SINGLE SOURCE SHORTEST PATHS

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CS 140 – Fall 2022

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Admin

Assignment 8

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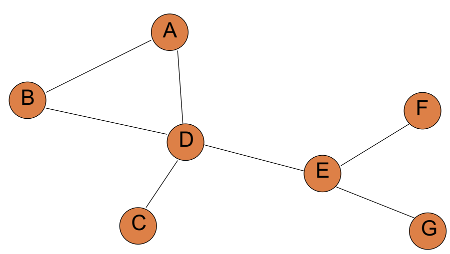
Graphs

A graph is a set of vertices V and a set of edges (u,v) $\in$ E where u,v $\in$ V



3

Connectedness

Given an undirected graph, for every node u $\in$ V, can we reach all other nodes in the graph? Algorithm + running time

Run BFS or DFS-Visit (one pass) and mark nodes as we visit them. If we visit all nodes, return true, otherwise false.

Running time: O(|V| + |E|)

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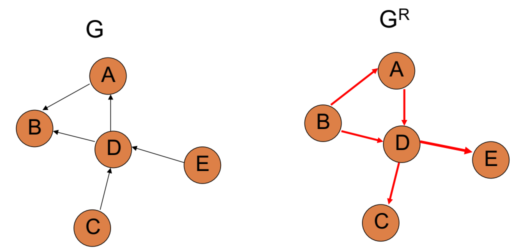
Strongly connected

Given a directed graph, can we reach any node v from any other node u?

Can we do the same thing?

Transpose of a graph

Given a graph G, we can calculate the transpose of a graph G^R by reversing the direction of all the edges



Running time to calculate G^R? $\theta(|V| + |E|)$

Strongly connected

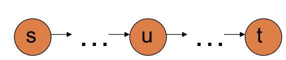
Strongly-Connected(G)
- Run DFS-Visit or BFS from some node u
- If not all nodes are visited: return false
- Create graph G^R
- Run DFS-Visit or BFS on G^R from node u
- If not all nodes are visited: return false
- return true

Is it correct?

What do we know after the first pass?
- Starting at u, we can reach every node

What do we know after the second pass?
- All nodes can reach u. Why?
- We can get from u to every node in G^R, therefore, if we reverse the edges (i.e. G), then we have a path from every node to u

Which means that any node can reach any other node. Given any two nodes s and t we can create a path through u



Runtime?

Strongly-Connected(G)

- Run DFS-Visit or BFS from some node u — O(|V| + |E|)
- If not all nodes are visited: return false — O(|V|)
- Create graph G^R — θ(|V| + |E|)
- Run DFS-Visit or BFS on G^R from node u — O(|V| + |E|)
- If not all nodes are visited: return false — O(|V|)
- return true

O(|V| + |E|)

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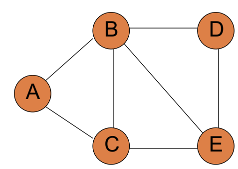
Shortest paths

What is the shortest path from a to d?



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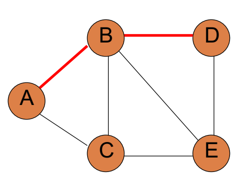
Shortest paths

BFS



11

Shortest paths

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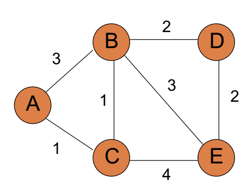

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Shortest paths

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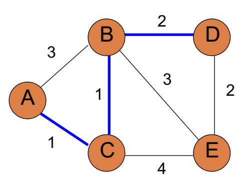

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Dijkstra's algorithm

What is dist?

What is prev?

How does it work?

What is the run-time?

How do we get the shortest path?

```
DIJKSTRA(G, s)
for all v ∈ V
    dist[v] ← ∞
    prev[v] ← null
dist[s] ← 0
Q ← MAKEHEAP(V)
while !EMPTY(Q)
    u ← EXTRACTMIN(Q)
    for all edges (u, v) ∈ E
        if dist[v] > dist[u] + w(u, v)
            dist[v] ← dist[u] + w(u, v)
            DECREASEKEY(Q, v, dist[v])
            prev[v] ← u
```

14

Dijkstra's algorithm

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```
BFS(G, s)
for each v ∈ V
    dist[v] = ∞
dist[s] = 0
ENQUEUE(Q, s)
while !EMPTY(Q)
    u ← DEQUEUE(Q)
    VISIT(u)
    for each edge (u, v) ∈ E
        if dist[v] = ∞
            ENQUEUE(Q, v)
            dist[v] ← dist[u] + 1
```

15

Dijkstra's algorithm

prev keeps track of
the shortest path

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16

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while !Empty(Q)
        u ← ExtractMin(Q)
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
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for each v ∈ V
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Enqueue(Q, s)
while !Empty(Q)
        u ← Dequeue(Q)
        Visit(u)
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17

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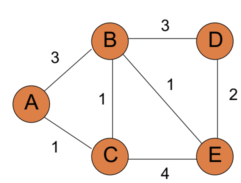

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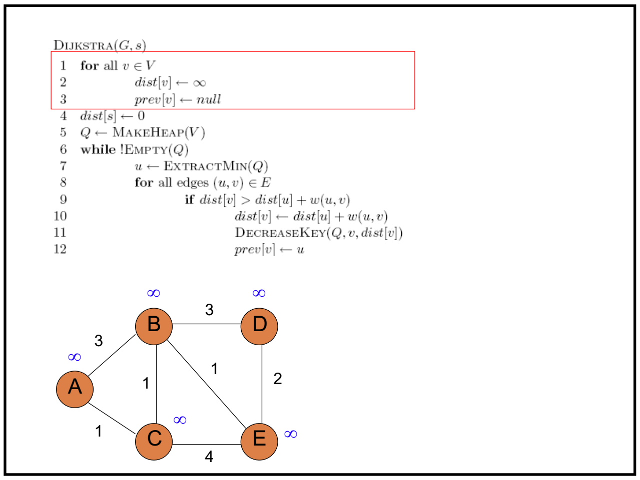

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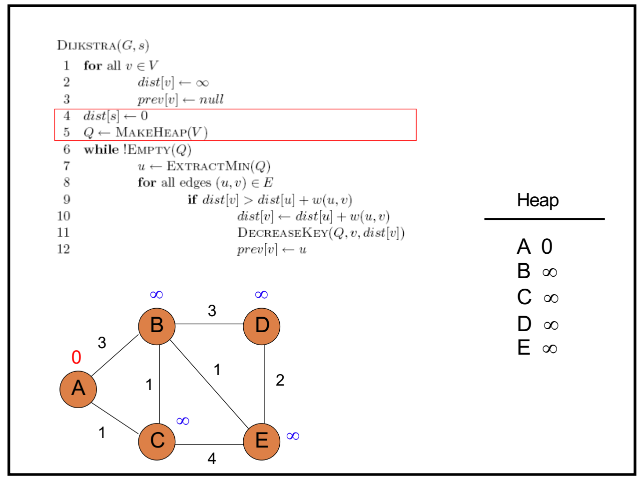
A
B
C
D
E

21



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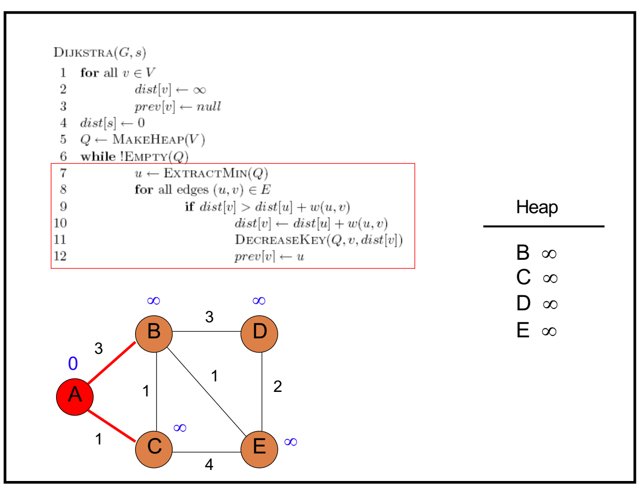
Heap
A 0
B ∞
C ∞
D ∞
E ∞

22



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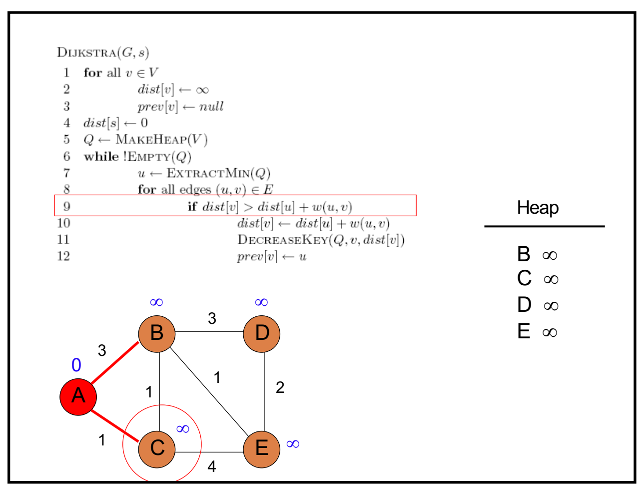
B ∞
C ∞
D ∞
E ∞

23



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B ∞
C ∞
D ∞
E ∞

24

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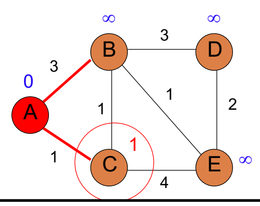

Heap

C 1
B ∞
D ∞
E ∞

25

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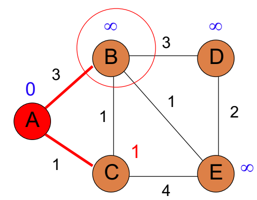

Heap

C 1
B ∞
D ∞
E ∞

26

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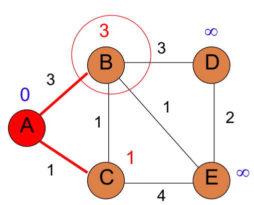

Heap

C 1
B 3
D ∞
E ∞

27

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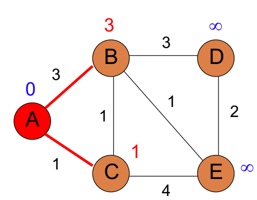
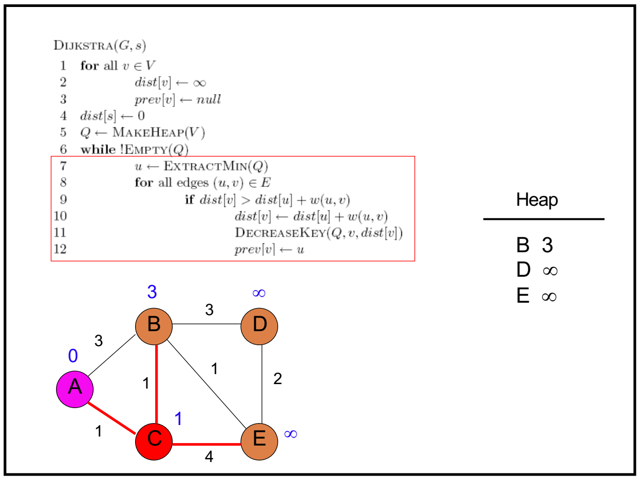


Heap

C 1
B 3
D ∞
E ∞

28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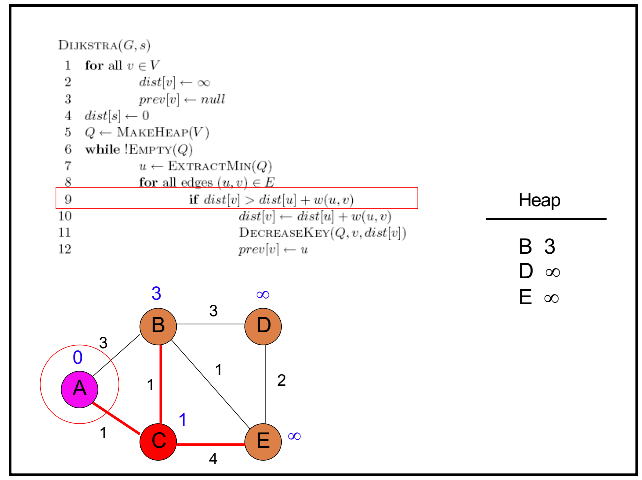
Heap
B 3
D ∞
E ∞
A B C D E
0 3 1 ∞ ∞
3 3 1 1 2 1 4

29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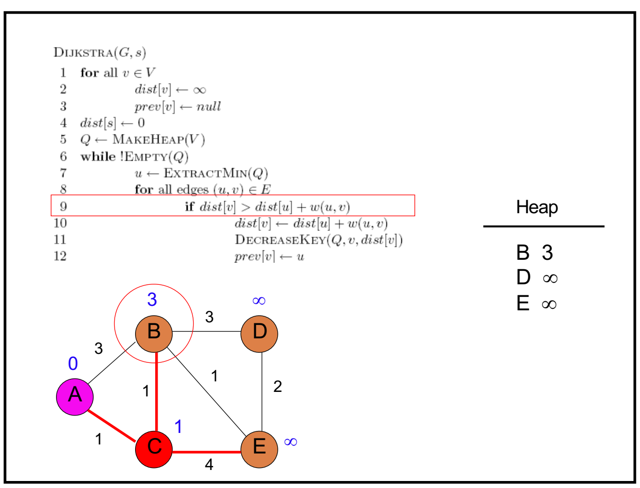
B 3
D ∞
E ∞
A B C D E
0 3 1 ∞ ∞
3 3 1 1 2 1 4

30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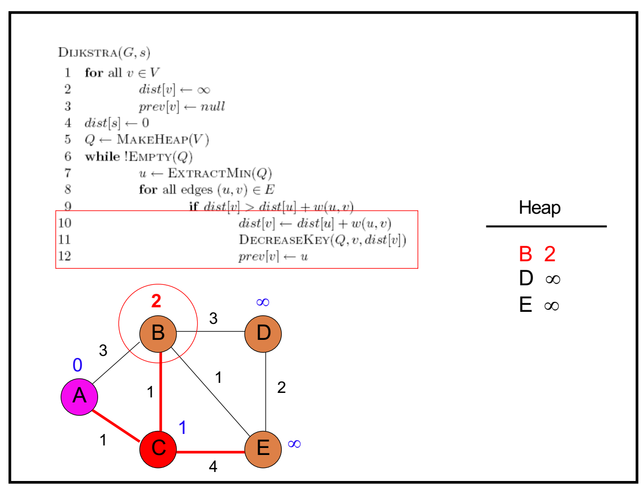
Heap
B 3
D ∞
E ∞
A B C D E
0 3 1 ∞ ∞
3 3 1 1 2 1 4

31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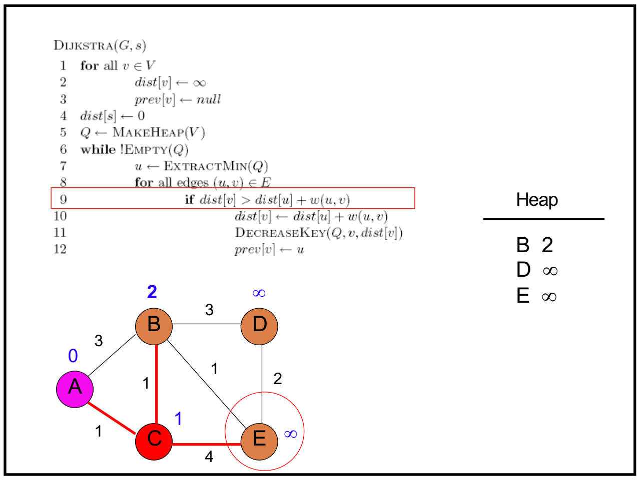
Heap
B 2
D ∞
E ∞
A B C D E
0 2 1 ∞ ∞
3 3 1 1 2 1 4

32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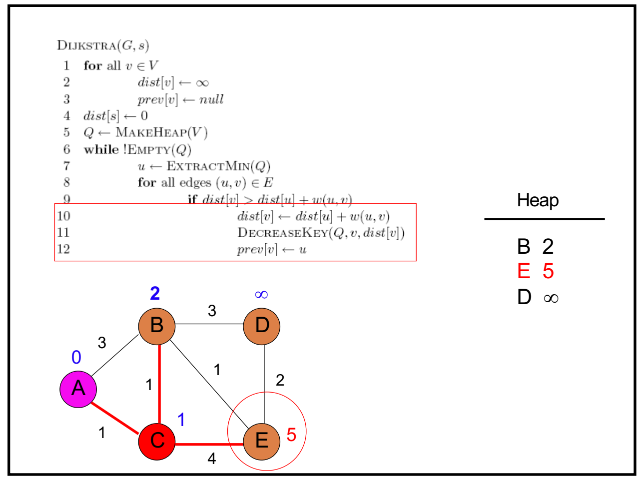
Heap
B 2
D ∞
E ∞
A B C D E
0 2 1 ∞ ∞
3 1 1 3 1 2 4

33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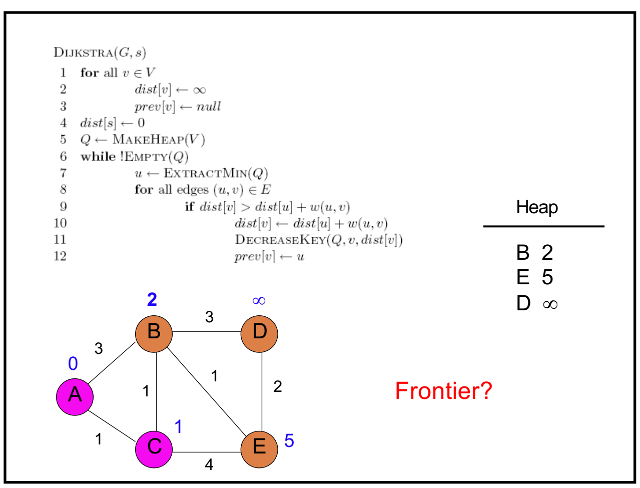
Heap
B 2
E 5
D ∞
A B C D E
0 2 1 ∞ 5
3 1 1 3 1 2 4

34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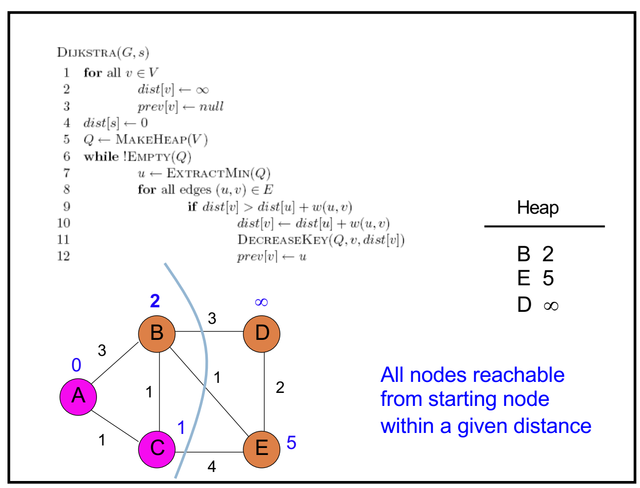
Heap
B 2
E 5
D ∞
Frontier?
A B C D E
0 2 1 ∞ 5
3 1 1 3 1 2 4

35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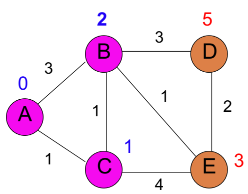
Heap
B 2
E 5
D ∞
All nodes reachable
from starting node
within a given distance
A B C D E
0 2 1 ∞ 5
3 1 1 3 1 2 4

36

```
DIJKSTRA(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
Q ← MAKEHEAP(V)
while !EMPTY(Q)
        u ← EXTRACTMIN(Q)
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        DECREASEKEY(Q, v, dist[v])
                        prev[v] ← u
```

Heap

E 3
D 5

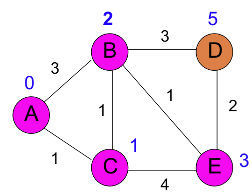


37

```
DIJKSTRA(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
Q ← MAKEHEAP(V)
while !EMPTY(Q)
        u ← EXTRACTMIN(Q)
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        DECREASEKEY(Q, v, dist[v])
                        prev[v] ← u
```

Heap

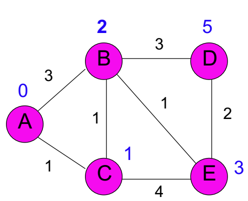
D 5



38

```
DIJKSTRA(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
Q ← MAKEHEAP(V)
while !EMPTY(Q)
        u ← EXTRACTMIN(Q)
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        DECREASEKEY(Q, v, dist[v])
                        prev[v] ← u
```

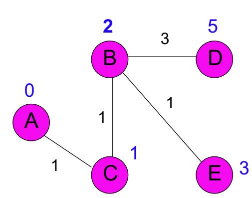
Heap



39

```
DIJKSTRA(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
Q ← MAKEHEAP(V)
while !EMPTY(Q)
        u ← EXTRACTMIN(Q)
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        DECREASEKEY(Q, v, dist[v])
                        prev[v] ← u
```

Heap



40

Is Dijkstra's algorithm correct?

Invariant: For every vertex removed from the heap, dist[v] is the actual shortest distance from s to v

```
Dijkstra(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
Q ← MakeHeap(V)
while !Empty(Q)
        u ← ExtractMin(Q)
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        DecreaseKey(Q, v, dist[v])
                        prev[v] ← u
```

proof?

41

Is Dijkstra's algorithm correct?

Invariant: For every vertex removed from the heap, dist[v] is the actual shortest distance from s to v

- The only time a vertex gets visited is when the distance from s to that vertex is smaller than the distance to any remaining vertex
- Therefore, there cannot be any other path that hasn't been visited already that would result in a shorter path

42

Running time?

```
Dijkstra(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
Q ← MakeHeap(V)
while !Empty(Q)
        u ← ExtractMin(Q)
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        DecreaseKey(Q, v, dist[v])
                        prev[v] ← u
```

43

Running time?

```
Dijkstra(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
Q ← MakeHeap(V)                          1 call to MakeHeap
while !Empty(Q)
        u ← ExtractMin(Q)
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        DecreaseKey(Q, v, dist[v])
                        prev[v] ← u
```

44

Running time?

```
Dijkstra(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
Q ← MakeHeap(V)
while !Empty(Q)                          |V| iterations
        u ← ExtractMin(Q)
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        DecreaseKey(Q, v, dist[v])
                        prev[v] ← u
```

45

Running time?

```
Dijkstra(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
Q ← MakeHeap(V)
while !Empty(Q)
        u ← ExtractMin(Q)                |V| calls
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        DecreaseKey(Q, v, dist[v])
                        prev[v] ← u
```

46

Running time?

```
Dijkstra(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
Q ← MakeHeap(V)
while !Empty(Q)
        u ← ExtractMin(Q)
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        DecreaseKey(Q, v, dist[v])   O(|E|) calls
                        prev[v] ← u
```

47

Running time?

Depends on the heap implementation

	1 MakeHeap	|V| ExtractMin	|E| DecreaseKey	Total
Array	O(|V|)	O(|V|^2)	O(|E|)	O(|V|^2)
Bin heap	O(|V|)	O(|V| log |V|)	O(|E| log |V|)	O((|V|+|E|) log |V|) O(|E| log |V|)

48

Running time?

Depends on the heap implementation

	1 MakeHeap	|V| ExtractMin	|E| DecreaseKey	Total
Array	O(|V|)	O(|V|^2)	O(|E|)	O(|V|^2)
Bin heap	O(|V|)	O(|V| log |V|)	O(|E| log |V|)	O((|V|+|E|) log |V|) O(|E| log |V|)

Is this an improvement? If |E| < |V|^2 / log |V|

49

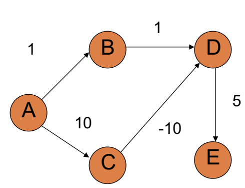
Running time?

Depends on the heap implementation

	1 MakeHeap	|V| ExtractMin	|E| DecreaseKey	Total
Array	O(|V|)	O(|V|^2)	O(|E|)	O(|V|^2)
Bin heap	O(|V|)	O(|V| log |V|)	O(|E| log |V|)	O((|V|+|E|) log |V|) O(|E| log |V|)
Fib heap	O(|V|)	O(|V| log |V|)	O(|E|)	O(|V| log |V| + |E|)

50

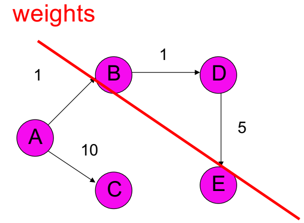
What about Dijkstra's on…?



51

What about Dijkstra's on…?

Dijkstra's algorithm only works for positive edge weights



52

Is Dijkstra's algorithm correct?

Invariant: For every vertex removed from the heap, dist[v] is the actual shortest distance from s to v

- The only time a vertex gets visited is when the distance from s to that vertex is smaller than the distance to any remaining vertex
- Therefore, there cannot be any other path that hasn't been visited already that would result in a shorter path

We relied on having positive edge weights for correctness!

53

Bounding the distance

Another invariant: For each vertex v, dist[v] is an upper bound on the actual shortest distance

```
DIJKSTRA(G, s)
for all v ∈ V
    dist[v] ← ∞
    prev[v] ← null
dist[s] ← 0
Q ← MAKEHEAP(V)
while !EMPTY(Q)
    u ← EXTRACTMIN(Q)
    for all edges (u, v) ∈ E
        if dist[v] > dist[u] + w(u, v)
            dist[v] ← dist[u] + w(u, v)
            DECREASEKEY(Q, v, dist[v])
            prev[v] ← u
```

Is this a valid invariant?

54

Bounding the distance

Another invariant: For each vertex v, dist[v] is an upper bound on the actual shortest distance

- start off at ∞
- only update the value if we find a shorter distance

An update procedure

$$dist[v] = \min\{dist[v], dist[u] + w(u,v)\}$$

55

$dist[v] = \min\{dist[v], dist[u] + w(u,v)\}$

Can we ever go wrong applying this update rule?

- We can apply this rule as many times as we want and will never underestimate dist[v]

When will dist[v] be right?

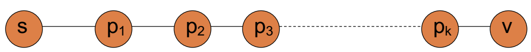
- If u is along the shortest path to v and dist[u] is correct

56

$$dist[v] = \min\{dist[v], dist[u] + w(u,v)\}$$

dist[v] will be right if u is along the shortest path to v and dist[u] is correct

Consider the shortest path from s to v

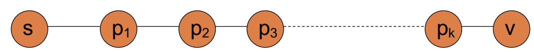


57

$$dist[v] = \min\{dist[v], dist[u] + w(u,v)\}$$

dist[v] will be right if u is along the shortest path to v and dist[u] is correct

What happens if we update all of the vertices with the above update?

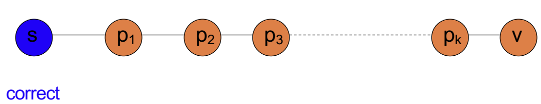


58

$$dist[v] = \min\{dist[v], dist[u] + w(u,v)\}$$

dist[v] will be right if u is along the shortest path to v and dist[u] is correct

What happens if we update all of the vertices with the above update?

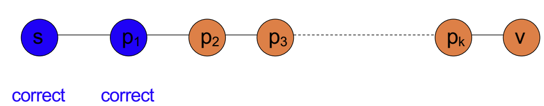


59

$$dist[v] = \min\{dist[v], dist[u] + w(u,v)\}$$

dist[v] will be right if u is along the shortest path to v and dist[u] is correct

What happens if we update all of the vertices with the above update?

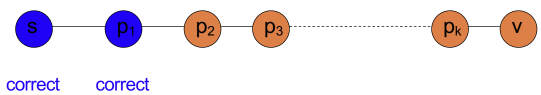


60

$$dist[v] = \min\{dist[v], dist[u] + w(u,v)\}$$

dist[v] will be right if u is along the shortest path to v and dist[u] is correct

Does the order that we update the vertices matter?



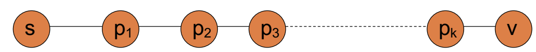
61

$$dist[v] = \min\{dist[v], dist[u] + w(u,v)\}$$

dist[v] will be right if u is along the shortest path to v and dist[u] is correct

How many times do we have to do this for vertex p_i to have the correct shortest path from s?

- i times



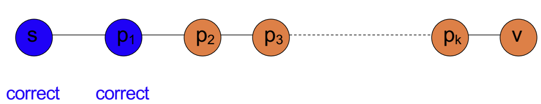
62

$$dist[v] = \min\{dist[v], dist[u] + w(u,v)\}$$

dist[v] will be right if u is along the shortest path to v and dist[u] is correct

How many times do we have to do this for vertex p_i to have the correct shortest path from s?

- i times



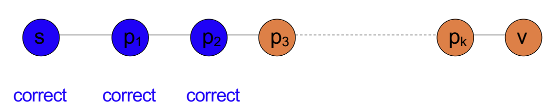
63

$$dist[v] = \min\{dist[v], dist[u] + w(u,v)\}$$

dist[v] will be right if u is along the shortest path to v and dist[u] is correct

How many times do we have to do this for vertex p_i to have the correct shortest path from s?

- i times



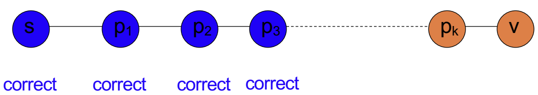
64

$$dist[v] = \min\{dist[v], dist[u] + w(u,v)\}$$

dist[v] will be right if u is along the shortest path to v and dist[u] is correct

How many times do we have to do this for vertex p_i to have the correct shortest path from s?

- i times



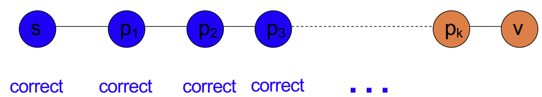
65

$$dist[v] = \min\{dist[v], dist[u] + w(u,v)\}$$

dist[v] will be right if u is along the shortest path to v and dist[u] is correct

How many times do we have to do this for vertex p_i to have the correct shortest path from s?

- i times



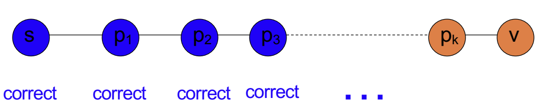
66

$$dist[v] = \min\{dist[v], dist[u] + w(u,v)\}$$

dist[v] will be right if u is along the shortest path to v and dist[u] is correct

What is the longest (vertex-wise) the path from s to any node v can be?

- |V| - 1 edges/vertices



67

Bellman-Ford algorithm

```
BELLMAN-FORD(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
for i ← 1 to |V| − 1
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        prev[v] ← u
for all edges (u, v) ∈ E
        if dist[v] > dist[u] + w(u, v)
                return false
```

68

Bellman-Ford algorithm

```
BELLMAN-FORD(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
for i ← 1 to |V| − 1
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        prev[v] ← u
for all edges (u, v) ∈ E
        if dist[v] > dist[u] + w(u, v)
                return false
```

Initialize all the distances

do it |V| -1 times

iterate over all edges/vertices and apply update rule

69

Bellman-Ford algorithm

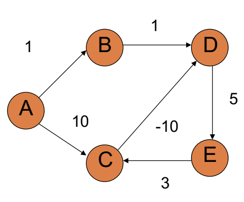
```
BELLMAN-FORD(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
for i ← 1 to |V| − 1
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        prev[v] ← u
for all edges (u, v) ∈ E
        if dist[v] > dist[u] + w(u, v)
                return false
```

check for negative cycles

70

Negative cycles

What is the shortest path from a to e?



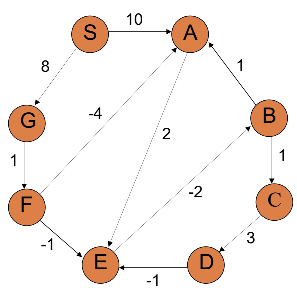
71

Bellman-Ford algorithm

```
BELLMAN-FORD(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
for i ← 1 to |V| − 1
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        prev[v] ← u
for all edges (u, v) ∈ E
        if dist[v] > dist[u] + w(u, v)
                return false
```

72

Bellman-Ford algorithm



How many edges is the shortest path from s to:

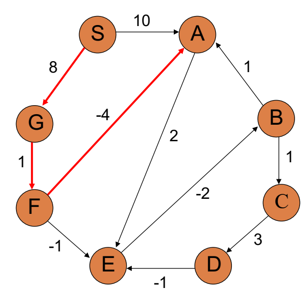
A:

B:

D:

73

Bellman-Ford algorithm

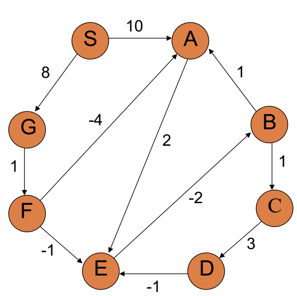


How many edges is the shortest path from s to:

A: 3

74

Bellman-Ford algorithm



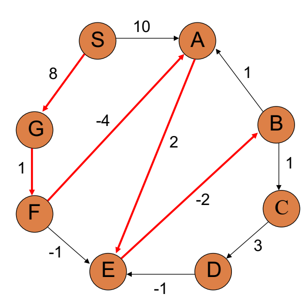
How many edges is the shortest path from s to:

A: 3

B:

75

Bellman-Ford algorithm



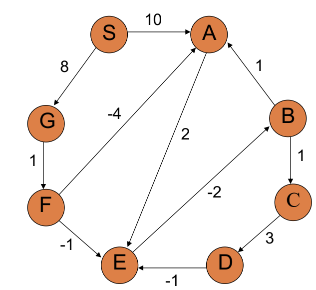
How many edges is the shortest path from s to:

A: 3

B: 5

76

Bellman-Ford algorithm



How many edges is the shortest path from s to:

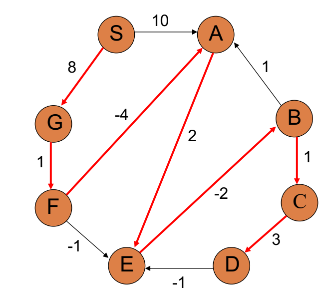
A: 3

B: 5

D:

77

Bellman-Ford algorithm



How many edges is the shortest path from s to:

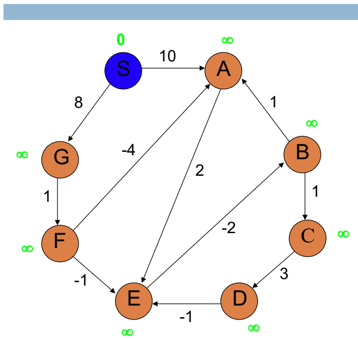
A: 3

B: 5

D: 7

78

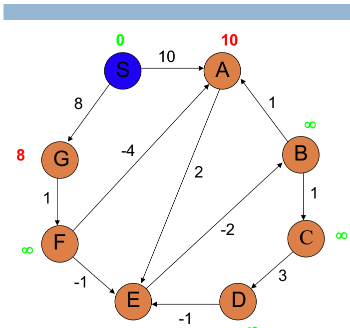
Bellman-Ford algorithm



Iteration: 0

79

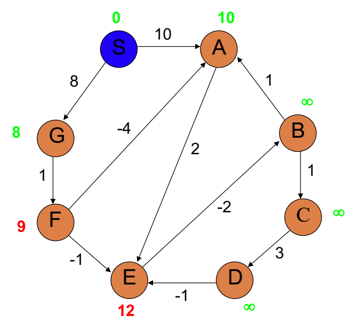
Bellman-Ford algorithm



Iteration: 1

80

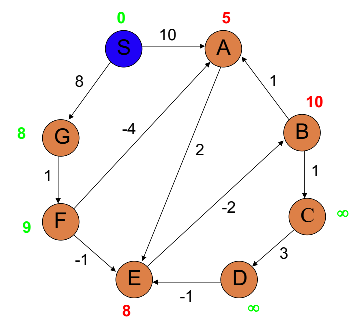
Bellman-Ford algorithm



Iteration: 2

81

Bellman-Ford algorithm

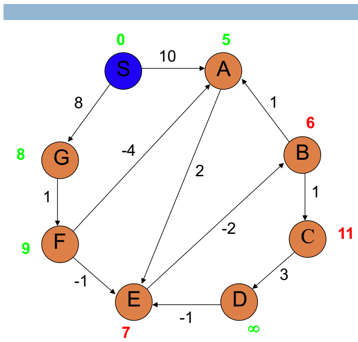


Iteration: 3

A has the correct distance and path

82

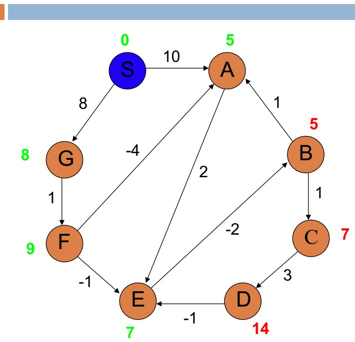
Bellman-Ford algorithm



Iteration: 4

83

Bellman-Ford algorithm

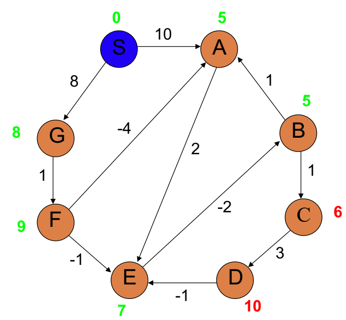


Iteration: 5

B has the correct distance and path

84

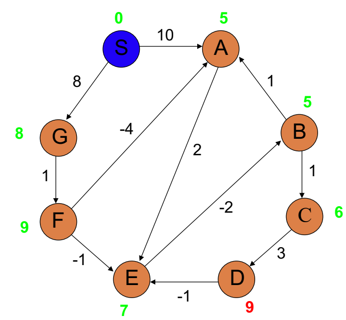
Bellman-Ford algorithm



Iteration: 6

85

Bellman-Ford algorithm



Iteration: 7

D (and all other nodes) have the correct distance and path

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Correctness of Bellman-Ford

Loop invariant: After iteration i, all vertices with shortest paths from s of length i edges or less have correct distances

```
BELLMAN-FORD(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
for i ← 1 to |V| − 1
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        prev[v] ← u
for all edges (u, v) ∈ E
        if dist[v] > dist[u] + w(u, v)
                return false
```

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Runtime of Bellman-Ford

```
BELLMAN-FORD(G, s)
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for i ← 1 to |V| − 1
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                        dist[v] ← dist[u] + w(u, v)
                        prev[v] ← u
for all edges (u, v) ∈ E
        if dist[v] > dist[u] + w(u, v)
                return false
```

$O(|V|\,|E|)$

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Runtime of Bellman-Ford

```
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                        prev[v] ← u
for all edges (u, v) ∈ E
        if dist[v] > dist[u] + w(u, v)
                return false
```

Can you modify the algorithm to run faster (in some circumstances)?

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Single source shortest paths

All of the shortest path algorithms we've looked at today are call "single source shortest paths" algorithms

Why?

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