More Recurrences

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cs140
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Administrative

Assignment 2

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Recurrence

A function that is defined with respect to itself on smaller inputs

$$T(n) = 2T(n/2) + n$$

$$T(n) = 16T(n/4) + n$$

$$T(n) = 2T(n-1) + n^2$$

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The challenge

Recurrences are often easy to define because they mimic the structure of the program

But… they do not directly express the computational cost, i.e. n, n^2, …

We want to remove self-recurrence and find a more understandable form for the function

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Three approaches

Substitution method: when you have a good guess of the solution, prove that it's correct

Recursion-tree method: If you don't have a good guess, the recursion tree can help. Then solve with substitution method.

Master method: Provides solutions for recurrences of the form:

$$T(n) = aT(n/b) + f(n)$$

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Substitution method

Guess the form of the solution

Then prove it's correct by induction

$$T(n) = T(n/2) + d$$

Halves the input then constant amount of work

Similar to binary search:

Guess: $O(\log_2 n)$

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$$T(n) = T(n/2) + d$$

Assume $T(k) = O(\log_2 k)$ for all $k < n$

Show that $T(n) = O(\log_2 n)$

From our assumption, $T(n/2) = O(\log_2 n/2)$:

$$O(g(n)) = \left\{ f(n) : \begin{array}{l} \text{there exists positive constants } c \text{ and } n \text{ such that} \\ 0 \le f(n) \le cg(n) \text{ for all } n \ge n_0 \end{array} \right\}$$

From the definition of big-O: $T(n/2) \le c \log_2(n/2)$

How do we now prove T(n) = O(log n)?

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$$T(n) = T(n/2) + d$$

To prove that $T(n) = O(\log_2 n)$ identify the appropriate constants:

$$O(g(n)) = \left\{ f(n) : \begin{array}{l} \text{there exists positive constants } c \text{ and } n \text{ such that} \\ 0 \le f(n) \le cg(n) \text{ for all } n \ge n_0 \end{array} \right\}$$

i.e. some constant c' such that $T(n) \le c' \log_2 n$

$$\begin{aligned} T(n) &= T(n/2) + d \\ &\le c \log_2(n/2) + d \quad \text{from our inductive hypothesis} \\ &\le c \log_2 n - c \log_2 2 + d \\ &\le c \log_2 n - c + d \quad \text{residual} \end{aligned}$$

Key question: does a constant exist such that:
$T(n) \le c' log_2 n$

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$$T(n) = T(n/2) + d$$

To prove that $T(n) = O(\log_2 n)$ identify the appropriate constants:

$$O(g(n)) = \left\{ f(n) : \begin{array}{l} \text{there exists positive constants } c \text{ and } n \text{ such that} \\ 0 \le f(n) \le cg(n) \text{ for all } n \ge n_0 \end{array} \right\}$$

i.e. some constant c' such that $T(n) \le c' \log_2 n$

Key question: does a constant exist such that:
$T(n) \le c' \log_2 n$

$$T(n) \le c \log_2 n - c + d$$

if $c \ge d$, then, yes!
(if not, just let c' = d)

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$$T(n) = T(n-1) + n$$

Guess the solution?

At each iteration, does a linear amount of work (i.e. iterate over the data) and reduces the size by one at each step

$O(n^2)$

Assume $T(k) = O(k^2)$ for all $k < n$

- again, this implies that $T(n-1) \le c(n-1)^2$

Show that $T(n) = O(n^2)$, i.e. $T(n) \le c' n^2$

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$$\begin{aligned} T(n) &= T(n-1) + n \\ &\le c(n-1)^2 + n && \text{from our inductive hypothesis} \\ &= c(n^2 - 2n + 1) + n \\ &= cn^2 - 2cn + c + n && \text{residual} \end{aligned}$$

if $-2cn + c + n \le 0$

then, we can let c' = c and there exists a constant such that $T(n) \le c' n^2$

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$$\begin{aligned} T(n) &= T(n-1) + n \\ &\le c(n-1)^2 + n && \text{from our inductive hypothesis} \\ &= c(n^2 - 2n + 1) + n \\ &= cn^2 - 2cn + c + n && \text{residual} \end{aligned}$$

$$-2cn + c + n \le 0$$
$$-2cn + c \le -n$$
$$c(-2n + 1) \le -n$$
$$c \ge \frac{n}{2n-1}$$
$$c \ge \frac{1}{2 - 1/n}$$

which holds for any c ≥1 for n ≥1

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$$T(n) = 2T(n/2) + n$$

Guess the solution?

Recurses into 2 sub-problems that are half the size and performs some operation on all the elements

O(n log n)

What if we guess wrong, e.g. O(n^2)?

Assume $T(k) = O(k^2)$ for all $k < n$

- again, this implies that $T(n/2) \le c(n/2)^2$

Show that $T(n) = O(n^2)$

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$$T(n) = 2T(n/2) + n$$
$$\le 2c(n/2)^2 + n \quad \text{from our inductive hypothesis}$$
$$= 2cn^2/4 + n$$
$$= 1/2cn^2 + n$$
$$= cn^2 - (1/2cn^2 - n) \quad \text{residual}$$

if

$$-(1/2cn^2 - n) \le 0$$
$$-1/2cn^2 + n \le 0$$
$$cn \ge 2$$

overkill?

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$$T(n) = 2T(n/2) + n$$

What if we guess wrong, e.g. O(n)?

Assume $T(k) = O(k)$ for all $k < n$

- again, this implies that $T(n/2) \le c(n/2)$

Show that $T(n) = O(n)$

$$T(n) = 2T(n/2) + n$$
$$\le 2cn/2 + n$$
$$= cn + n$$
$$\le cn$$

factor of n so we can just roll it in?

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$$T(n) = 2T(n/2) + n$$

What if we guess wrong, e.g. O(n)?

Assume $T(k) = O(k)$ for all $k < n$

- again, this implies that $T(n/2) \le c(n/2)$

Show that $T(n) = O(n)$

$$T(n) = 2T(n/2) + n$$
$$\le 2cn/2 + n$$
$$= cn + n$$
$$\le cn$$

~~factor of n so we can just roll it in?~~

Must prove the exact form!

$cn+n \le cn$??

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$$T(n) = 2T(n/2) + n$$

Prove $T(n) = O(n \log_2 n)$

Assume $T(k) = O(k \log_2 k)$ for all $k < n$

- again, this implies that $T(k) = ck \log_2 k$

Show that $T(n) = O(n \log_2 n)$

$$\begin{aligned} T(n) &= 2T(n/2) + n \\ &\le 2cn/2 \log(n/2) + n \\ &\le cn(\log_2 n - \log_2 2) + n \\ &\le cn \log_2 n - cn + n \quad \text{residual} \\ &\le cn \log_2 n \end{aligned}$$

if $cn \ge n$, $c > 1$

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Recursion Tree

Guessing the answer can be difficult

$$T(n) = 3T(n/4) + n^2$$

$$T(n) = T(n/3) + 2T(2n/3) + cn$$

The recursion tree approach

- Draw out the cost of the tree at each level of recursion
- Sum up the cost of the levels of the tree
  - Find the cost of each level with respect to the depth
  - Figure out the depth of the tree
  - Figure out (or bound) the number of leaves
- Verify your answer using the substitution method

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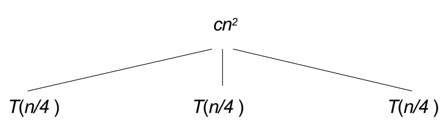
$$T(n) = 3T(n/4) + n^2$$

cost



cn^2

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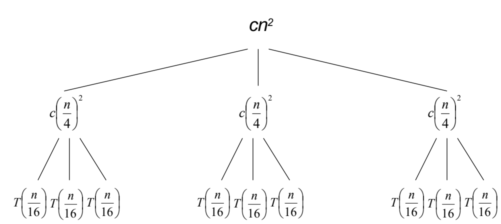
$$T(n) = 3T(n/4) + n^2$$

cost



cn^2

$3/16cn^2$

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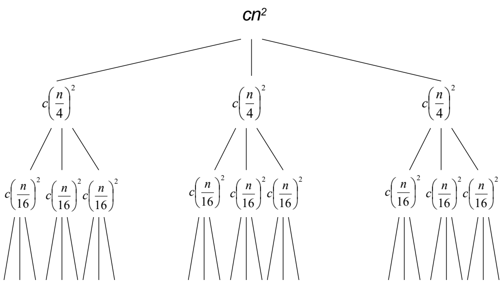
$$T(n) = 3T(n/4) + n^2$$

cost



cn^2

$3/16cn^2$

$(3/16)^2cn^2$

What is the cost at each level? $\left(\frac{3}{16}\right)^d cn^2$

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What is the depth of the tree?

At each level, the size of the data is divided by 4

$$\frac{n}{4^d} = 1$$

$$\log\left(\frac{n}{4^d}\right) = 0$$

$$\log n - \log 4^d = 0$$

$$d \log 4 = \log n$$

$$d = \log_4 n$$

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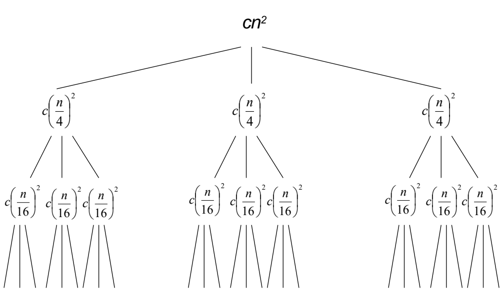
$$T(n) = 3T(n/4) + n^2$$



$$T(1)$$

How many leaves are there?

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How many leaves?

How many leaves are there in a complete ternary tree of depth d?

$$3^d = 3^{\log_4 n}$$

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Total cost

$$T(n) = cn^2 + \frac{3}{16}cn^2 + \left(\frac{3}{16}\right)^2 cn^2 + \ldots + \left(\frac{3}{16}\right)^{d-1} cn^2 + \Theta(3^{\log_4 n})$$

$$= cn^2 \sum_{i=0}^{\log_4 n - 1} \left(\frac{3}{16}\right)^i + \Theta(3^{\log_4 n})$$

$$< cn^2 \sum_{i=0}^{\infty} \left(\frac{3}{16}\right)^i + \Theta(3^{\log_4 n})$$

$$= \frac{1}{1-(3/16)} cn^2 + \Theta(3^{\log_4 n})$$

$$= \frac{16}{13} cn^2 + \Theta(3^{\log_4 n})$$?

$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$

let x = 3/16

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Total cost $T(n) = \frac{16}{13}cn^2 + \Theta(3^{\log_4 n})$

$$3^{\log_4 n} = 4^{\log_4 3^{\log_4 n}}$$

$$= 4^{\log_4 n \log_4 3}$$

$$= 4^{\log_4 n^{\log_4 3}}$$

$$= n^{\log_4 3}$$

Assignment 1!

$$T(n) = \frac{16}{13}cn^2 + \Theta(n^{\log_4 3})$$

$$T(n) = O(n^2)$$

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Verify solution using substitution

$$T(n) = 3T(n/4) + n^2$$

Assume $T(k) = O(k^2)$ for all $k < n$

Show that $T(n) = O(n^2)$

Given that $T(n/4) = O((n/4)^2)$, then

$$O(g(n)) = \left\{ f(n) : \begin{array}{l} \text{there exists positive constants } c \text{ and } n \text{ such that} \\ 0 \le f(n) \le cg(n) \text{ for all } n \ge n_0 \end{array} \right\}$$

$T(n/4) \le c(n/4)^2$

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$$T(n) = 3T(n/4) + n^2$$

To prove that Show that $T(n) = O(n^2)$ we need to identify the appropriate constants:

$$O(g(n)) = \left\{ f(n) : \begin{array}{l} \text{there exists positive constants } c \text{ and } n \text{ such that} \\ 0 \le f(n) \le cg(n) \text{ for all } n \ge n_0 \end{array} \right\}$$

i.e. some constant c such that $T(n) \le cn^2$

$$T(n) = 3T(n/4) + n^2$$

$$\le 3c(n/4)^2 + n^2$$

$$= cn^2 3/16 + n^2$$

$$= cn^2 - cn^2 * \frac{13}{16} + n^2$$ residual

a constant exists if, if $-cn^2 * \frac{13}{16} + n^2 \le 0$

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$$T(n) = 3T(n/4) + n^2$$

To prove that Show that $T(n) = O(n^2)$ we need to identify the appropriate constants:

$$O(g(n)) = \left\{ f(n) : \begin{array}{l} \text{there exists positive constants } c \text{ and } n \text{ such that} \\ 0 \le f(n) \le cg(n) \text{ for all } n \ge n_0 \end{array} \right\}$$

i.e. some constant c such that $T(n) \le cn^2$

$$-cn^2 * \frac{13}{16} + n^2 \le 0$$

$$cn^2 * \frac{13}{16} \ge n^2$$

$$c \ge \frac{16}{13}$$

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Master Method

Provides solutions to the recurrences of the form:

$$T(n) = aT(n/b) + f(n)$$

if $f(n) = O(n^{\log_b a - \varepsilon})$ for $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$

if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$

if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for $\varepsilon > 0$ and $af(n/b) \le cf(n)$ for $c < 1$

then $T(n) = \Theta(f(n))$

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$$T(n) = 16T(n/4) + n$$

if $f(n) = O(n^{\log_b a - \varepsilon})$ for $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$

if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$

if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for $\varepsilon > 0$ and $af(n/b) \le cf(n)$ for $c < 1$

then $T(n) = \Theta(f(n))$

a = 16
b = 4
f(n) = *n*

$$n^{\log_b a} = n^{\log_4 16} = n^2$$

is $n = O(n^{2-\varepsilon})$?
is $n = \Theta(n^2)$?
is $n = \Omega(n^{2+\varepsilon})$?

Case 1: $\Theta(n^2)$

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$$T(n) = T(n/2) + 2^n$$

if $f(n) = O(n^{\log_b a - \varepsilon})$ for $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$

if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$

if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for $\varepsilon > 0$ and $af(n/b) \le cf(n)$ for $c < 1$

then $T(n) = \Theta(f(n))$

a = 1
b = 2
f(n) = 2^n

$$n^{\log_b a} = n^{\log_2 1} = n^0$$

is $2^n = O(n^{0-\varepsilon})$?
is $2^n = \Theta(n^0)$?
is $2^n = \Omega(n^{0+\varepsilon})$?

Case 3?

is $2^{n/2} \le c2^n$ for $c < 1$?

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$$T(n) = T(n/2) + 2^n$$

if $f(n) = O(n^{\log_b a - \varepsilon})$ for $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$
if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$
if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for $\varepsilon > 0$ and $af(n/b) \le cf(n)$ for $c < 1$
then $T(n) = \Theta(f(n))$

is $2^{n/2} \le c2^n$ for $c < 1$?

Let c = 1/2

$2^{n/2} \le (1/2)2^n$
$2^{n/2} \le 2^{-1}2^n$
$2^{n/2} \le 2^{n-1}$

T(n) = Θ(2^n)

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$$T(n) = 2T(n/2) + n$$

if $f(n) = O(n^{\log_b a - \varepsilon})$ for $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$
if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$
if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for $\varepsilon > 0$ and $af(n/b) \le cf(n)$ for $c < 1$
then $T(n) = \Theta(f(n))$

a = 2
b = 2
f(n) = *n*

$n^{\log_b a} = n^{\log_2 2} = n^1$

is $n = O(n^{1-\varepsilon})$?
is $n = \Theta(n^1)$?
is $n = \Omega(n^{1+\varepsilon})$?

Case 2: Θ(n log n)

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$$T(n) = 16T(n/4) + n!$$

if $f(n) = O(n^{\log_b a - \varepsilon})$ for $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$
if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$
if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for $\varepsilon > 0$ and $af(n/b) \le cf(n)$ for $c < 1$
then $T(n) = \Theta(f(n))$

a = 16
b = 4
f(n) = *n!*

$n^{\log_b a} = n^{\log_4 16} = n^2$

is $n! = O(n^{2-\varepsilon})$?
is $n! = \Theta(n^2)$?
is $n! = \Omega(n^{2+\varepsilon})$?

Case 3?

is $16(n/4)! \le cn!$ for $c < 1$?

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$$T(n) = 16T(n/4) + n!$$

if $f(n) = O(n^{\log_b a - \varepsilon})$ for $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$
if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$
if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for $\varepsilon > 0$ and $af(n/b) \le cf(n)$ for $c < 1$
then $T(n) = \Theta(f(n))$

is $16(n/4)! \le cn!$ for $c < 1$?

Let c = 1/2

$cn! = 1/2n!$
$> (n/2)!$

therefore,

$16(n/4)! \le (n/2)! < 1/2n!$

T(n) = Θ($n!$)

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$$T(n) = \sqrt{2}T(n/2) + \log n$$

if $f(n) = O(n^{\log_b a - \varepsilon})$ for $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$
if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$
if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for $\varepsilon > 0$ and $af(n/b) \le cf(n)$ for $c < 1$
then $T(n) = \Theta(f(n))$

a = $\sqrt{2}$
b = 2
f(n) = log*n*

$$n^{\log_b a} = n^{\log_2 \sqrt{2}} = n^{\log_2 2^{1/2}} = \sqrt{n}$$

is $\log n = O(n^{1/2-\varepsilon})$?
is $\log n = \Theta(n^{1/2})$?
is $\log n = \Omega(n^{1/2+\varepsilon})$?

Case 1: $\Theta(\sqrt{n})$

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$$T(n) = 4T(n/2) + n$$

if $f(n) = O(n^{\log_b a - \varepsilon})$ for $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$
if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$
if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for $\varepsilon > 0$ and $af(n/b) \le cf(n)$ for $c < 1$
then $T(n) = \Theta(f(n))$

a = 4
b = 2
f(n) = *n*

$$n^{\log_b a} = n^{\log_2 4} = n^2$$

is $n = O(n^{2-\varepsilon})$?
is $n = \Theta(n^2)$?
is $n = \Omega(n^{2+\varepsilon})$?

Case 1: $\Theta(n^2)$

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Recurrences

$$T(n) = 2T(n/3) + d \qquad T(n) = 7T(n/7) + n$$

if $f(n) = O(n^{\log_b a - \varepsilon})$ for $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$
if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$
if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for $\varepsilon > 0$ and $af(n/b) \le cf(n)$ for $c < 1$
then $T(n) = \Theta(f(n))$

$$T(n) = T(n-1) + \log n \qquad T(n) = 8T(n/2) + n^3$$

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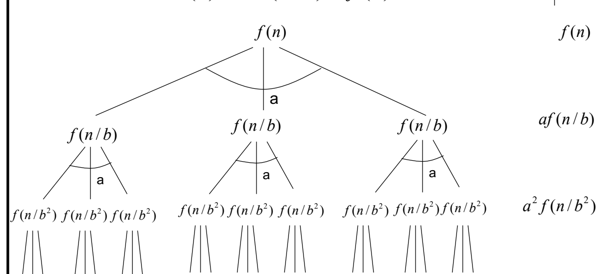
Why does the master method work?

$$T(n) = aT(n/b) + f(n)$$



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What is the depth of the tree?

At each level, the size of the data is divided by b

$$\frac{n}{b^d} = 1$$

$$\log\left(\frac{n}{b^d}\right) = 0$$

$$\log n - \log 4^b = 0$$

$$d \log b = \log n$$

$$d = \log_b n$$

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How many leaves?

How many leaves are there in a complete *a*-ary tree of depth *d*?

$$a^d = a^{\log_b n}$$

$$= n^{\log_b a}$$

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Total cost

if $f(n) = O(n^{\log_b a - \varepsilon})$ for $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$

if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$

if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for $\varepsilon > 0$ and $af(n/b) \le cf(n)$ for $c < 1$ then $T(n) = \Theta(f(n))$

$$T(n) = cf(n) + af(n/b) + a^2 f(n/b^2) + ... + a^{n-1} f(n/b^{n-1}) + \Theta(n^{\log_b a3})$$

$$= \sum_{i=0}^{\log_b n - 1} a^i f(n/b^i) + \Theta(n^{\log_b a})$$

Case 1: cost is dominated by the cost of the leaves

$$= \sum_{i=0}^{\log_b n - 1} a^i f(n/b^i) < \Theta(n^{\log_b a})$$

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Total cost

if $f(n) = O(n^{\log_b a - \varepsilon})$ for $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$

if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$

if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for $\varepsilon > 0$ and $af(n/b) \le cf(n)$ for $c < 1$ then $T(n) = \Theta(f(n))$

$$T(n) = cf(n) + af(n/b) + a^2 f(n/b^2) + ... + a^{n-1} f(n/b^{n-1}) + \Theta(n^{\log_b a3})$$

$$= \sum_{i=0}^{\log_b n - 1} a^i f(n/b^i) + \Theta(n^{\log_b a})$$

Case 2: cost is evenly distributed across tree

As we saw with mergesort, log *n* levels to the tree and at each level *f(n)* work

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Total cost

$$\text{if } f(n) = O(n^{\log_b a-\varepsilon}) \text{ for } \varepsilon > 0, \text{ then } T(n) = \Theta(n^{\log_b a})$$

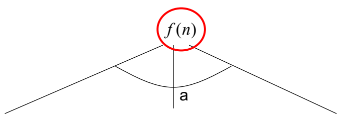
$$\text{if } f(n) = \Theta(n^{\log_b a}), \text{ then } T(n) = \Theta(n^{\log_b a} \log n)$$

$$\text{if } f(n) = \Omega(n^{\log_b a+\varepsilon}) \text{ for } \varepsilon > 0 \text{ and } af(n/b) \le cf(n) \text{ for } c < 1$$
$$\text{then } T(n) = \Theta(f(n))$$

$$T(n) = cf(n) + af(n/b) + a^2 f(n/b^2) + \ldots + a^{d-1} f(n/b^{d-1}) + \Theta(n^{\log_b a3})$$

$$= \sum_{i=0}^{\log_b n-1} a^i f(n/b^i) + \Theta(n^{\log_b a})$$

Case 3: cost is dominated by the cost of the root



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Other forms of the master method

$$T(n) = aT(n/b) + O(n^d)$$

$$T(n) = \begin{cases} O(n^d) & \text{if } d > \log_b a \\ O(n^d \log n) & \text{if } d = \log_b a \\ O(n^{\log_b a}) & \text{if } d < \log_b a \end{cases}$$

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