Approximation Algorithms

https://cs.pomona.edu/classes/cs140/

Outline

Topics and Learning Objectives

- Discuss strategies for finding solutions to difficult problems
- Apply an approximation algorithm to an NP-Hard problem

Exercise

- None

Good Enough

NP-Complete

<u>What does it mean if your problem is NP-Complete?</u>

1. It belongs to NP, and
2. It belongs to NP-Hard.

<u>What does it mean to belong to NP?</u>

- We can verify a solution as correct or incorrect in polynomial time.

<u>What does it mean to belong to NP-Hard?</u>

- We do not know an algorithm to solve it in polynomial-time.

So, your problem is NP-Hard…

- This **does not** mean you cannot solve your problem.
- This **does not** mean that you cannot get an optimal solution.
- It does mean that you should set your expectations appropriately.
- You are probably not going to accidentally prove that P = NP.

Strategies

1. Focus on solving a special case that is tractable
    - The general Knapsack problem is NP-Complete, but we solved it by looking at problems where the total capacity W was O(nW).

1. Solve the problem in exponential time (but faster than brute-force)
    - We looked an algorithm for TSP that runs in $O(n^2 2^n)$ instead of $O(n!)$

2. Solve the problem using some heuristics
    - These algorithms are not guaranteed to give optimal solutions,
    - but they are (generally) fast.

The Traveling Salesman Problem

Given a list of cities and the distances between each pair of cities, what is the shortest possible route that visits each city exactly once and returns to the origin city?

- Input: a complete, undirected graph with non-negative edge costs
- Output: a minimum cost tour (a cycle that visits each vertex once)

Solving the TSP

- There are n! total possible tours.

Input Size	Brute-Force n!	Exponential $O(n^2 2^n)$
14	87 billion 178 million …	~ 3 million
15	1 trillion 307 billion …	~ 7 million
16	20 trillion 922 billion …	~ 16 million …
30	265 nonillion 252 octillion 859 septillion 812 sextillion 191 quintillion 58 quadrillion 636 trillion …	~ 966 billion 367 million …

Very long time

Why is TSP so difficult?

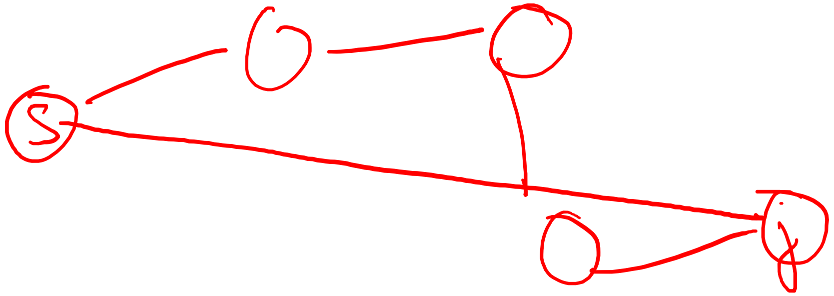
Doesn't it seem like it is just a special case of SSSP, with one extra edge back to the start vertex?

Remember our SSSP sub-problems (Bellman-Ford):

For every edge edge budget (`FOR num_edges IN [0 ..= n]`)

Let L_{ij} = the length of the shortest path from 1 to j that uses at most i edges

Why is TSP so difficult?



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How are they different?

- Subproblems of SSSP do not solve the original TSP problem (SSSP does not **require** the use of i edges).
- SSSP doesn’t enforce that we cannot visit a vertex more than once.
- If we change SSSP to enforce the use of i edges with no repeats, we lose the ability to solve larger problems from smaller problems.

Dynamic Programming for TSP

For every destination j in {1, 2, …, n}, and
for every subset S of {1, 2, ..., n}
$L_{s,j}$ = the minimum length of a path from 1 to j that visits all of the vertices in S

How does this improve on brute-force?

- It does not care about the order in which we visit the vertices in S.
- But, there are still an exponential number of choices for S → $O(2^n)$

Optimal Substructure Lemma

- Let P be a shortest path from 1 to j that visits S.
- If the last hop of P is (k, j)

- Then P' is the shortest path from 1 to k

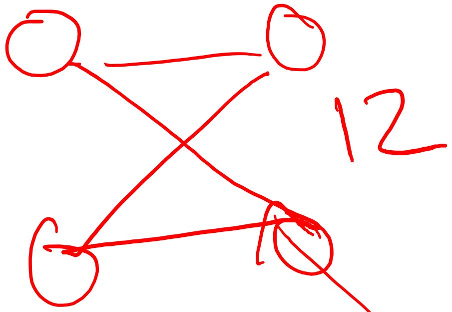
$$L_{i,j} = \min_{k \in S, k \neq j} L_{S-\{j\},k} + C_{kj}$$

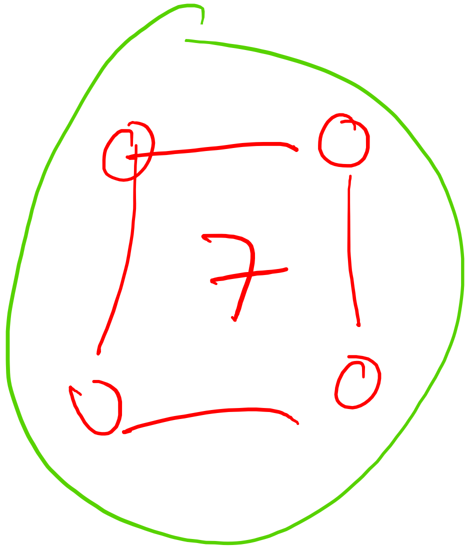
What if we don't need the optimal path?
Just one that is "good enough"?

Local Search Heuristic for Hard Problems

```
FUNCTION LocalSearch(numTrials, solutionFcn, evaluationFcn)
    bestSolution = solutionFcn()
    bestPerformance = evaluationFcn(bestSolution)

    FOR trial IN [0 ..< numTrials]
        newSolution = solutionFcn(bestSolution)
        newPerformance = evaluationFcn(newSolution)

        IF newPerformance > bestPerformance
            bestPerformance = newPerformance
            bestSolution = newSolution

    RETURN bestSolution
```

Local Search

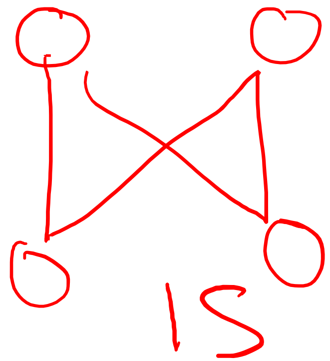


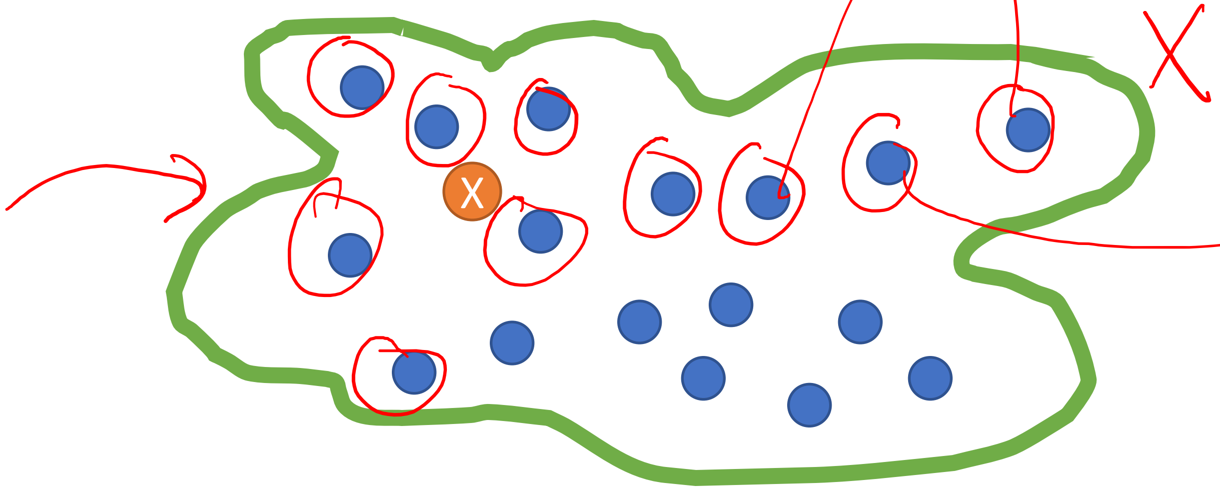


- Let X be a set of candidate solutions to a problem
- For example, let it be all possible tours of a graph

The key to local search to to define a neighborhood:

- For each x in X, specify which y in X are its “neighbors”





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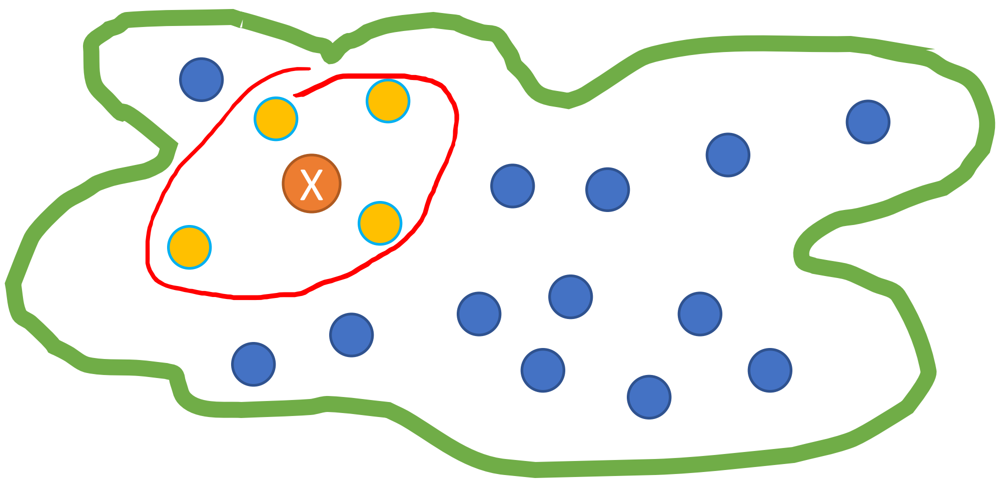

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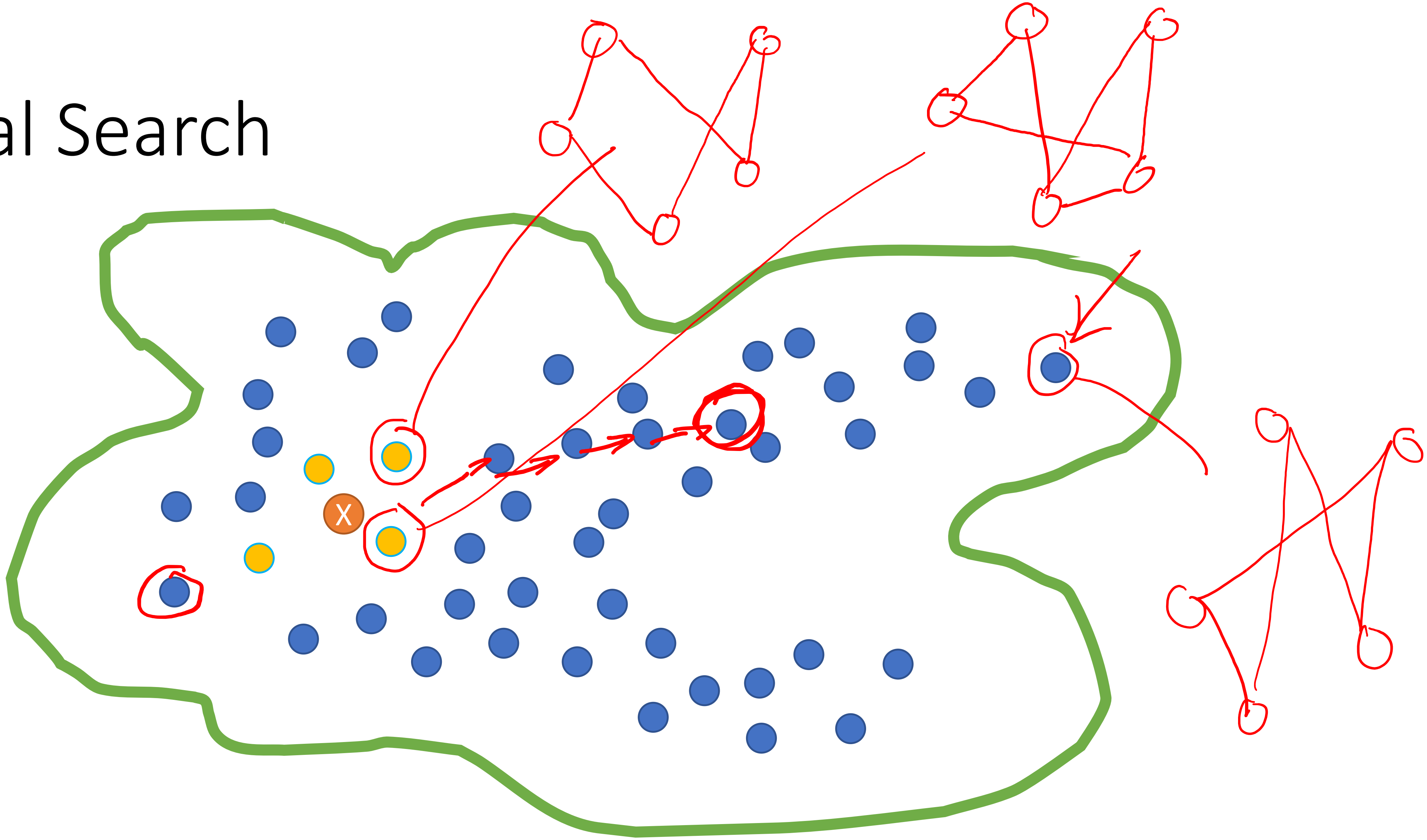

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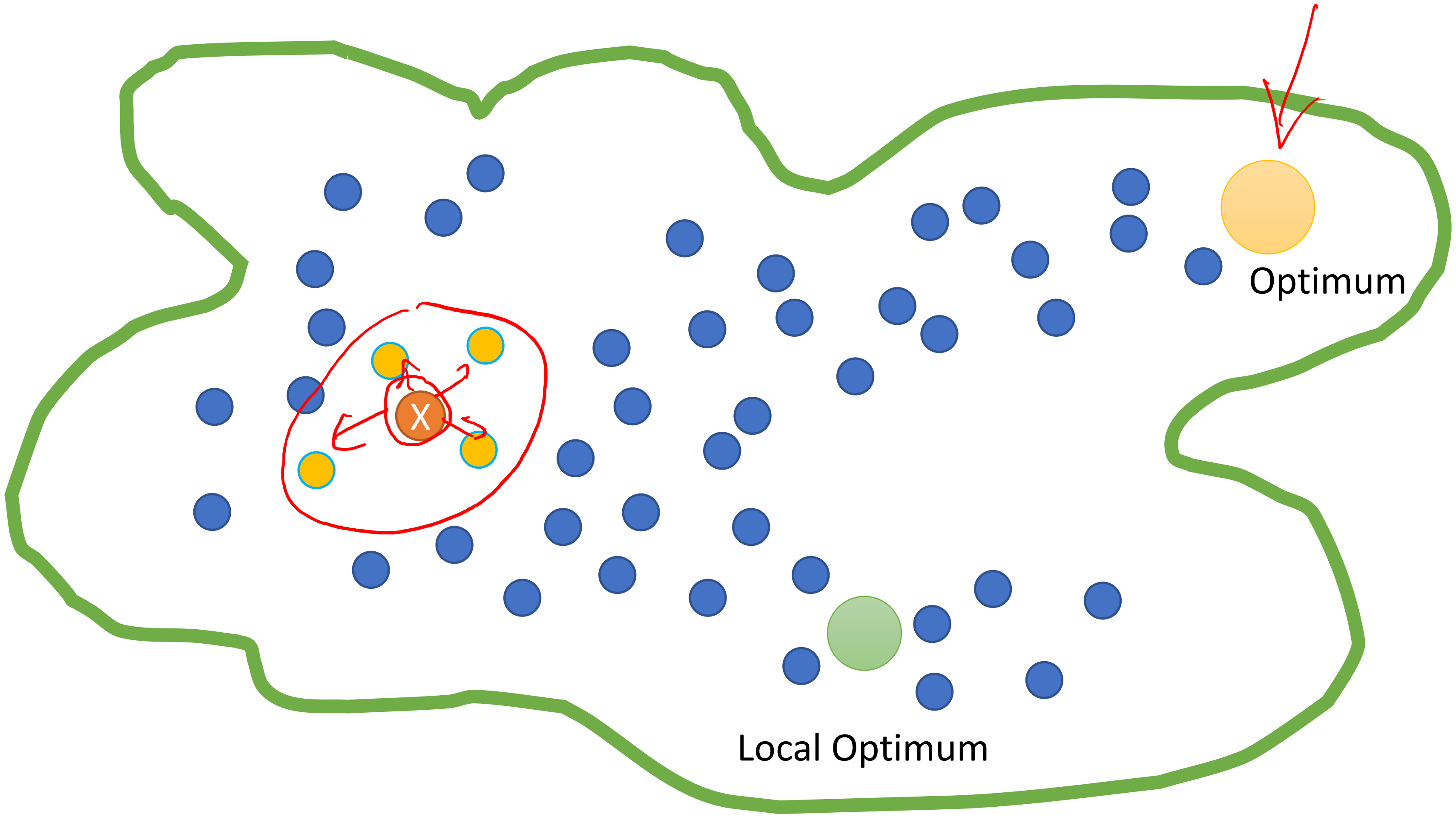

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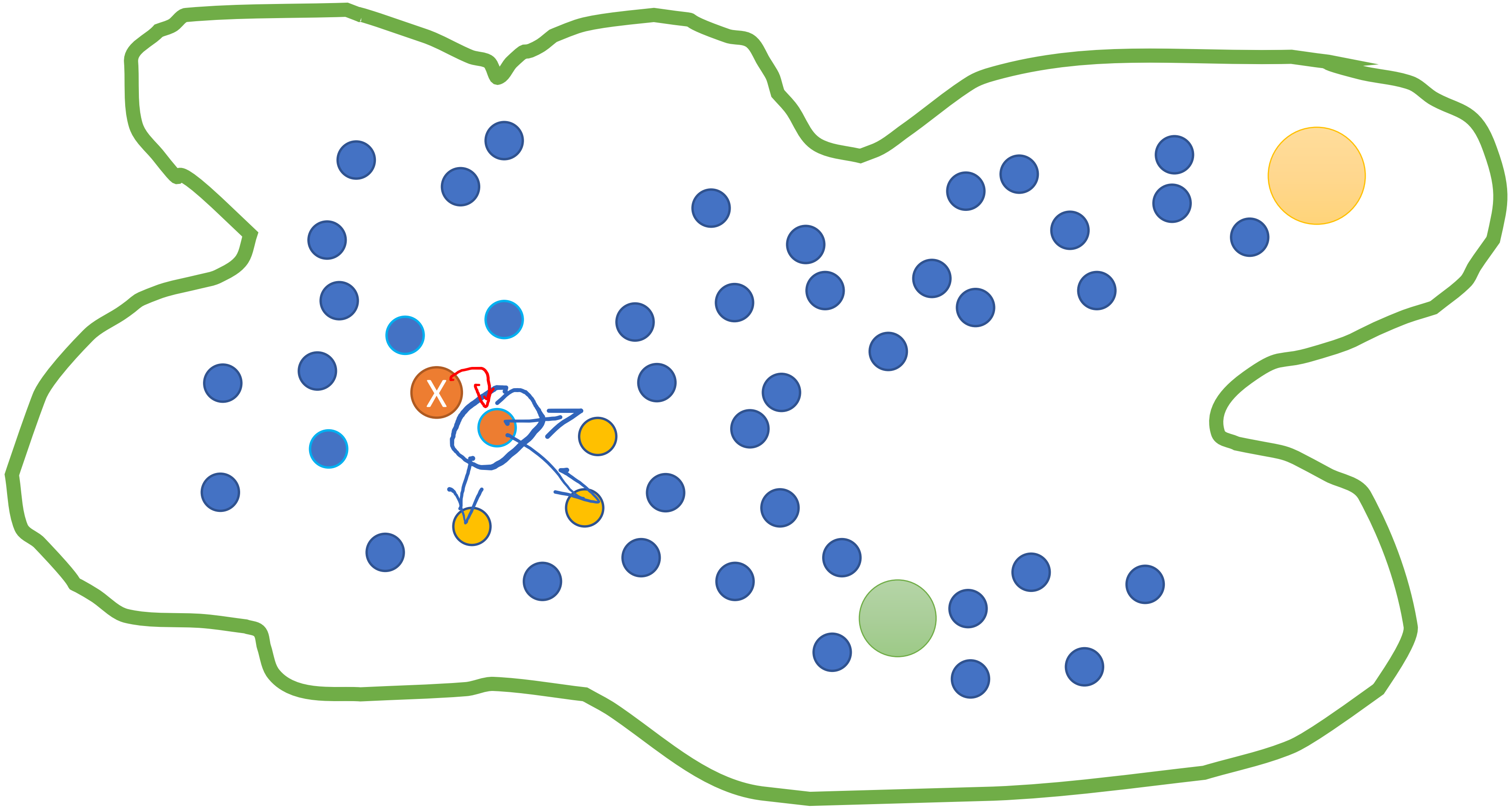

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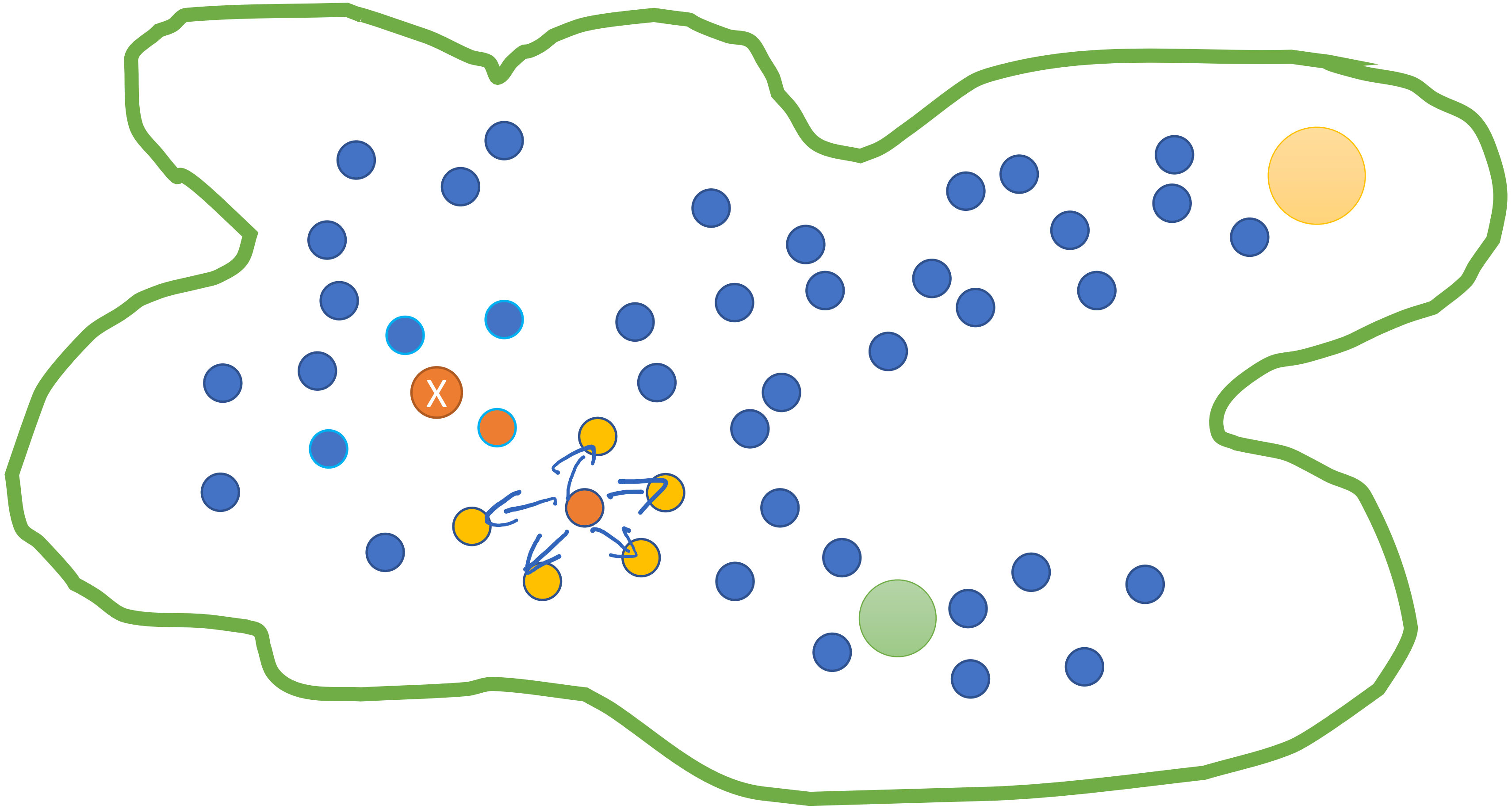

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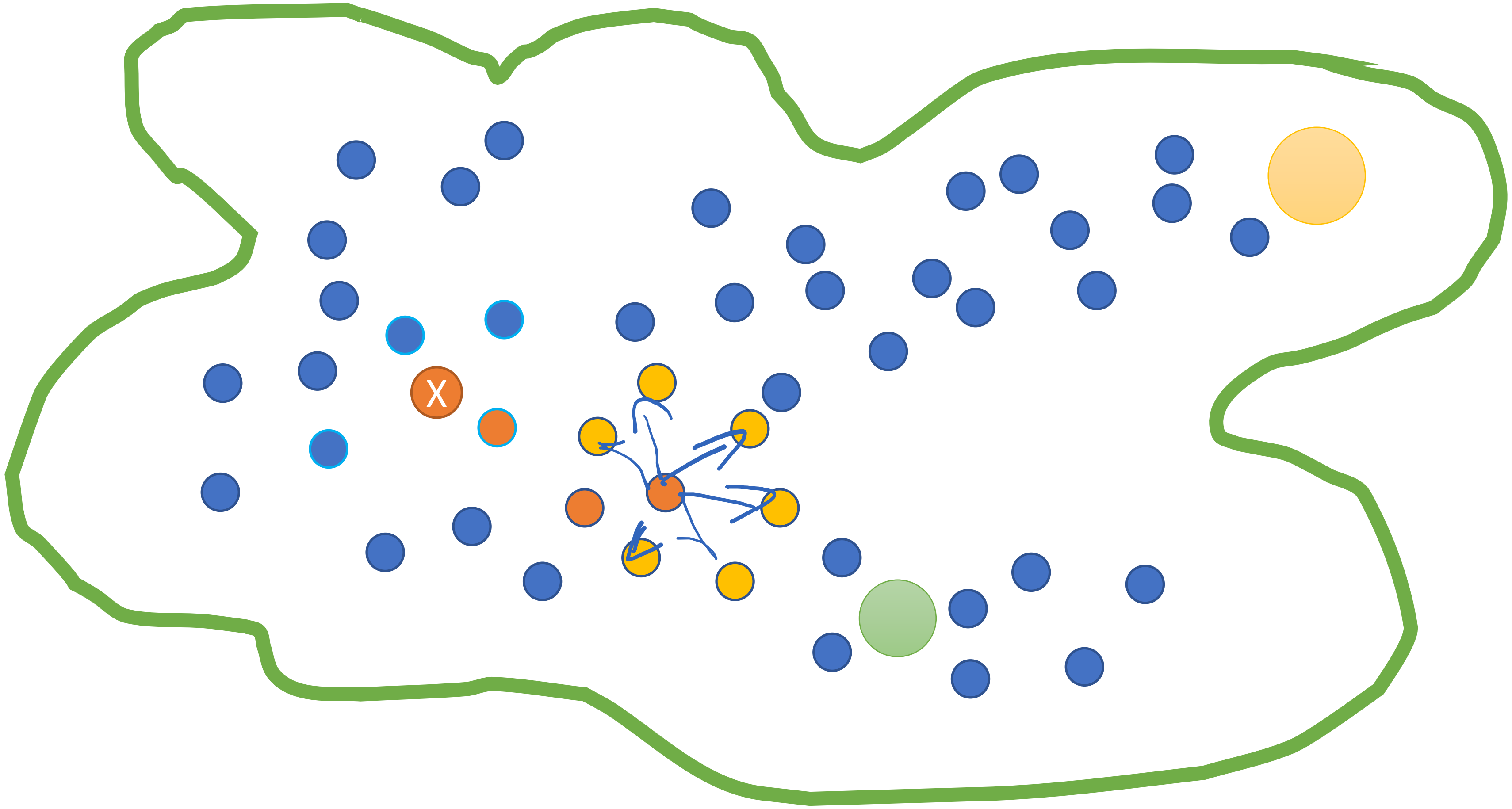

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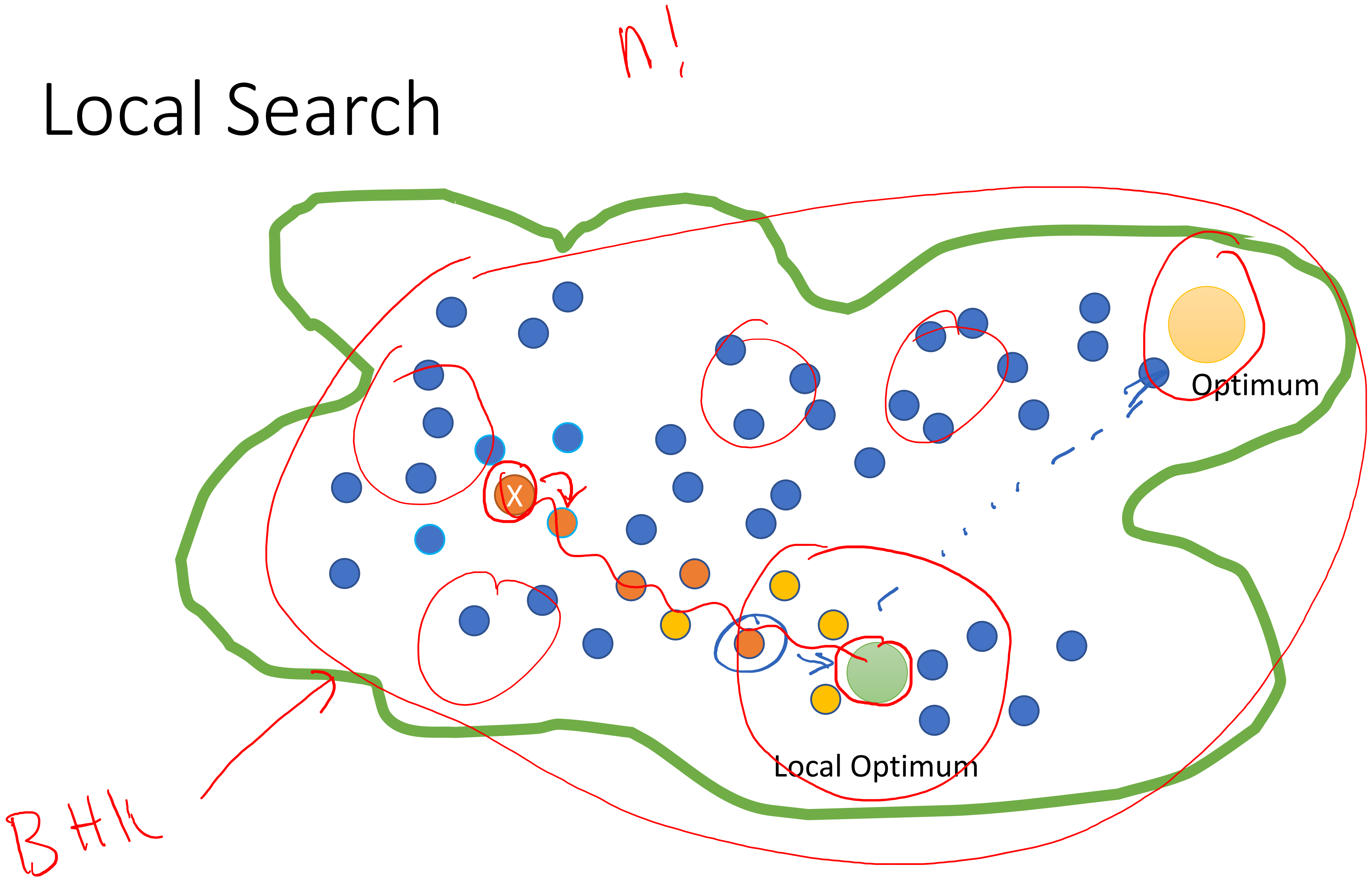
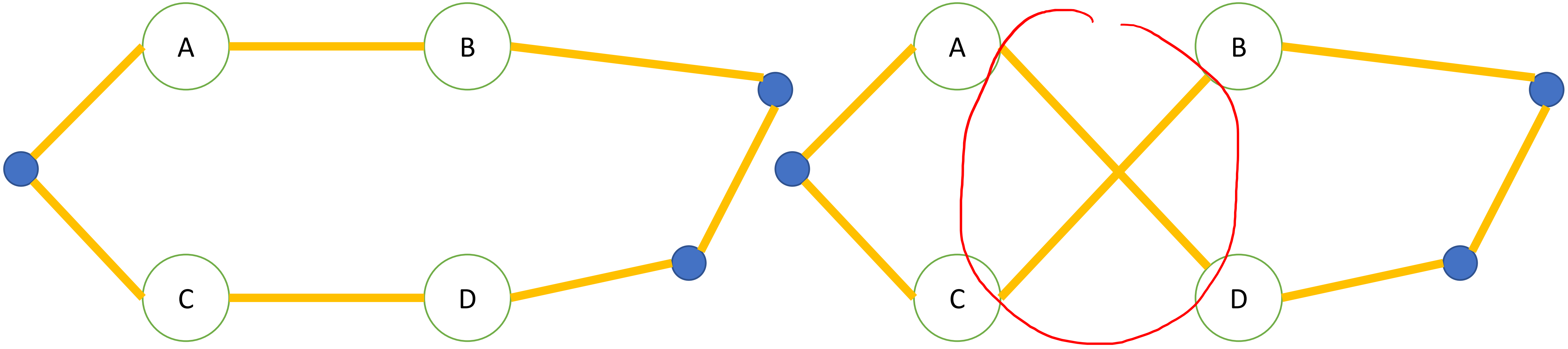


Neighborhood for TSP



Let’s say that two tours are neighbors if they differ by a minimal number of edges.

```
FUNCTION LocalSearch(numTrials, solutionFcn, evaluationFcn)
  bestSolution = solutionFcn()
  bestPerformance = evaluationFcn(bestSolution)

  FOR trial IN [0 ..< numTrials]
    newSolution = solutionFcn(bestSolution)
    newPerformance = evaluationFcn(newSolution)

    IF newPerformance > bestPerformance
      bestPerformance = newPerformance
      bestSolution = newSolution

  RETURN bestSolution
```

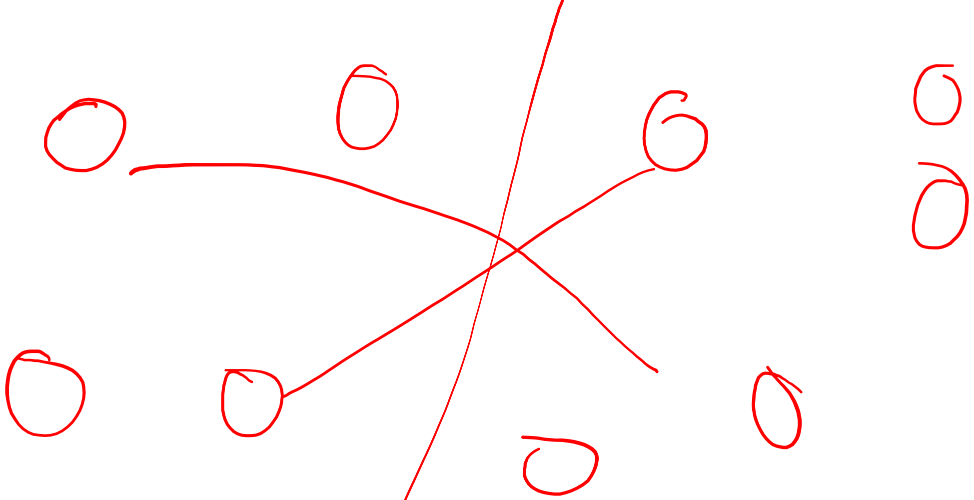
x ← any permutation of the cities

compute tour cost

y ← Take current solution and flip two edges

x = y

The Max-Cut Problem



- Input: an undirected graph
- Output: a cut (A,B) that maximizes the number of crossing edges

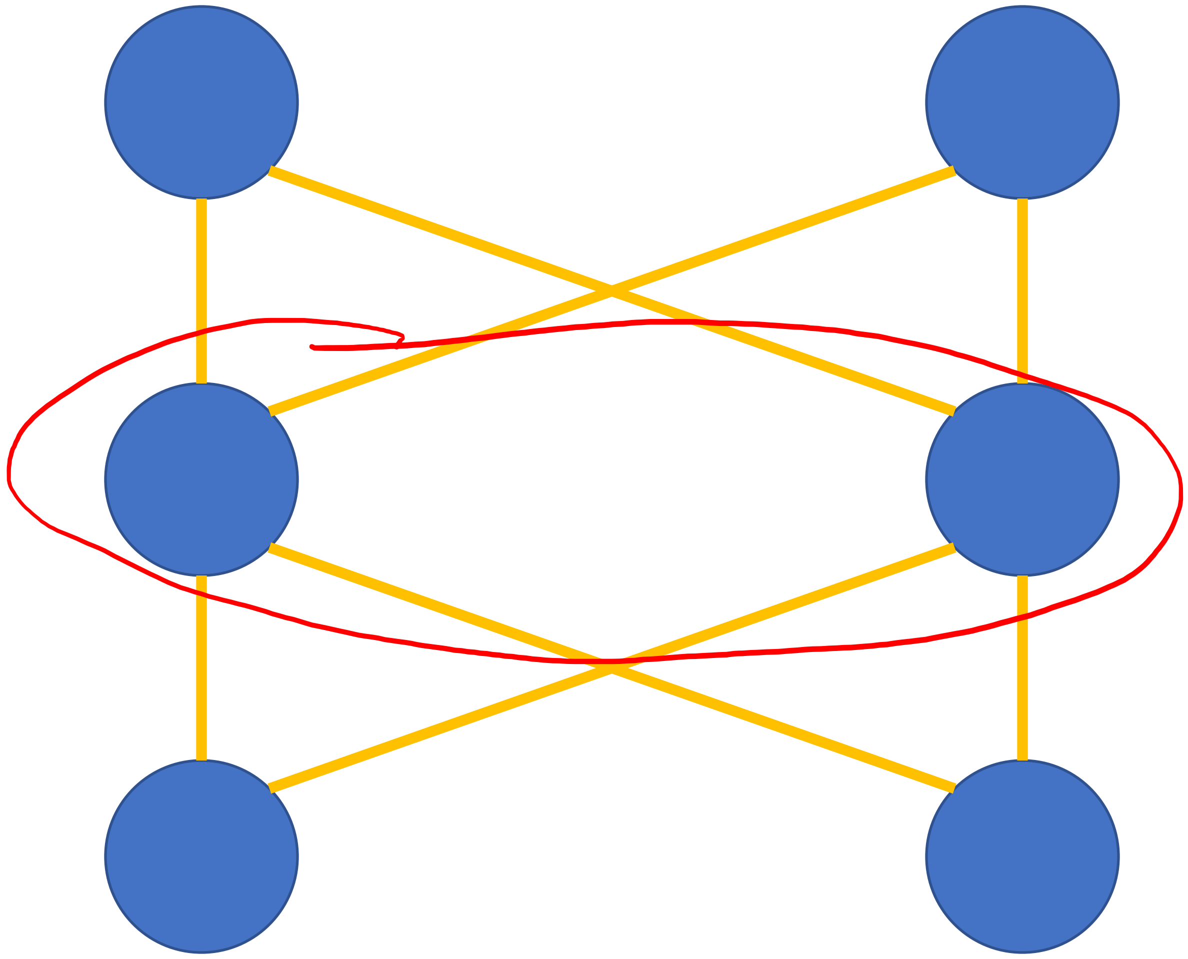
- Reminder: a cut is a partition of the vertices into two non-empty sets
- How many possible cuts are there?

It turns out that:

- The min-cut problem is tractable (we have a polynomial time algorithm)
- The max-cut problem is NP-Complete

How many edges cross the max-cut?

a. 4
b. 6
c. 8
d. 10



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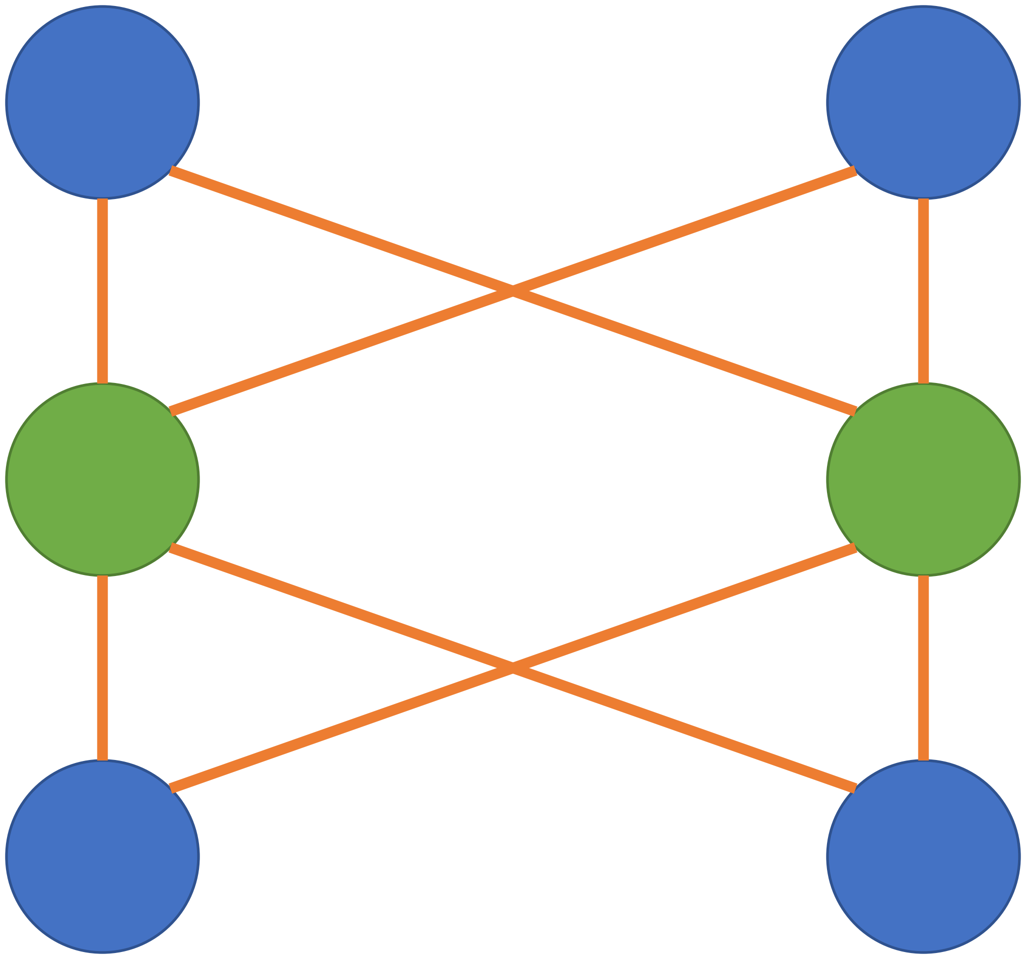
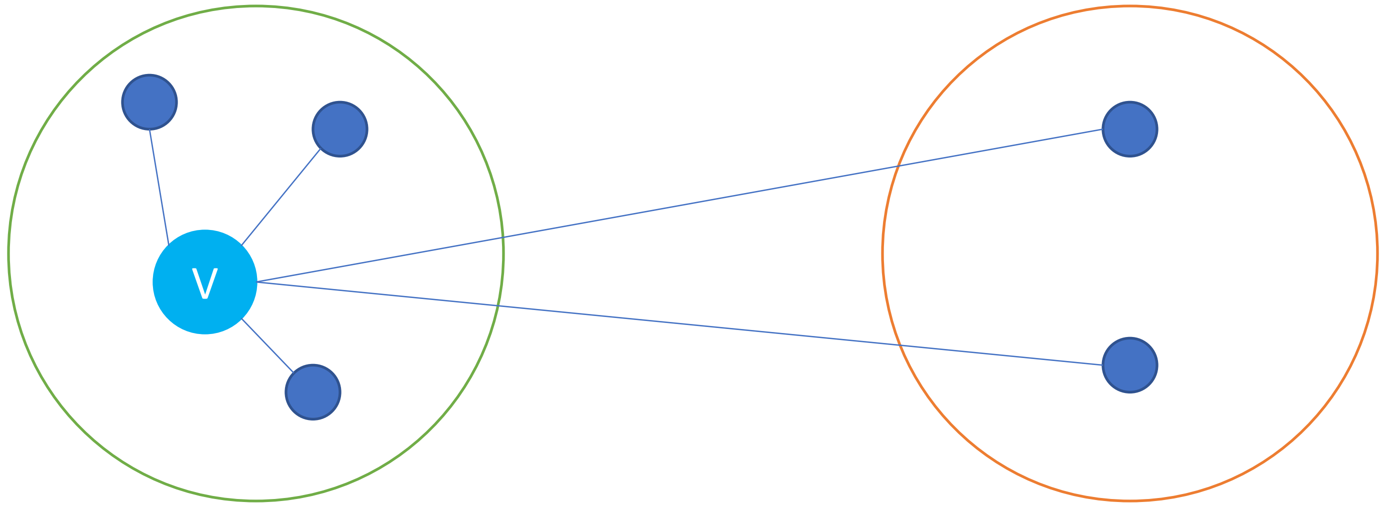


Local Search for Max-Cut

Notation: for a cut (A, B) and a vertex v:

- $C_v(A,B)$ = the number of edges incident on v that cross (A,B)
- $D_v(A,B)$ = the number of edges incident on v that don't cross (A,B)



$C_v = 2$

$D_v = 3$

Local Search for Max-Cut

1. Let (A,B) be some arbitrary cut of the graph G

2. While there is a vertex v with $D_v(A,B) > C_v(A,B)$
   1. move v to the other side of the cut

3. Return the final cut (A,B)

About this algorithm

- This algorithm runs in polynomial time (quadratic)
- This algorithm is **not** guaranteed to give the optimal cut
- This algorithm outputs a cut which is at least 50% of the maximum possible

About Local Search Algorithms

How do you pick the initial solution?

- Use a heuristic
- "this type of solution is usually a good place to start"
- Use a random choice

Which superior neighbor should you choose?

- Use a heuristic
- Choose the neighbor at random
- Choose the neighbor that yields the most improvement
- How do you define the neighborhood?

Can you think of some simple techniques for improving local search?

- Run the algorithm multiple times with some random choices!
- Independent trials.
- Combine good solutions.

Gradient-Free Optimization