.

$h(x)$: height of node x: number of edges in longest path from x to a leaf

82

Red-black trees: BST (plus some)

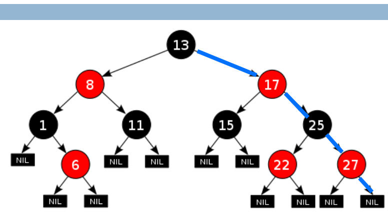


$h(x)$: height of node x: number of edges in longest path from x to a leaf

What is the height of the root node?

83

Red-black trees: BST (plus some)



$h(x)$: height of node x: number of edges in longest path from x to a leaf

4

84

Red-black trees: BST (plus some)

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5. for every node, all paths from the node to descendant leaves contain the same number of black nodes.

$bh(x)$: black height of node x: number of black nodes on a path from x to leaf (**not** including x)

Why don't we say "path with the most…"?

85

Red-black trees: BST (plus some)

1. every node is either red or black
2. root is black
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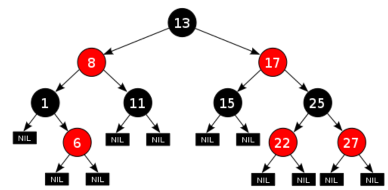
Why don't we say "path with the most…"?

86

Red-black trees: BST (plus some)

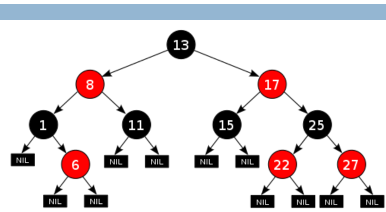


$bh(x)$: black height of node x: number of black nodes on a path from x to leaf (**not** including x)

What is the black height of the root node?

87

Red-black trees: BST (plus some)



$bh(x)$: black height of node x: number of black nodes on a path from x to leaf (**not** including x)

2

88

Bounding the height

1. every node is either red or black
2. root is black
3. leaves (NIL) are black
4. if a node is red, both children are black
5. for every node, all paths from the node to descendant leaves contain the same number of black nodes.

$h(x)$: height of node x: number of edges in longest path from x to a leaf

$bh(x)$: black height of node x: number of black nodes on a path from x to leaf (**not** including x)

Claim 1: For every node x, $bh(x) \geq h(x)/2$

Proof?

89

Bounding the height

1. every node is either red or black
2. root is black
3. leaves (NIL) are black
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$bh(x)$: black height of node x: number of black nodes on a path from x to leaf (**not** including x)

Claim 1: For every node x, $bh(x) \geq h(x)/2$

Worst case: nodes alternate red/black
- root is black
- leaf is black

In terms of h(x): How many black nodes are there on this path?

90

Bounding the height

1. every node is either red or black
2. root is black
3. leaves (NIL) are black
4. if a node is red, both children are black
5. for every node, all paths from the node to descendant leaves contain the same number of black nodes.

$h(x)$: height of node x: number of edges in longest path from x to a leaf

$bh(x)$: black height of node x: number of black nodes on a path from x to leaf (**not** including x)

Claim 1: For every node x, $bh(x) \geq h(x)/2$

minimum black nodes on path: $\frac{h(x)}{2} + 1$

$\mathrm{b}h(x) \geq \frac{h(x)}{2}$

bh does NOT include x, i.e., the root in this case

91

Bounding the height

Claim 2: The subtree rooted at any node x contains at least $2^{bh(x)} - 1$ internal (non-leaf) nodes

Proof?

92

Bounding the height

Claim 2: The subtree rooted at any node x contains at least $2^{bh(x)} - 1$ internal (non-leaf) nodes

Base case:

93

Bounding the height

Claim 2: The subtree rooted at any node x contains at least $2^{bh(x)} - 1$ internal (non-leaf) nodes

Base case: leaf ($h(x) = 0$)

$$bh(x) = 0$$
$$2^0 - 1 = 0$$

94

Bounding the height

Claim 2: The subtree rooted at any node x contains at least $2^{bh(x)} - 1$ internal (non-leaf) nodes

Inductive case: $h(x) > 0$

IH: $2^{bh(y)} - 1$ for all y that are subtrees of x

What is $bh(child(x))$ wrt $bh(x)$?

$bh(x)$: black height of node x: number of black nodes on a path from x to leaf (**not** including x)

95

Bounding the height

Claim 2: The subtree rooted at any node x contains at least $2^{bh(x)} - 1$ internal (non-leaf) nodes

Inductive case: $h(x) > 0$

IH: $2^{bh(y)} - 1$ for all y that are subtrees of x

x is red: $bh(child(x)) = bh(x) - 1$
x is black: $bh(child(x)) = bh(x) \text{ or } bh(x) - 1$

$bh(x)$: black height of node x: number of black nodes on a path from x to leaf (**not** including x)

96

Bounding the height

Claim 2: The subtree rooted at any node x contains at least $2^{bh(x)} - 1$ internal (non-leaf) nodes

Inductive case: $h(x) > 0$

IH: $2^{bh(y)} - 1$ for all y that are subtrees of x

x is red: $bh(child(x)) = bh(x) - 1$
x is black: $bh(child(x)) = bh(x) \text{ or } bh(x) - 1$

$$bh(child(x)) \geq bh(x) - 1$$

97

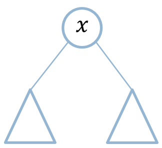
Bounding the height

Claim 2: The subtree rooted at any node x contains at least $2^{bh(x)} - 1$ internal (non-leaf) nodes

Inductive case: $h(x) > 0$

IH: $2^{bh(y)} - 1$ for all y that are subtrees of x

$bh(child(x)) \geq bh(x) - 1$



How many (internal nodes are in this tree (at least)?

98

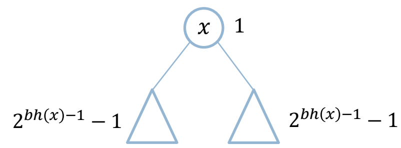
Bounding the height

Claim 2: The subtree rooted at any node x contains at least $2^{bh(x)} - 1$ internal (non-leaf) nodes

Inductive case: $h(x) > 0$

IH: $2^{bh(y)} - 1$ for all y that are subtrees of x

$bh(child(x)) \geq bh(x) - 1$



99

Bounding the height

Claim 2: The subtree rooted at any node x contains at least $2^{bh(x)} - 1$ internal (non-leaf) nodes

Inductive case: $h(x) > 0$

IH: $2^{bh(y)} - 1$ for all y that are subtrees of x

$bh(child(x)) \geq bh(x) - 1$

$$(2^{bh(x)-1} - 1) + (2^{bh(x)-1} - 1) + 1 = 2^{bh(x)} - 1$$

100

Bounding the height (*almost there!)*

Claim 1: For every node x, $bh(x) \leq \frac{h(x)}{2}$

Claim 2: The subtree rooted at any node x contains at least $2^{bh(x)} - 1$ internal (non-leaf) nodes

How does this help us?

101

Bounding the height

Claim 1: For every node x, $bh(x) \geq \frac{h(x)}{2}$

Claim 2: The subtree rooted at any node x contains at least $2^{bh(x)} - 1$ internal (non-leaf) nodes

$n \geq 2^{bh(x)} - 1$	Claim 2
$n \geq 2^{h(x)/2} - 1$	Claim 1
$n + 1 \geq 2^{h(x)/2}$	math
$h(x) \leq 2\log(n+1)$	math

What does this mean?

102

Bounding the height

1. every node is either red or black
2. root is black
3. leaves (NIL) are black
4. if a node is red, both children are black
5. for every node, all paths from the node to descendant leaves contain the same number of black nodes.

If we can maintain these properties: height $O(\log n)$

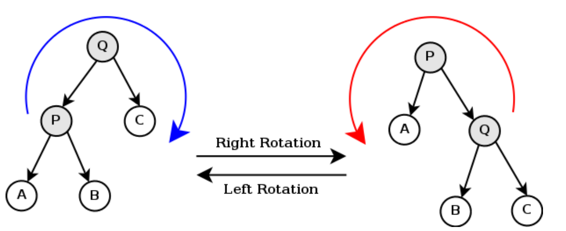
Search
Insert
Delete
Maximum

These all become $O(\log n)$

103

Can it be done?

Can we maintain the red-black tree properties without making insertion and deletion more expensive?



https://en.wikipedia.org/wiki/Tree_rotation#/media/File:Tree_rotation.png

104

A quick example

https://www.youtube.com/watch?v=vDHFF4wjWYU

105