

Graph Representations

<https://cs.pomona.edu/classes/cs140/>

Outline

Topics and Learning Objectives

- Graphs with weights
- Directed graphs
- Edge lists
- Adjacency matrices
- Adjacency lists

Exercise

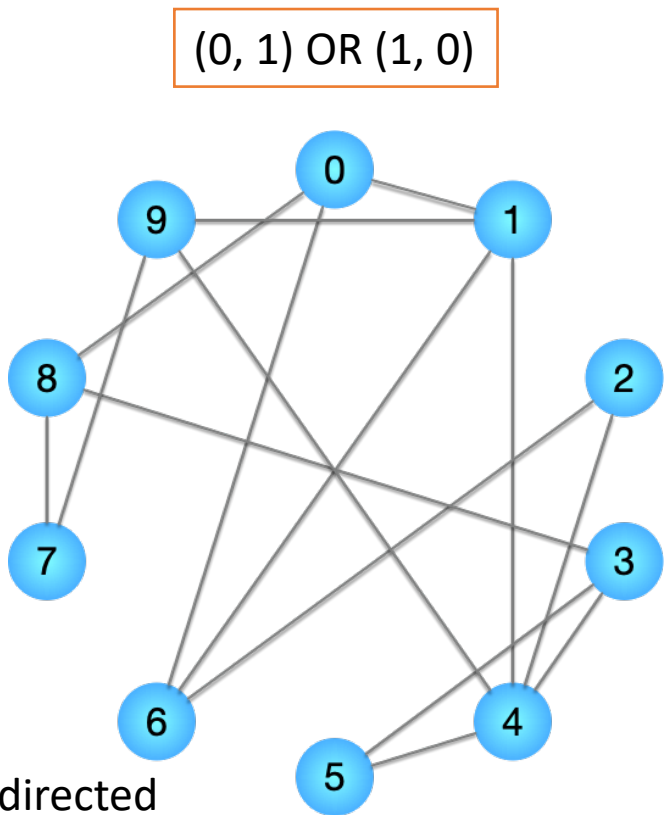
- Friend Circles

Comparisons

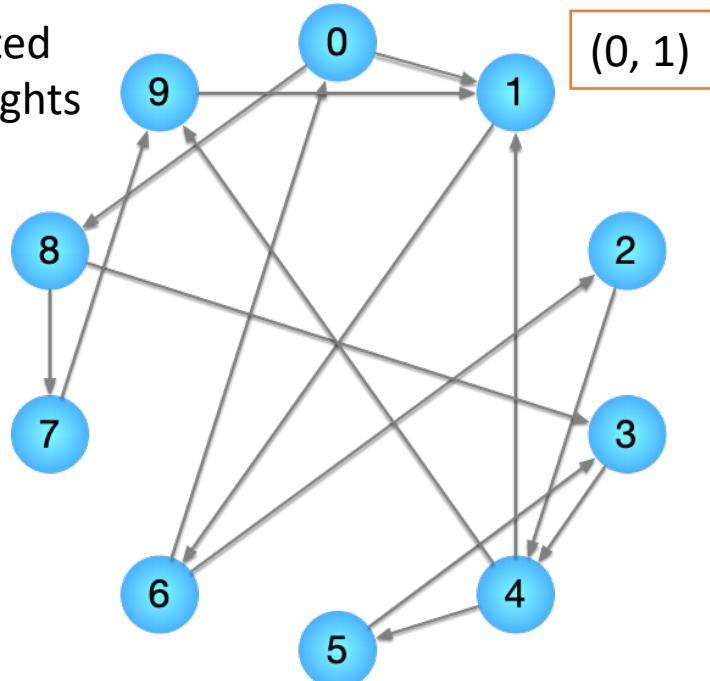
For each representation, we are going to ask the following questions:

1. How do we count the **number of vertices**, and how long does it take?
2. How do we count the **number of edges**, and how long does it take?
3. How do we **add vertices**, and how long does it take?
4. How do we **add edges**, and how long does it take?
5. How do you check for the **existence of an edge**, and how long does it take?
6. How do you find all **neighbors** of a vertex, and how long does it take?
7. How much **memory** is needed to store the graph?

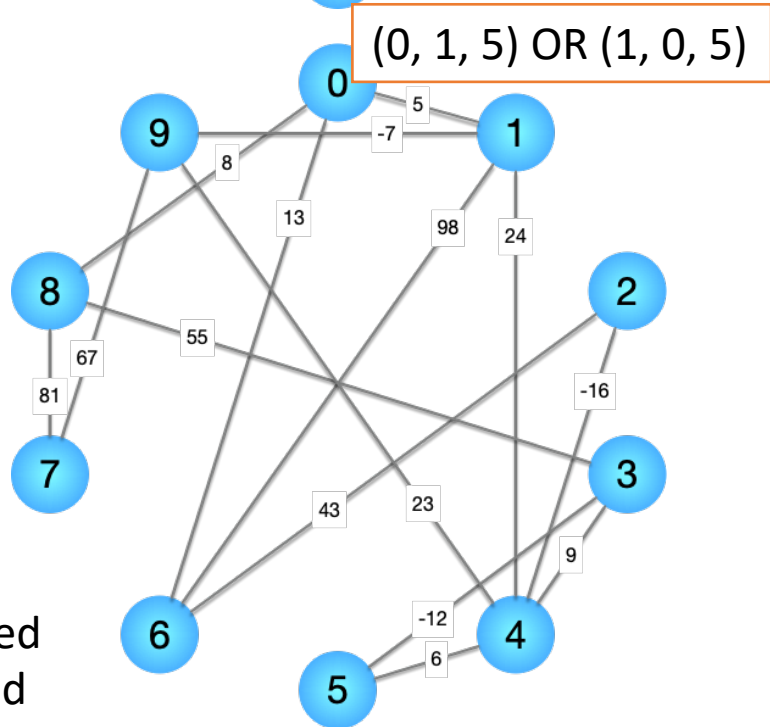
Undirected
No Weights



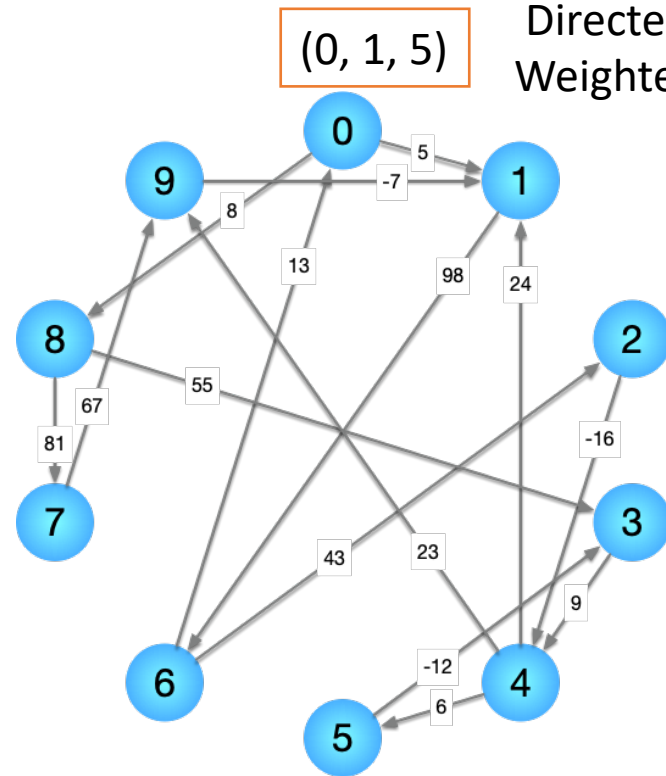
Directed
No Weights



Undirected
Weighted



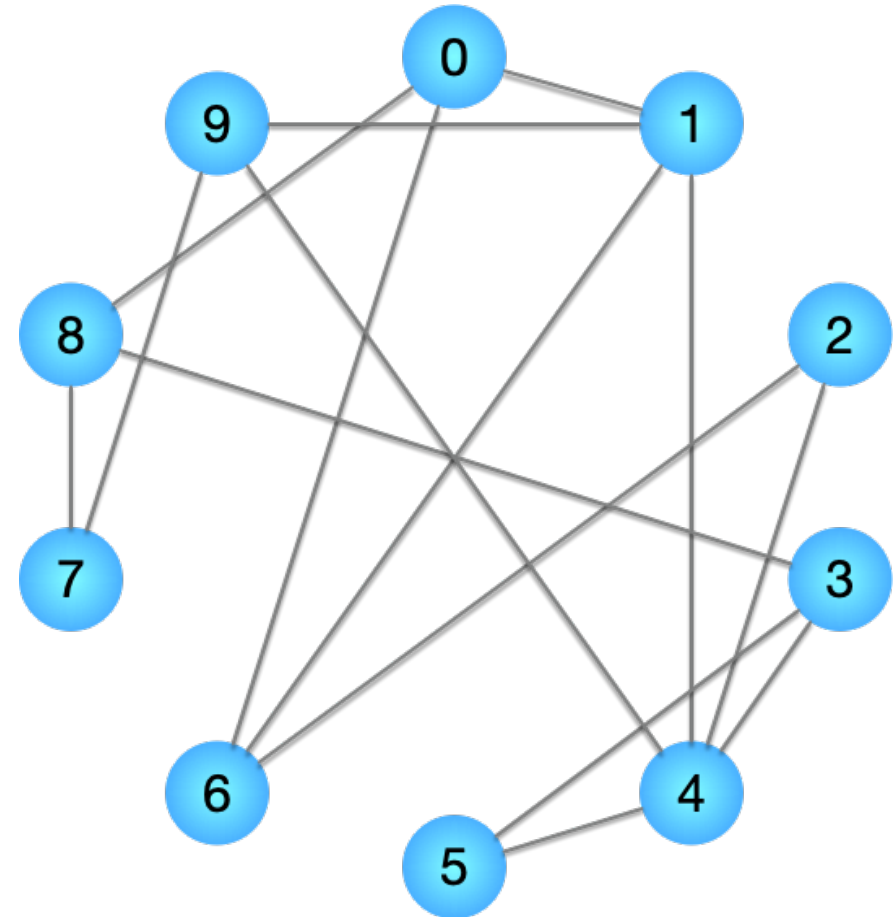
Directed
Weighted



Weights? Directedness?

Edge List Representation

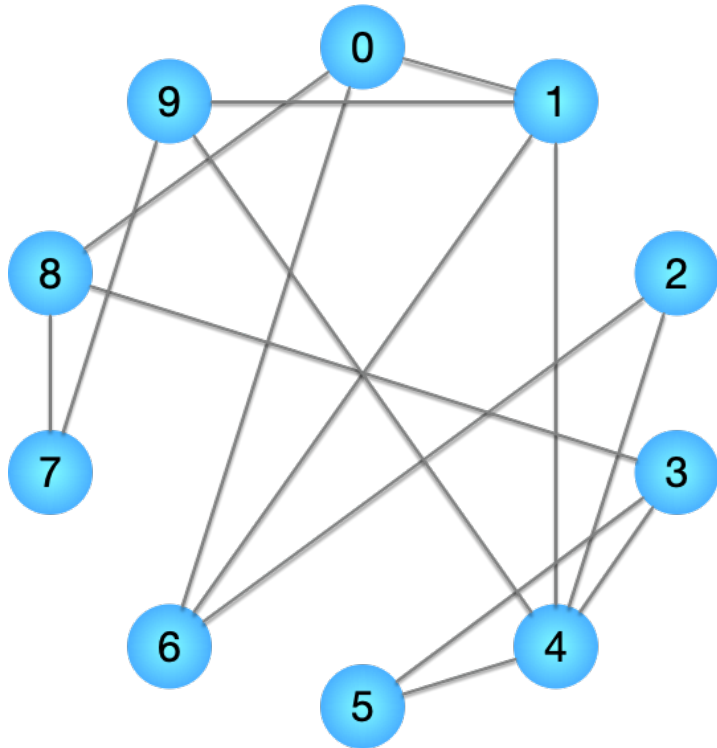
```
edge_list = [  
    (0, 1), (0, 6), (0, 8),  
    (1, 4), (1, 6), (1, 9),  
    (2, 4), (2, 6),  
    (3, 4), (3, 5), (3, 8),  
    (4, 5), (4, 9),  
    (7, 8), (7, 9),  
]
```



```

edge_list = [
    (0, 1), (0, 6), (0, 8),
    (1, 4), (1, 6), (1, 9),
    (2, 4), (2, 6),
    (3, 4), (3, 5), (3, 8),
    (4, 5), (4, 9),
    (7, 8), (7, 9),
]

```



1. How do we count the number of vertices, and how long does it take?

```

def count_vertices_el(el):
    '''Return the number of vertices in an edge list.

```

Directedness does not matter.

This procedure loops over all m vertices. So:

$T(n, m) = O(m)$

```

    '''

```

```

V = set()

```

```

for e in el:

```

```

    v1, v2 = e[0], e[1]

```

*# We have to add both v1 and v2 since v2
might never show up as the first vertex in
an edge.*

```

V.add(v1)

```

```

V.add(v2)

```

```

return len(V)

```

```
edge_list = [  
    (0, 1), (0, 6), (0, 8),  
    (1, 4), (1, 6), (1, 9),  
    (2, 4), (2, 6),  
    (3, 4), (3, 5), (3, 8),  
    (4, 5), (4, 9),  
    (7, 8), (7, 9),  
]
```

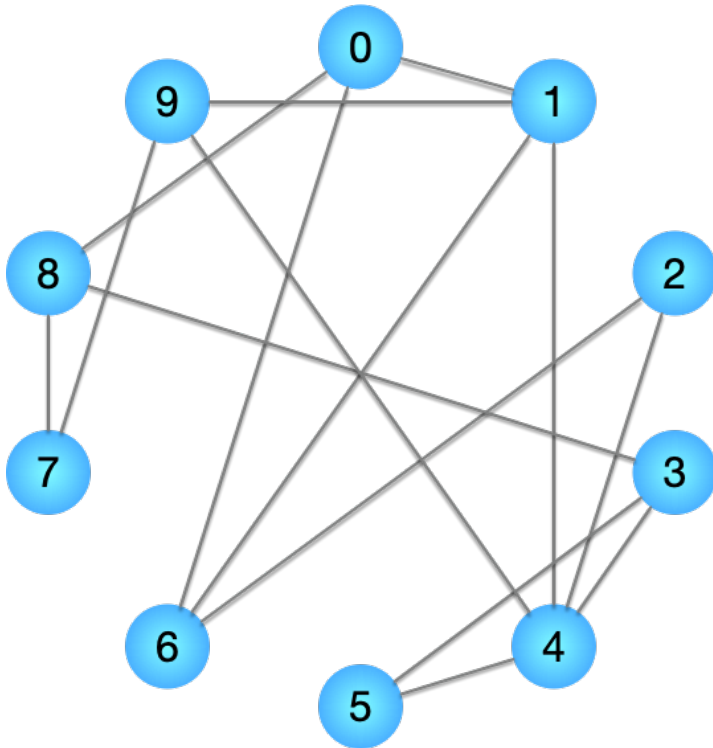
2. How do we count the number of edges, and how long does it take?

```
def count_edges_el(el):  
    '''Return the number of edges in an edge list.
```

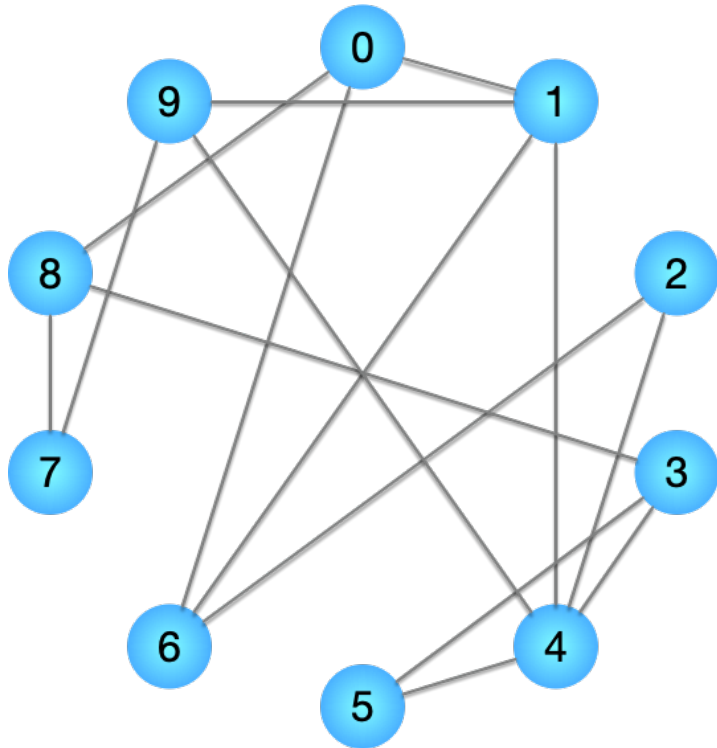
This function assumes that the edge list does not contain any self-connections (an edge connecting a vertex to itself).

```
T(n, m) = O(1)  
'''
```

```
return len(el)
```



```
edge_list = [
    (0, 1), (0, 6), (0, 8),
    (1, 4), (1, 6), (1, 9),
    (2, 4), (2, 6),
    (3, 4), (3, 5), (3, 8),
    (4, 5), (4, 9),
    (7, 8), (7, 9),
]
```



3. How do we add vertices, and how long does it take?

```
def add_vertex_el(el, v, is_weighted=False):
    '''Add a new vertex to an edge list.
```

This function assumes that v is not already in the edge list.

```
T(n, m) = O(1)
'''
```

```
new_edge = (v, v, 0) if is_weighted else (v, v)
return add_edge_el(new_edge)
```

```
edge_list = [  
    (0, 1), (0, 6), (0, 8),  
    (1, 4), (1, 6), (1, 9),  
    (2, 4), (2, 6),  
    (3, 4), (3, 5), (3, 8),  
    (4, 5), (4, 9),  
    (7, 8), (7, 9),  
]
```

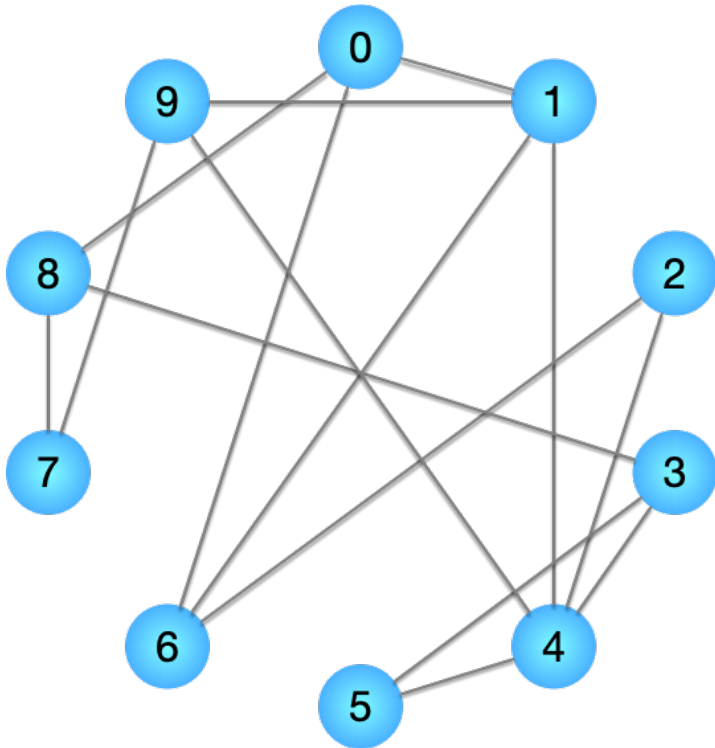
4. How do we add edges, and how long does it take?

```
def add_edge_el(el, e):  
    '''Add a new edge to an edge list.
```

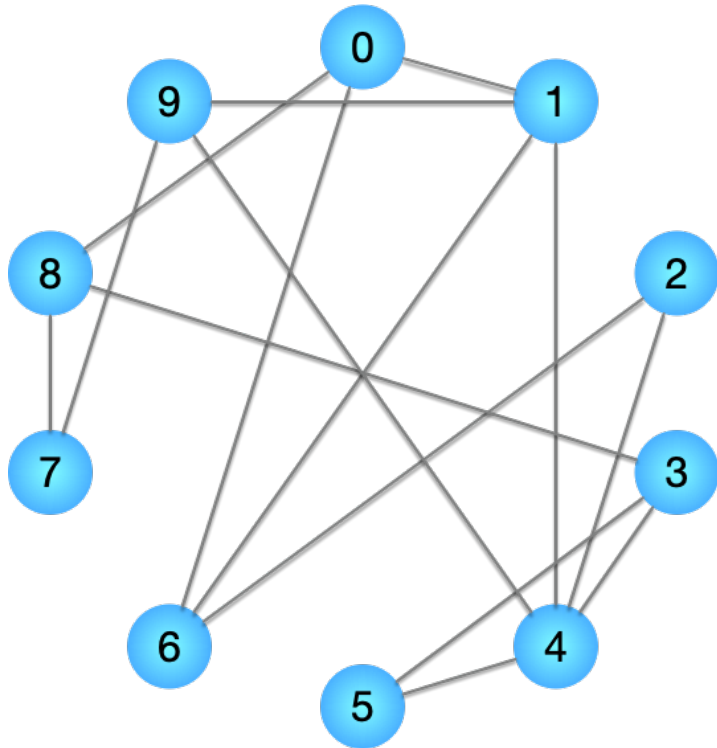
This function assumes that e is not already in the edge list.

```
T(n, m) = O(1)  
'''
```

```
el.append(e)  
return el
```



```
edge_list = [
    (0, 1), (0, 6), (0, 8),
    (1, 4), (1, 6), (1, 9),
    (2, 4), (2, 6),
    (3, 4), (3, 5), (3, 8),
    (4, 5), (4, 9),
    (7, 8), (7, 9),
]
```



5. How do you check for the existence of an edges, and how long does it take?

```
def find_edge_el(el, e, is_ordered=False):
    '''Check for existence of edge in edge list.
```

If is_ordered is True, then all edges are in sorted order.

This function does not assume directed or undirected edges, so the order (v1, v2) vs (v2, v1) DOES matter.

This function will not work well with floating-point weights.

Must search through all edges in order, or using a binary search.

$T(n, m) = O(m)$, or

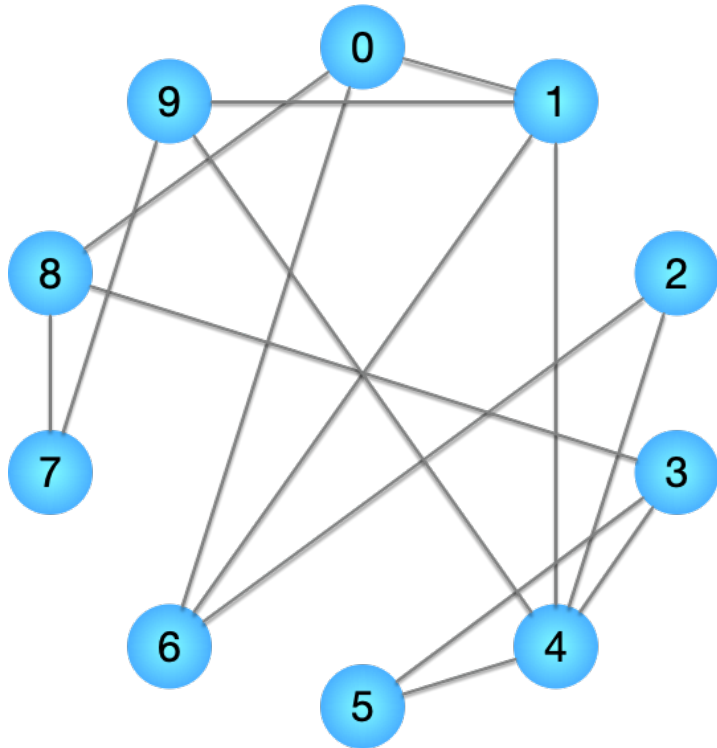
$T(n, m) = O(\lg(m))$

```
'''
```

```

edge_list = [
    (0, 1), (0, 6), (0, 8),
    (1, 4), (1, 6), (1, 9),
    (2, 4), (2, 6),
    (3, 4), (3, 5), (3, 8),
    (4, 5), (4, 9),
    (7, 8), (7, 9),
]

```



```

# Binary search
if is_ordered:
    left, right = 0, len(el) - 1
    while left <= right:
        mid = (left + right) // 2

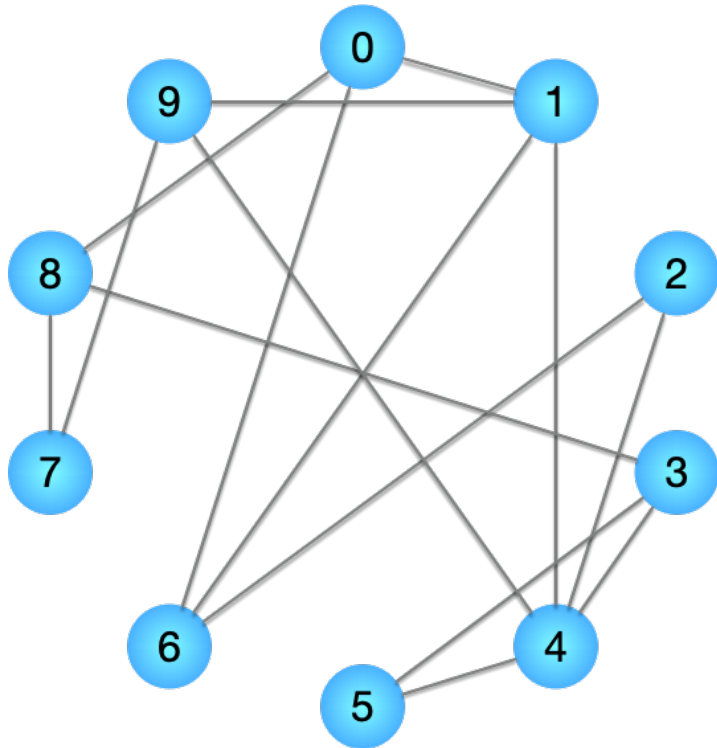
        if e == el[mid]:
            return True
        elif e < el[mid]:
            right = mid - 1
        else:
            left = mid + 1

# Linear search
else:
    for edge in el:
        if e == edge:
            return True

return False

```

```
edge_list = [
    (0, 1), (0, 6), (0, 8),
    (1, 4), (1, 6), (1, 9),
    (2, 4), (2, 6),
    (3, 4), (3, 5), (3, 8),
    (4, 5), (4, 9),
    (7, 8), (7, 9),
]
```



6. How do you find all neighbors of a vertex, and how long does it take?

```
def find_neighbors_el(el, v, is_directed=False):
    '''Return all neighbors of a given vertex.
```

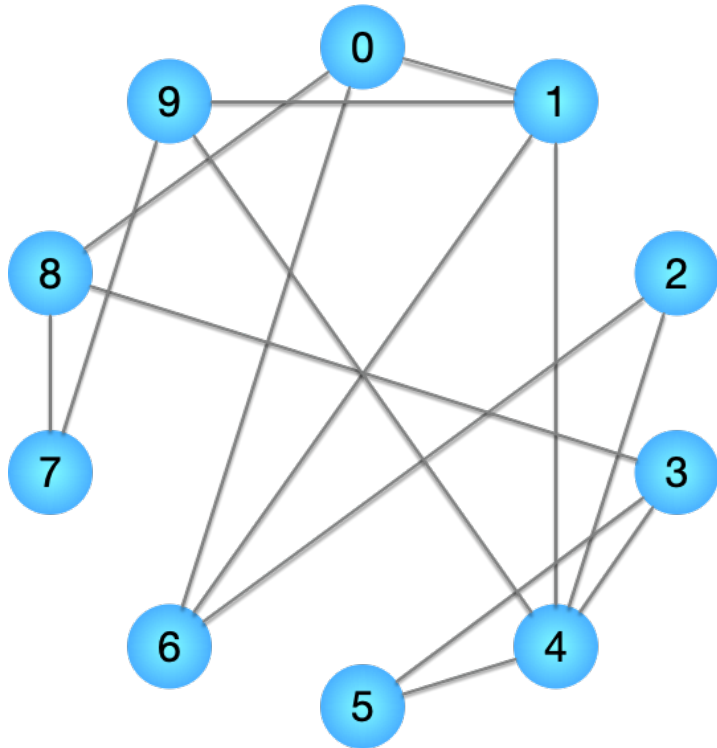
This function does not assume edges are in sorted order.

```
T(n, m) = O(m)
'''
```

```
neighbors = []
for edge in el:
    v1, v2 = edge[0], edge[1]
    if v1 == v:
        neighbors.append(v2)
    elif not is_directed and v2 == v:
        neighbors.append(v1)

return neighbors
```

```
edge_list = [
    (0, 1), (0, 6), (0, 8),
    (1, 4), (1, 6), (1, 9),
    (2, 4), (2, 6),
    (3, 4), (3, 5), (3, 8),
    (4, 5), (4, 9),
    (7, 8), (7, 9),
]
```



7. How much memory is needed to store the graph?

```
def calc_memory_el(el):
    '''Calculate the approximate amount of memory
        used by el in bytes.
```

Edge lists use 2 or 3 numbers for representing undirected, or directed graphs, respectively.

```
M(n, m) = O(m)
```

```
'''
```

```
m = count_edges_el(el)
```

Memory used for storing vertices

```
vertices_per_edge = 2
```

```
bytes_per_int = 4
```

```
memory_usage = m * vertices_per_edge * bytes_per_int
```

Check for edge weights

```
if len(el[0]) == 3:
```

```
    bytes_per_float = 8
```

```
    memory_usage += m * bytes_per_float
```

```
return memory_usage
```

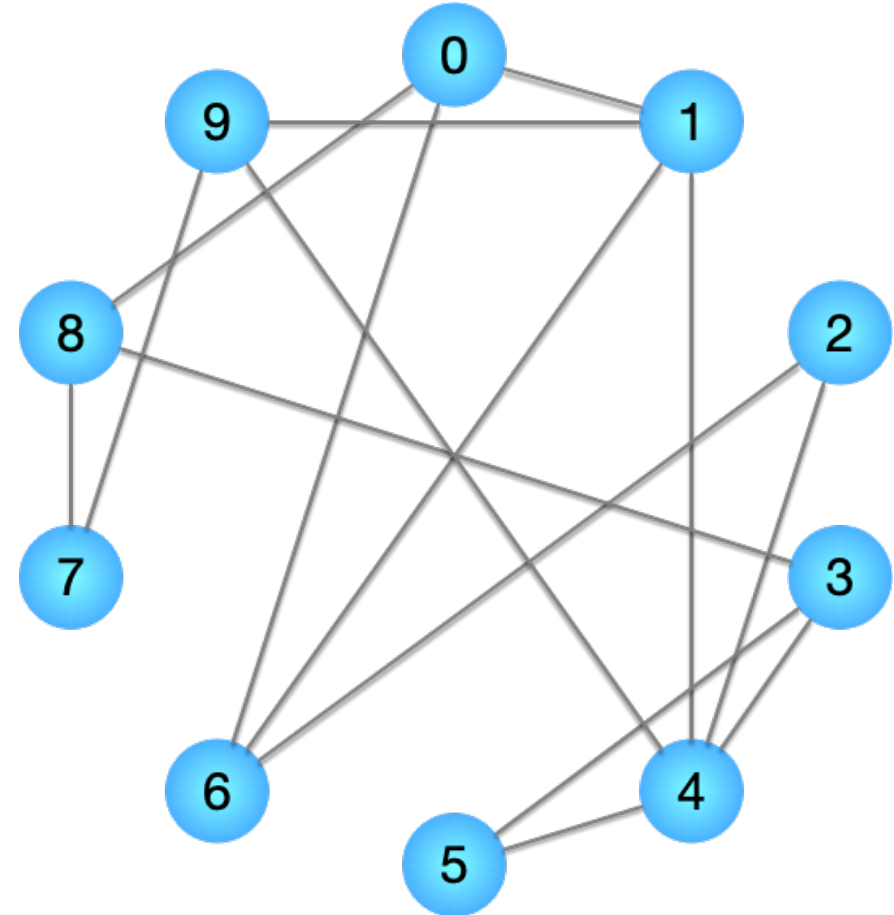
Weights? Directedness?

Adjacency Matrix Representation

```
adjacency_matrix = [
    [0, 1, 0, 0, 0, 0, 1, 0, 1, 0],
    [1, 0, 0, 0, 1, 0, 1, 0, 0, 1],
    [0, 0, 0, 0, 1, 0, 1, 0, 0, 0],
    [0, 0, 0, 0, 1, 1, 0, 0, 1, 0],
    [0, 1, 1, 1, 0, 1, 0, 0, 0, 1],
    [0, 0, 0, 1, 1, 0, 0, 0, 0, 0],
    [1, 1, 1, 0, 0, 0, 0, 0, 0, 0],
    [0, 0, 0, 0, 0, 0, 0, 0, 1, 1],
    [1, 0, 0, 1, 0, 0, 0, 1, 0, 0],
    [0, 1, 0, 0, 1, 0, 0, 1, 0, 0],
]

# Or T/F

# Symmetric matrix for undirected graphs
```



```

adjacency_matrix = [
    [0, 1, 0, 0, 0, 0, 1, 0, 1, 0],
    [1, 0, 0, 0, 1, 0, 1, 0, 0, 1],
    [0, 0, 0, 0, 1, 0, 1, 0, 0, 0],
    [0, 0, 0, 0, 1, 1, 0, 0, 1, 0],
    [0, 1, 1, 1, 0, 1, 0, 0, 0, 1],
    [0, 0, 0, 1, 1, 0, 0, 0, 0, 0],
    [1, 1, 1, 0, 0, 0, 0, 0, 0, 0],
    [0, 0, 0, 0, 0, 0, 0, 0, 1, 1],
    [1, 0, 0, 1, 0, 0, 0, 1, 0, 0],
    [0, 1, 0, 0, 1, 0, 0, 1, 0, 0],
]

```

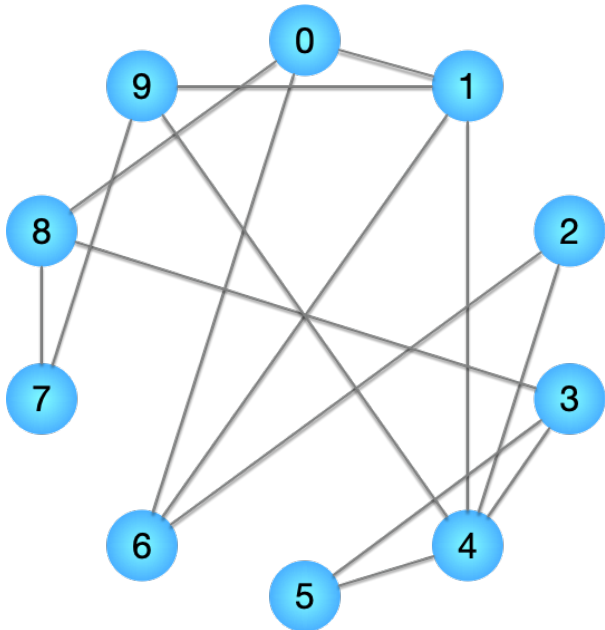
1. How do we count the number of vertices, and how long does it take?

```

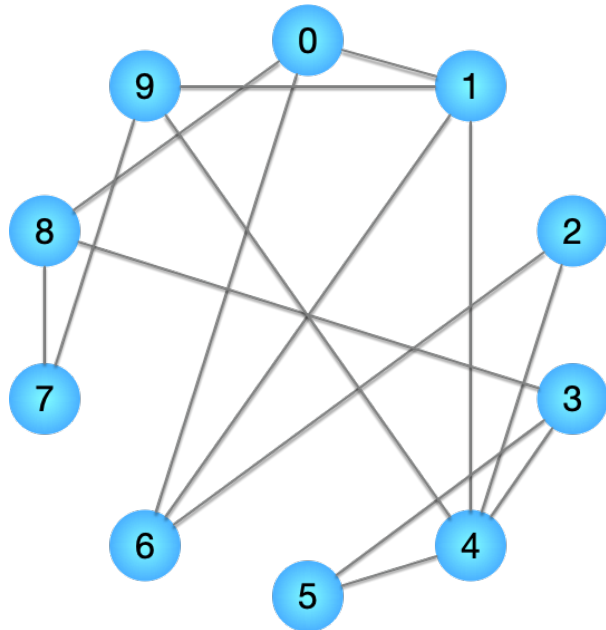
def count_vertices_am(am):
    '''Return the number of vertices in an adjacency
        matrix.

    T(n, m) = O(1)
    '''
    return len(am)

```



```
adjacency_matrix = [
    [0, 1, 0, 0, 0, 0, 1, 0, 1, 0],
    [1, 0, 0, 0, 1, 0, 1, 0, 0, 1],
    [0, 0, 0, 0, 1, 0, 1, 0, 0, 0],
    [0, 0, 0, 0, 1, 1, 0, 0, 1, 0],
    [0, 1, 1, 1, 0, 1, 0, 0, 0, 1],
    [0, 0, 0, 1, 1, 0, 0, 0, 0, 0],
    [1, 1, 1, 0, 0, 0, 0, 0, 0, 0],
    [0, 0, 0, 0, 0, 0, 0, 0, 1, 1],
    [1, 0, 0, 1, 0, 0, 0, 1, 0, 0],
    [0, 1, 0, 0, 1, 0, 0, 1, 0, 0],
]
```



2. How do we count the number of edges, and how long does it take?

```
def count_edges_am(am, is_weighted=False, is_directed=False):
    '''Return the number of.
```

We must loop over the entire matrix and look for non-zero entries.

If `is_weighted` is `True`, then assume all entries are `None` or a floating-point number.

If `is_directed` is `False`, then the matrix is symmetric.

```
T(n, m) = O(n^2)
'''
```

```
if is_weighted:
```

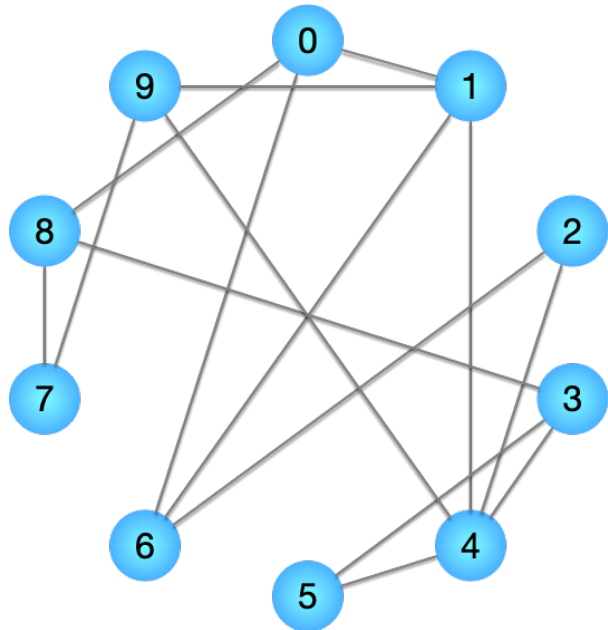
```
    num_edges = sum(sum(1 for val in row
                        if val is not None) for row in am)
```

```
else:
```

```
    num_edges = sum(sum(row) for row in am)
```

```
return num_edges if is_directed else num_edges // 2
```

```
adjacency_matrix = [
    [0, 1, 0, 0, 0, 0, 1, 0, 1, 0],
    [1, 0, 0, 0, 1, 0, 1, 0, 0, 1],
    [0, 0, 0, 0, 1, 0, 1, 0, 0, 0],
    [0, 0, 0, 0, 1, 1, 0, 0, 1, 0],
    [0, 1, 1, 1, 0, 1, 0, 0, 0, 1],
    [0, 0, 0, 1, 1, 0, 0, 0, 0, 0],
    [1, 1, 1, 0, 0, 0, 0, 0, 0, 0],
    [0, 0, 0, 0, 0, 0, 0, 0, 1, 1],
    [1, 0, 0, 1, 0, 0, 0, 1, 0, 0],
    [0, 1, 0, 0, 1, 0, 0, 1, 0, 0],
]
```



3. How do we add vertices, and how long does it take?

```
def add_vertex_am(am, is_weighted=False):
    '''Add a new vertex to an adjacency matrix.
```

Use the next available index.

We must create a new row and a new column

```
T(n, m) = O(n)
'''
```

```
default_value = None if is_weighted else 0
```

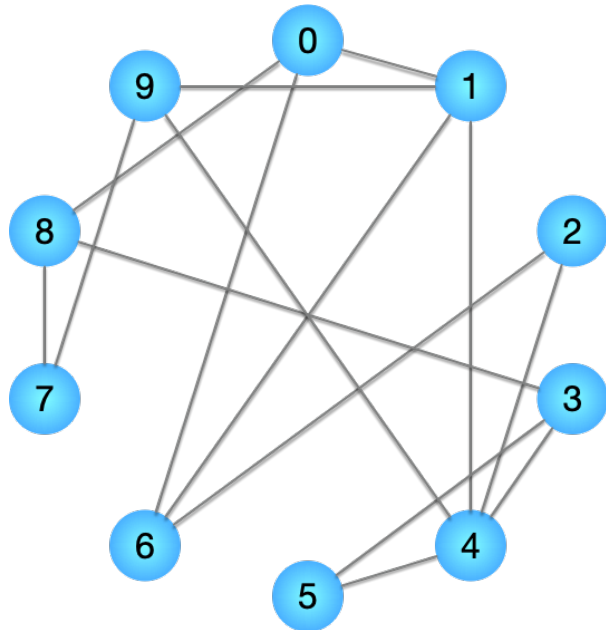
Add the new column

```
for row in am:
    row.append(default_value)
```

Add the new row

```
new_n = count_edges_am(am)
am.append([default_value] * new_n)
return am
```

```
adjacency_matrix = [
    [0, 1, 0, 0, 0, 0, 1, 0, 1, 0],
    [1, 0, 0, 0, 1, 0, 1, 0, 0, 1],
    [0, 0, 0, 0, 1, 0, 1, 0, 0, 0],
    [0, 0, 0, 0, 1, 1, 0, 0, 1, 0],
    [0, 1, 1, 1, 0, 1, 0, 0, 0, 1],
    [0, 0, 0, 1, 1, 0, 0, 0, 0, 0],
    [1, 1, 1, 0, 0, 0, 0, 0, 0, 0],
    [0, 0, 0, 0, 0, 0, 0, 0, 1, 1],
    [1, 0, 0, 1, 0, 0, 0, 1, 0, 0],
    [0, 1, 0, 0, 1, 0, 0, 1, 0, 0],
]
```



4. How do we add edges, and how long does it take?

```
def add_edge_am(am, e, is_directed=False):
    '''Add a new edge to an adjacency matrix.
```

If `is_directed` is `False`, then the matrix is symmetric.

$T(n, m) = O(1)$

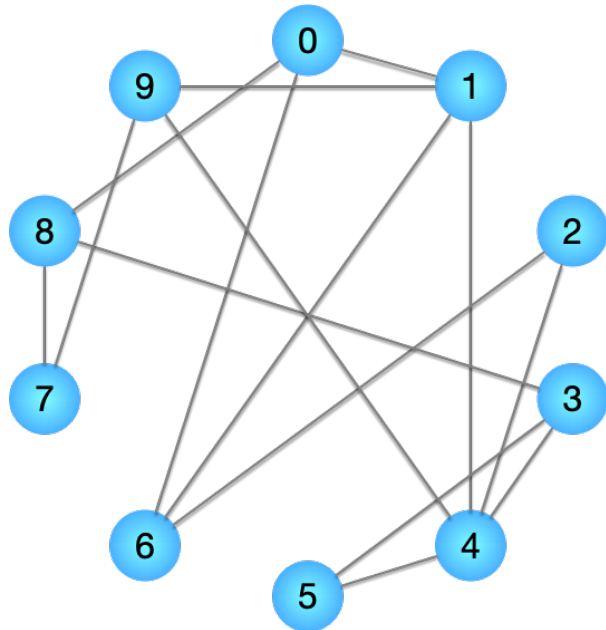
Check for edge weight

```
if len(e) == 2:
    e = (e[0], e[1], 1)
```

```
v1, v2, w = e
am[v1][v2] = w
if not is_directed:
    am[v2][v1] = w
```

```
return am
```

```
adjacency_matrix = [
    [0, 1, 0, 0, 0, 0, 1, 0, 1, 0],
    [1, 0, 0, 0, 1, 0, 1, 0, 0, 1],
    [0, 0, 0, 0, 1, 0, 1, 0, 0, 0],
    [0, 0, 0, 0, 1, 1, 0, 0, 1, 0],
    [0, 1, 1, 1, 0, 1, 0, 0, 0, 1],
    [0, 0, 0, 1, 1, 0, 0, 0, 0, 0],
    [1, 1, 1, 0, 0, 0, 0, 0, 0, 0],
    [0, 0, 0, 0, 0, 0, 0, 0, 1, 1],
    [1, 0, 0, 1, 0, 0, 0, 1, 0, 0],
    [0, 1, 0, 0, 1, 0, 0, 1, 0, 0],
]
```



5. How do you check for the existence of an edges, and how long does it take?

```
def find_edge_am(am, e, is_weighted=False):
    '''Check for existence of edge in adjacency
        matrix.
```

This function assumes that edges contain only v1 and v2.

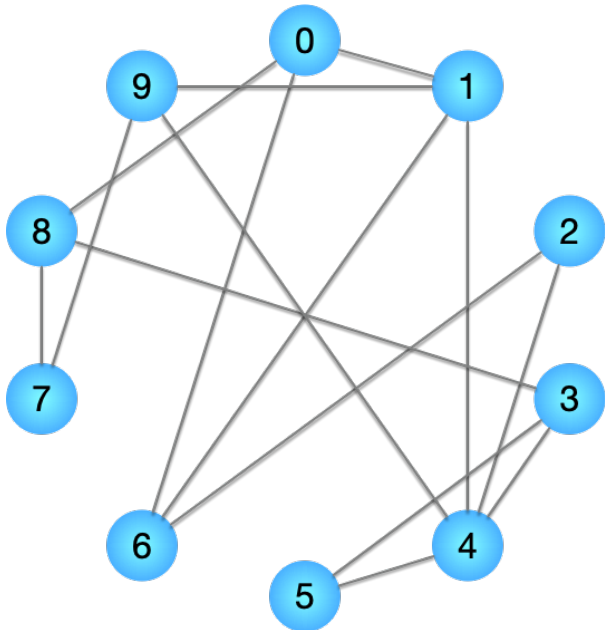
```
T(n, m) = O(1)
'''
```

```
if is_weighted:
    v1, v2, _ = e
    return am[v1][v2] != None
else:
    v1, v2 = e
    return am[v1][v2] == 1
```

```

adjacency_matrix = [
    [0, 1, 0, 0, 0, 0, 1, 0, 1, 0],
    [1, 0, 0, 0, 1, 0, 1, 0, 0, 1],
    [0, 0, 0, 0, 1, 0, 1, 0, 0, 0],
    [0, 0, 0, 0, 1, 1, 0, 0, 1, 0],
    [0, 1, 1, 1, 0, 1, 0, 0, 0, 1],
    [0, 0, 0, 1, 1, 0, 0, 0, 0, 0],
    [1, 1, 1, 0, 0, 0, 0, 0, 0, 0],
    [0, 0, 0, 0, 0, 0, 0, 0, 1, 1],
    [1, 0, 0, 1, 0, 0, 0, 1, 0, 0],
    [0, 1, 0, 0, 1, 0, 0, 1, 0, 0],
]

```



6. How do you find all neighbors of a vertex, and how long does it take?

```

def find_neighbors_am(am, v, is_weighted=False):
    '''Return all neighbors of a given vertex.

```

```

    T(n, m) = O(n)
    '''

```

```

    default_value = None if is_weighted else 0
    neighbors = []

```

```

    for v2, w in enumerate(am[v]):
        if w != default_value:
            neighbors.append(v2)

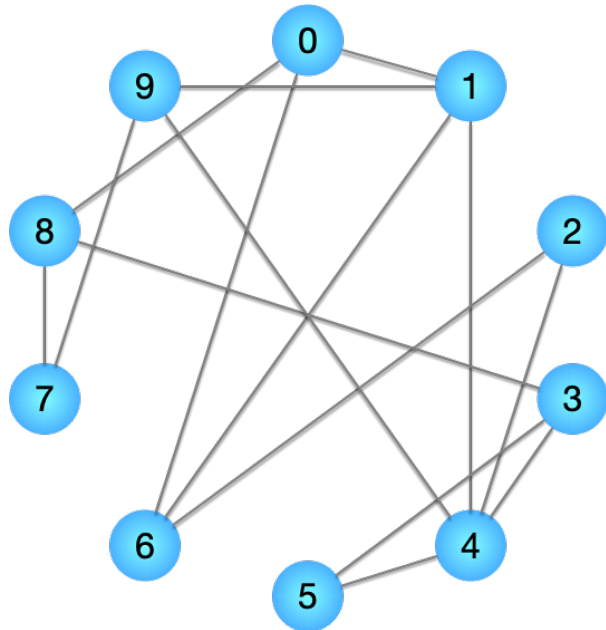
```

```

    return neighbors

```

```
adjacency_matrix = [
    [0, 1, 0, 0, 0, 0, 1, 0, 1, 0],
    [1, 0, 0, 0, 1, 0, 1, 0, 0, 1],
    [0, 0, 0, 0, 1, 0, 1, 0, 0, 0],
    [0, 0, 0, 0, 1, 1, 0, 0, 1, 0],
    [0, 1, 1, 1, 0, 1, 0, 0, 0, 1],
    [0, 0, 0, 1, 1, 0, 0, 0, 0, 0],
    [1, 1, 1, 0, 0, 0, 0, 0, 0, 0],
    [0, 0, 0, 0, 0, 0, 0, 0, 1, 1],
    [1, 0, 0, 1, 0, 0, 0, 1, 0, 0],
    [0, 1, 0, 0, 1, 0, 0, 1, 0, 0],
]
```



7. How much memory is needed to store the graph?

```
def calc_memory_am(am, is_weighted=False):
    '''Calculate the approximate amount of memory
        used by am in bytes.
```

Matrices store booleans or floats for representing un-weighted, or weighted graphs, respectively.

```
M(n, m) = O(n^2)
'''
```

```
n = count_vertices_am(am)
```

```
bytes_per_bool = 1
```

```
bytes_per_float = 8
```

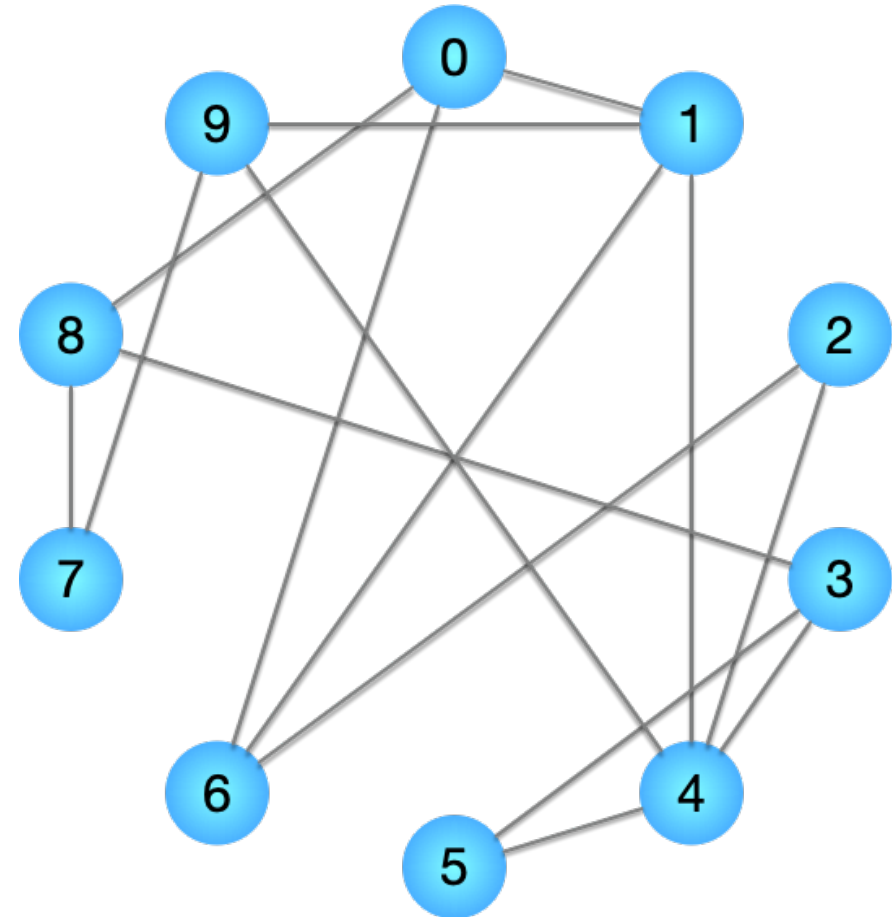
```
bytes_per_cell = bytes_per_float if is_weighted
                  else bytes_per_bool
```

```
return n * n * bytes_per_cell
```

Weights? Directedness?

Adjacency List Representation

```
adjacency_list = [  
    [1, 6, 8],  
    [0, 4, 6, 9],  
    [4, 6],  
    [4, 5, 8],  
    [1, 2, 3, 5, 9],  
    [3, 4],  
    [0, 1, 2],  
    [8, 9],  
    [0, 3, 7],  
    [1, 4, 7],  
]
```



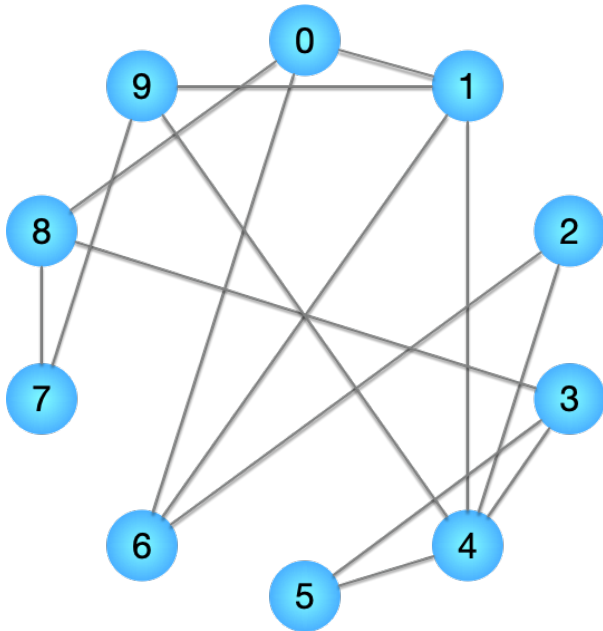
```
adjacency_list = [  
    [1, 6, 8],  
    [0, 4, 6, 9],  
    [4, 6],  
    [4, 5, 8],  
    [1, 2, 3, 5, 9],  
    [3, 4],  
    [0, 1, 2],  
    [8, 9],  
    [0, 3, 7],  
    [1, 4, 7],  
]
```

1. How do we count the number of vertices, and how long does it take?

```
def count_vertices_al(al):  
    '''Return the number of vertices in an adjacency  
    list.
```

```
    T(n, m) = O(1)  
    '''
```

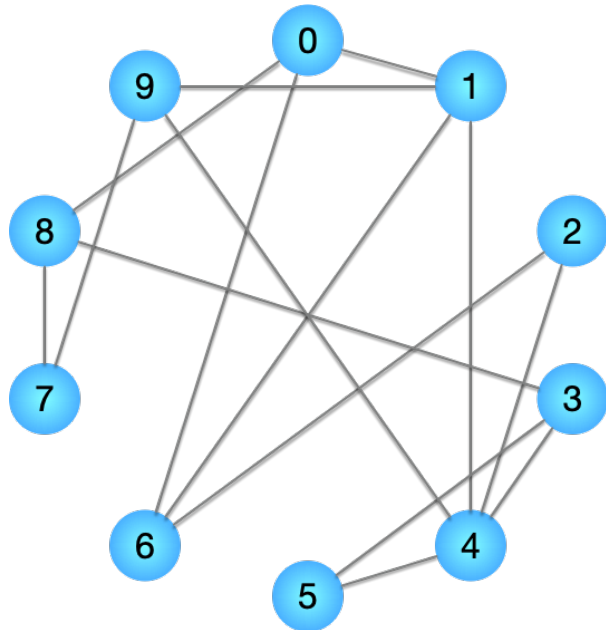
```
    return len(al)
```



```

adjacency_list = [
    [1, 6, 8],
    [0, 4, 6, 9],
    [4, 6],
    [4, 5, 8],
    [1, 2, 3, 5, 9],
    [3, 4],
    [0, 1, 2],
    [8, 9],
    [0, 3, 7],
    [1, 4, 7],
]

```



2. How do we count the number of edges, and how long does it take?

```

def count_edges_al(al, is_directed=False):
    '''Return the number of edges in an adjacency
    list.

```

We must loop over each vertex and count edges.

```

T(n, m) = O(n)
'''

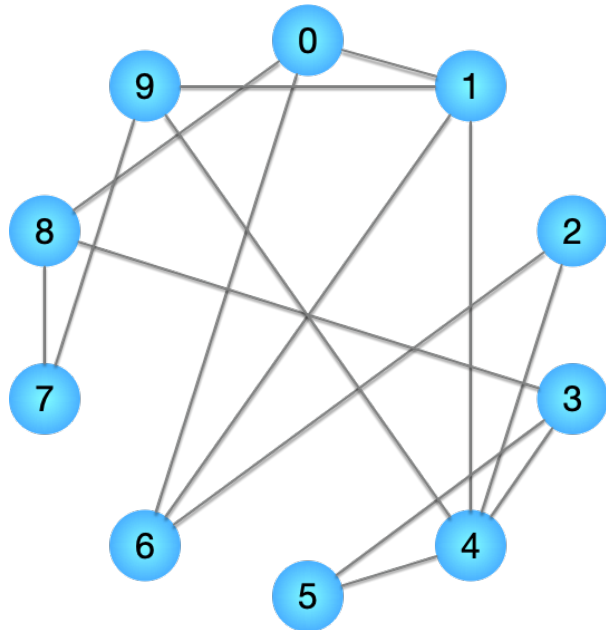
```

```

if is_directed:
    return sum(len(l) for l in al)
else:
    return sum(len(l) for l in al) // 2

```

```
adjacency_list = [
    [1, 6, 8],
    [0, 4, 6, 9],
    [4, 6],
    [4, 5, 8],
    [1, 2, 3, 5, 9],
    [3, 4],
    [0, 1, 2],
    [8, 9],
    [0, 3, 7],
    [1, 4, 7],
]
```



3. How do we add vertices, and how long does it take?

```
def add_vertex_al(al):
    '''Add a new vertex to an adjacency list.
```

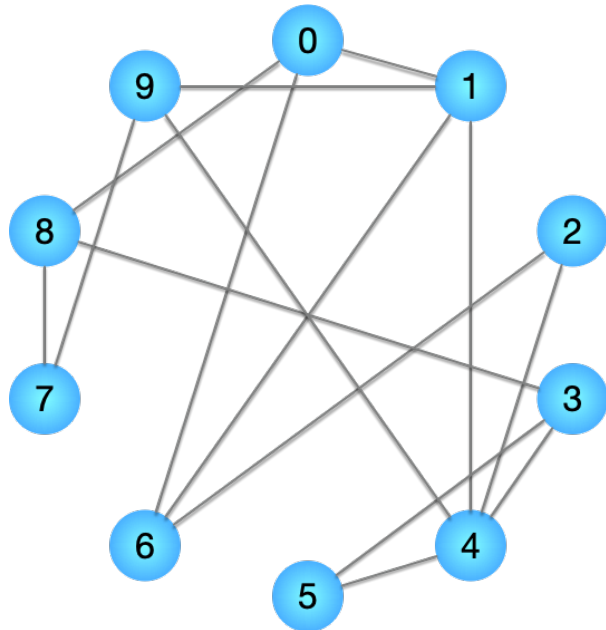
Use the next available index.

This function assumes that v is not already in the adjacency list.

```
T(n, m) = O(1)
'''
```

```
al.append([])
return al
```

```
adjacency_list = [
    [1, 6, 8],
    [0, 4, 6, 9],
    [4, 6],
    [4, 5, 8],
    [1, 2, 3, 5, 9],
    [3, 4],
    [0, 1, 2],
    [8, 9],
    [0, 3, 7],
    [1, 4, 7],
]
```



4. How do we add edges, and how long does it take?

```
def add_edge_al(al, e):
    '''Add a new edge to an adjacency list.
```

```
    T(n, m) = O(1)
    '''
```

```
    v1, v2 = e[0], e[1]
```

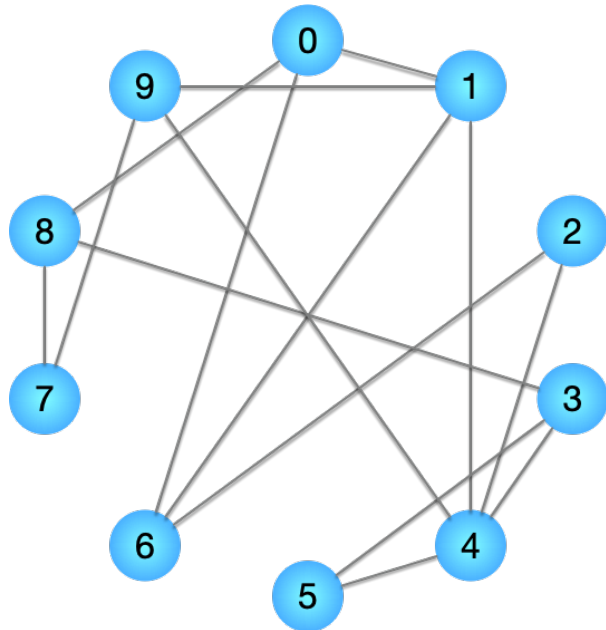
Check for edge weight

```
    if len(e) == 3:
        v2 = (v2, e[2])
```

```
    al[v1].append(v2)
```

```
    return al
```

```
adjacency_list = [
    [1, 6, 8],
    [0, 4, 6, 9],
    [4, 6],
    [4, 5, 8],
    [1, 2, 3, 5, 9],
    [3, 4],
    [0, 1, 2],
    [8, 9],
    [0, 3, 7],
    [1, 4, 7],
]
```



*# 5. How do you check for the existence of an edges,
and how long does it take?*

```
def find_edge_al(al, e, is_weighted=False):
    '''Check for existence of edge in adjacency list.
```

This function assumes that edges contain only v1
and v2.

$T(n, m) = O(m)$
'''

v1 = e[0]

vw = (e[1], e[2]) if is_weighted else e[1]
return vw in al[v1]

```

adjacency_list = [
    [1, 6, 8],
    [0, 4, 6, 9],
    [4, 6],
    [4, 5, 8],
    [1, 2, 3, 5, 9],
    [3, 4],
    [0, 1, 2],
    [8, 9],
    [0, 3, 7],
    [1, 4, 7],
]

```

6. How do you find all neighbors of a vertex, and how long does it take?

```

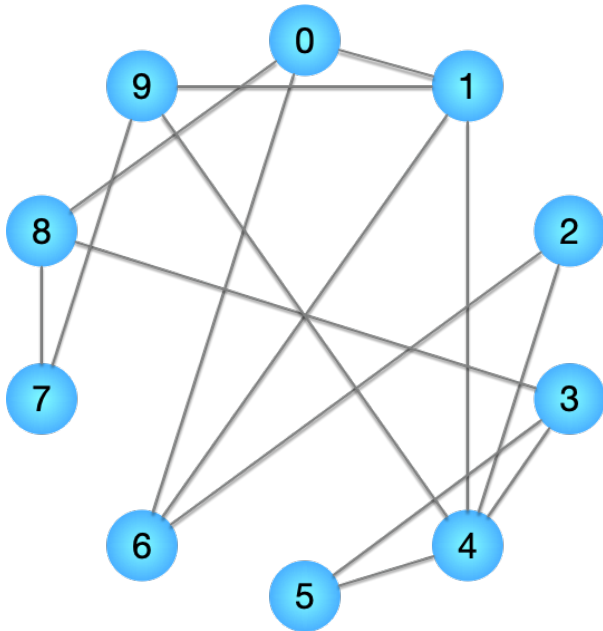
def find_neighbors_al(al, v, is_weighted=False):
    '''Return all neighbors of a given vertex.

```

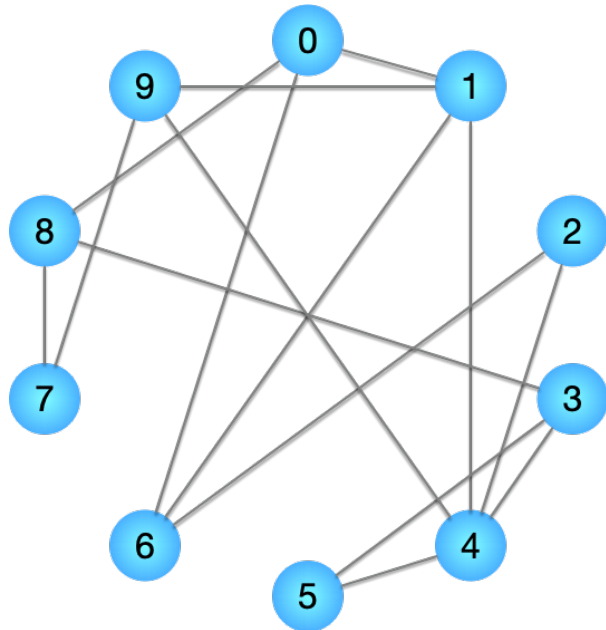
```

    T(n, m) = O(1)
    '''
    return al[v]

```



```
adjacency_list = [
    [1, 6, 8],
    [0, 4, 6, 9],
    [4, 6],
    [4, 5, 8],
    [1, 2, 3, 5, 9],
    [3, 4],
    [0, 1, 2],
    [8, 9],
    [0, 3, 7],
    [1, 4, 7],
]
```



7. How much memory is needed to store the graph?

```
def calc_memory_al(al, is_weighted=False):
    '''Calculate the approximate amount of memory
        used by al in bytes.
```

Adjacency lists use 1 or 2 numbers for representing each edge in weighted or unweighted graphs, respectively.

```
M(n, m) = O(n + m)
'''
```

```
m = count_edges_al(al)
```

Memory used for storing vertices

```
vertices_per_edge = 1
```

```
bytes_per_int = 4
```

```
memory_usage = m * vertices_per_edge * bytes_per_int
```

Check for edge weights

```
if is_weighted:
```

```
    bytes_per_float = 8
```

```
    memory_usage += m * bytes_per_float
```

```
return memory_usage
```

m ? n	Edge List	Adjacency Matrix	Adjacency List
Count Vertices	$O(m)$	$O(1)$	$O(1)$
Count Edges	$O(1)$	$O(n^2)$	$O(n)$
Add Vertex	$O(1)$	$O(n)$	$O(1)$
Add Edge	$O(1)$	(1)	$O(1)$
Check Edge	$O(m)$ or $O(\lg(m))$	$O(1)$	$O(m)$
Find Neighbors	$O(m)$	$O(n)$	$O(1)$
Memory Usage	$O(m)$	$O(n^2)$	$O(n + m)$



call
BFS
DFS
RFS

Exercise

Friend Circles Algorithm

on each node.

