Order Statistics

David Kauchak
cs140
Fall 2022

1

Administrative

Checkpoint 1

- 2 pages of notes
- cover everything up through today

2

Medians

The median of a set of numbers is the number such that half of the numbers are larger and half smaller

A = [50, 12, 1, 97, 30]

How might we calculate the median of a set?

Sort the numbers, then pick the n/2 element

A = [1, 12, 30, 50, 97]

runtime?

3

Medians

The median of a set of numbers is the number such that half of the numbers are larger and half smaller

A = [50, 12, 1, 97, 30]

How might we calculate the median of a set?

Sort the numbers, then pick the n/2 element

A = [1, 12, 30, 50, 97]

$\Theta(n \log n)$

4

Selection

More general problem:
find the *k*-th smallest element in an array

- i.e. element where exactly k-1 things are smaller than it
- aka the "selection" problem
- can use this to find the median if we want

Can we solve this in a similar way?

- Yes, sort the data and take the kth element
- Θ(n log n)

5

Can we do better?

Are we doing more work than we need to?

To get the k-th element (or the median) by sorting, we're finding *all* the k-th elements at once

We just want the one!

Often when you find yourself doing more work than you need to, there is a faster way (though not always)

6

selection problem

Our tools

- divide and conquer
- sorting algorithms
- other functions
  - merge
  - partition
  - binary search

7

Partition

Partition takes Θ(n) time and performs a similar operation

given an element A[q], Partition can be seen as dividing the array into three sets:

- < A[q]
- = A[q]
- > A[q]

Ideas?

8

An example

We're looking for the 5th smallest

5 2 34 9 17 2 1 34 18 5 3 2 1 6 5

If we called partition, what would be the in three sets?

< 5:

= 5:

> 5:

9

An example

We're looking for the 5th smallest

5 2 34 9 17 2 1 34 18 5 3 2 1 6 5

< 5: 2 2 1 3 2 1

= 5: 5 5 5

> 5: 34 9 17 34 18 6

Does this help us?

10

An example

We're looking for the 5th smallest

5 2 34 9 17 2 1 34 18 5 3 2 1 6 5

< 5: 2 2 1 3 2 1

We know the 5th smallest has to be in this set

= 5: 5 5 5

> 5: 34 9 17 34 18 6

11

```
Selection(A, k, p, r)
  q <- Partition(A,p,r)
  relq = q-p+1
  if k = relq
    Return A[q]
  else if k < relq
    Return Selection(A, k, p, q-1)
  else // k > relq
    Return Selection(A, k-relq, q+1, r)
```

A: array of data
k: find the kth smallest
p,r: current span we're exploring (initially 1, len(A))

12

Selection: divide and conquer

Call partition

- decide which of the three sets contains the answer we're looking for
- recurse

Like binary search on unsorted data

?

```
Selection(A, k, p, r)
  q <- Partition(A,p,r)
  relq = q-p+1
  if k = relq
    Return A[q]
  else if k < relq
    Return Selection(A, k, p, q-1)
  else // k > relq
    Return Selection(A, k-relq, q+1, r)
```

13

Selection: divide and conquer

Call partition

- decide which of the three sets contains the answer we're looking for
- recurse

Like binary search on unsorted data

```
Selection(A, k, p, r)
  q <- Partition(A,p,r)
  relq = q-p+1
  if k = relq
    Return A[q]
  else if k < relq
    Return Selection(A, k, p, q-1)
  else // k > relq
    Return Selection(A, k-relq, q+1, r)
```

Partition returns the absolute index, we want an index relative to the current p (window start)

14

Selection: divide and conquer

Call partition

- decide which of the three sets contains the answer we're looking for
- recurse

Like binary search on unsorted data

```
Selection(A, k, p, r)
  q <- Partition(A,p,r)
  relq = q-p+1
  if k = relq
    Return A[q]
  else if k < relq
    Return Selection(A, k, p, q-1)
  else // k > relq
    Return Selection(A, k-relq, q+1, r)
```

As we recurse, we may update the k that we're looking for because we update the lower end

15

Selection(A, 3, 1, 8)

1	2	3	4	5	6	7	8
5	7	1	4	8	3	2	6

```
Selection(A, k, p, r)
  q <- Partition(A,p,r)
  relq = q-p+1
  if k = relq
    Return A[q]
  else if k < relq
    Selection(A, k, p, q-1)
  else // k > relq
    Selection(A, k-relq, q+1, r)
```

16

Selection(A, 3, 1, 8)

1	2	3	4	5	6	7	8
5	1	4	3	2	6	8	7
					↑		

relq = 6 – 1 + 1 = 6

```
Selection(A, k, p, r)
  q <- Partition(A,p,r)
  relq = q-p+1
  if k = relq
    Return A[q]
  else if k < relq
    Selection(A, k, p, q-1)
  else // k > relq
    Selection(A, k-relq, q+1, r)
```

17

Selection(A, 3, 1, 8)

1	2	3	4	5	6	7	8
5	1	4	3	2	6	8	7
					↑		

relq = 6 – 1 + 1 = 6

```
Selection(A, k, p, r)
  q <- Partition(A,p,r)
  relq = q-p+1
  if k = relq
    Return A[q]
  else if k < relq
    Selection(A, k, p, q-1)
  else // k > relq
    Selection(A, k-relq, q+1, r)
```

18

Selection(A, 3, 1, 5)

1	2	3	4	5	6	7	8
5	1	4	3	2	6	8	7

At each call, discard part of the array

```
Selection(A, k, p, r)
  q <- Partition(A,p,r)
  relq = q-p+1
  if k = relq
    Return A[q]
  else if k < relq
    Selection(A, k, p, q-1)
  else // k > relq
    Selection(A, k-relq, q+1, r)
```

19

Selection(A, 3, 1, 5)

1	2	3	4	5	6	7	8
1	2	4	3	5	6	8	7
	↑						

relq = 2 – 1 + 1 = 2

```
Selection(A, k, p, r)
  q <- Partition(A,p,r)
  relq = q-p+1
  if k = relq
    Return A[q]
  else if k < relq
    Selection(A, k, p, q-1)
  else // k > relq
    Selection(A, k-relq, q+1, r)
```

20

Selection(A, 1, 3, 5)

1	2	3	4	5	6	7	8
1	2	4	3	5	6	8	7

```
Selection(A, k, p, r)
  q <- Partition(A,p,r)
  relq = q-p+1
  if k = relq
    Return A[q]
  else if k < relq
    Selection(A, k, p, q-1)
  else // k > relq
    Selection(A, k-relq, q+1, r)
```

21

Selection(A, 1, 3, 5)

1	2	3	4	5	6	7	8
1	2	4	3	5	6	8	7

```
Selection(A, k, p, r)
  q <- Partition(A,p,r)
  relq = q-p+1
  if k = relq
    Return A[q]
  else if k < relq
    Selection(A, k, p, q-1)
  else // k > relq
    Selection(A, k-relq, q+1, r)
```

22

Selection(A, 1, 3, 5)

1	2	3	4	5	6	7	8
1	2	4	3	5	6	8	7

relq = 5 – 3 + 1 = 3

```
Selection(A, k, p, r)
  q <- Partition(A,p,r)
  relq = q-p+1
  if k = relq
    Return A[q]
  else if k < relq
    Selection(A, k, p, q-1)
  else // k > relq
    Selection(A, k-relq, q+1, r)
```

23

Selection(A, 1, 3, 4)

1	2	3	4	5	6	7	8
1	2	4	3	5	6	8	7

```
Selection(A, k, p, r)
  q <- Partition(A,p,r)
  relq = q-p+1
  if k = relq
    Return A[q]
  else if k < relq
    Selection(A, k, p, q-1)
  else // k > relq
    Selection(A, k-relq, q+1, r)
```

24

Selection(A, 1, 3, 4)

1	2	3	4	5	6	7	8
1	2	4	3	5	6	8	7

```
Selection(A, k, p, r)
  q <- Partition(A,p,r)
  relq = q-p+1
  if k = relq
    Return A[q]
  else if k < relq
    Selection(A, k, p, q-1)
  else // k > relq
    Selection(A, k-relq, q+1, r)
```

25

Selection(A, 1, 3, 4)

1	2	3	4	5	6	7	8
1	2	3	4	5	6	8	7

```
Selection(A, k, p, r)
  q <- Partition(A,p,r)
  relq = q-p+1
  if k = relq
    Return A[q]
  else if k < relq
    Selection(A, k, p, q-1)
  else // k > relq
    Selection(A, k-relq, q+1, r)
```

26

Selection(A, 1, 3, 4)

1	2	3	4	5	6	7	8
1	2	3	4	5	6	8	7

relq = 3 – 3 + 1 = 1

```
Selection(A, k, p, r)
  q <- Partition(A,p,r)
  relq = q-p+1
  if k = relq
    Return A[q]
  else if k < relq
    Selection(A, k, p, q-1)
  else // k > relq
    Selection(A, k-relq, q+1, r)
```

27

Running time of Selection?

Best case?

We get lucky and the element at the end of the list is the kth smallest element!

One call to partition: $\theta(n)$

28

Running time of Selection?

Worst case?

Each call to Partition only reduces our search by 1

Recurrence?

$$T(n) = T(n-1) + \Theta(n)$$

$O(n^2)$

29

Running time of Selection?

Worst case?

Each call to Partition only reduces our search by 1

When does this happen?

- sorted
- reverse sorted
- others…

30

How can randomness help us?

```
RSelection(A, k, p, r)
  q <- RPartition(A,p,r)
  if k = q
    Return A[q]
  else if k < q
    Return Selection(A, k, p, q-1)
  else // k > q
    Return Selection(A, k, q+1, r)
```

31

Running time of RSelection?

Best case
- $\theta(n)$

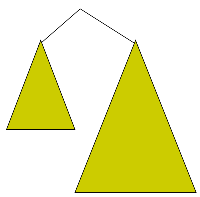
Worst case
- Still $O(n^2)$
- As with Quicksort, we can get unlucky

Average case?

32

Average case

Depends on how much data we throw away at each step



33

Average case

We'll call a partition "good" if the pivot falls within within the 25^{th} and 75^{th} percentile

- a "good" partition throws away at least a quarter of the data
- Or, each of the partitions contains at least 25% of the data

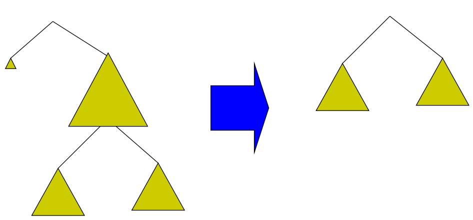
What is the probability of a "good" partition?

Half of the elements lie within this range and half outside, so 50% chance

34

Average case

Recall, that like Quicksort, we can absorb the cost of a number of "bad" partitions



35

Average case

On average, how many times will Partition need to be called before we get a good partition?

Let E be the number of times

Recurrence:

$$E = 1 + \frac{1}{2}E$$

half the time we get a good partition on the first try and half of the time, we have to try again

$$= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$$

$$= 2$$

36

Mathematicians and beer

An infinite number of mathematicians walk into a bar. The first one orders a beer. The second orders half a beer. The third, a quarter of a beer. The bartender says "You're all idiots", and pours two beers.

37

Average case

If on average we can get a "good" partition ever other time, what is the recurrence?

- recall the pivot of a "good" partition falls in the 25th and 75th percentile

$$T(n) = T(\frac{3}{4}n) + O(n)$$

We throw away at least ¼ of the data

roll in the cost of the "bad" partitions

39

Which is?

$$T(n) = T(3/4n) + \theta(n)$$

40

$$T(n) = T(3/4n) + \Theta(n)$$

if $f(n) = O(n^{\log_b a - \varepsilon})$ for $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$

if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$

if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for $\varepsilon > 0$ and $af(n/b) \le cf(n)$ for $c < 1$ then $T(n) = \Theta(f(n))$

a = 1
b = 4/3
f(n) = *n*

$n^{\log_b a} = n^{\log_{4/3} 1} = n^0$

is $n = O(n^{0-\varepsilon})$?
is $n = \Theta(n^0)$?
is $n = \Omega(n^{0+\varepsilon})$?

Case 3: Θ(n)

Average case running time!

41

An aside…

Notice a trend?

$$T(n) = T(n/2) + \Theta(n)$$ Θ(n)

$$T(n) = T(3/4n) + \Theta(n)$$ Θ(n)

42

$T(n) = T(pn) + f(n)$

for $0 < p < 1$ and

$f(n) \notin \Theta(1)$

if $f(n) = O(n^{\log_b a - \varepsilon})$ for $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$

if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$

if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for $\varepsilon > 0$ and $af(n/b) \le cf(n)$ for $c < 1$ then $T(n) = \Theta(f(n))$

a = 1
b = 1/p
f(n) = *f(n)*

$$n^{\log_b a} = n^{\log_{1/p} 1} = n^0$$

Case 3: Θ(f(n))

43