Floyd-Warshall Algorithm For Solving the All-Pairs Shortest Path Problem

https://cs.pomona.edu/classes/cs140/

Outline

Topics and Learning Objectives

- Discuss and analyze the Floyd-Warshall Algorithm

Exercise

- None

All-Pairs Shortest Path Problem

Compute the shortest path from every vertex to every other vertex

- Input: a weighted graph (no need for a start vertex)
- Output:
  - Shortest path from u → v for all values of u and v
  - Or report that a negative cycle has been discovered

- Can we solve this problem with what we know already?

SSSP → APSP

How do we turn a solution to the single-source shortest path (SSSP) problem into a solution for the all-pairs shortest path (APSP) problem?

- This is called a reduction!

- How many times do we need to run a SSSP procedure for APSP?

   a. 1
   b. $n - 1$
   c. n
   d. n^2

SSSP algorithms

Running time of APSP if we **don't** allow negative edges?

- n * O(Dijkstra's Algorithm) = $O(n\ m \lg n)$
- For sparse graphs: $O(n^2 \lg n)$
- For dense graphs: $O(n^3 \lg n)$

Running time of APSP if we do allow negative edges?

- n * O(Bellman-Ford) = $O(n^2\ m)$
- For sparse graphs: $O(n^3)$
- For dense graphs: $O(n^4)$

Consider APSP on dense graphs.

- How many values are we going to output? n^2
- What is the potential length of a shortest path? $n - 1$
- What is the lower bound on the running time of ASPS?
- It is tempting to say that the lower bound is n^3
- However, this lower bound has yet to be determined
- Consider the matrix multiplication procedure developed by Strassen

Specialized APSP Algorithm

- Although we can use Bellman-Ford and Dijkstra's algorithms, there are, in fact, specialized APSP algorithms

- The Floyd-Warshall algorithm solves the APSP problem deterministically in $O(n^3)$ on all types of graph

- It works with negative edge lengths
- Meaning that is is as good as Bellman-Ford for sparse graphs,
- And much better than Bellman-Ford for dense graphs.

Question

	Sparse Graphs	Dense Graphs
Dijkstra's n times	$O(n^2 \lg n)$	$O(n^3 \lg n)$
Bellman-Ford n times	$O(n^3)$	$O(n^4)$
Floyd-Warshall	$O(n^3)$	$O(n^3)$

- What algorithm would you choose for sparse graphs?
  - Dijkstra's n times if there are no nnegative edges, Floyd-Warshall otherwise
- What algorithm would you choose for dense graphs?
  - Always Floyd-Warshall

Optimal Substructure for APSP

Key concept:

- label the vertices 1 though n (giving them an arbitrary order),
- and then introduce the notation $V^{(k)} = \{1, 2, \ldots, k\}$

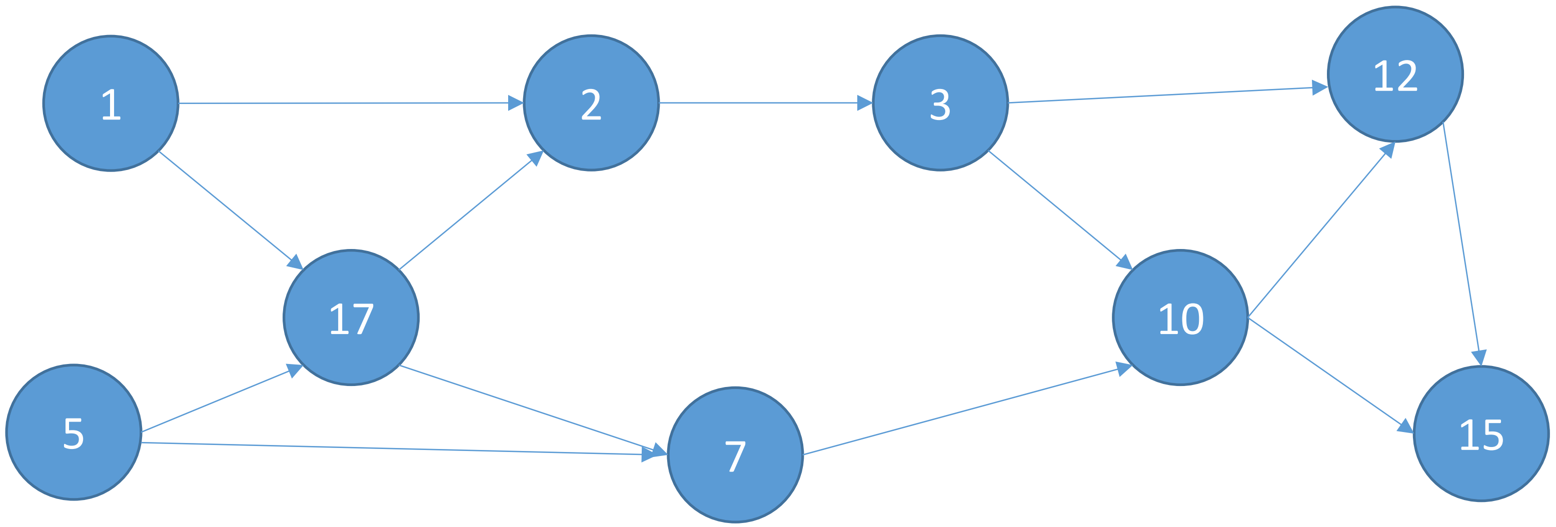
Optimal Substructure Lemma:

- Assume, for now, that the graph does **not** include a negative cycle
- Fix a source vertex i, a destination vertex j, and a value for k
- Then let P be the shortest $i \rightarrow j$ path with **internal** nodes from $V^{(k)}$

Example Substructure

Optimal Substructure Lemma:

- Fix a source vertex i, a destination vertex j, and a value for k
- Then let P be the shortest i → j path with **internal** nodes from $V^{(k)}$

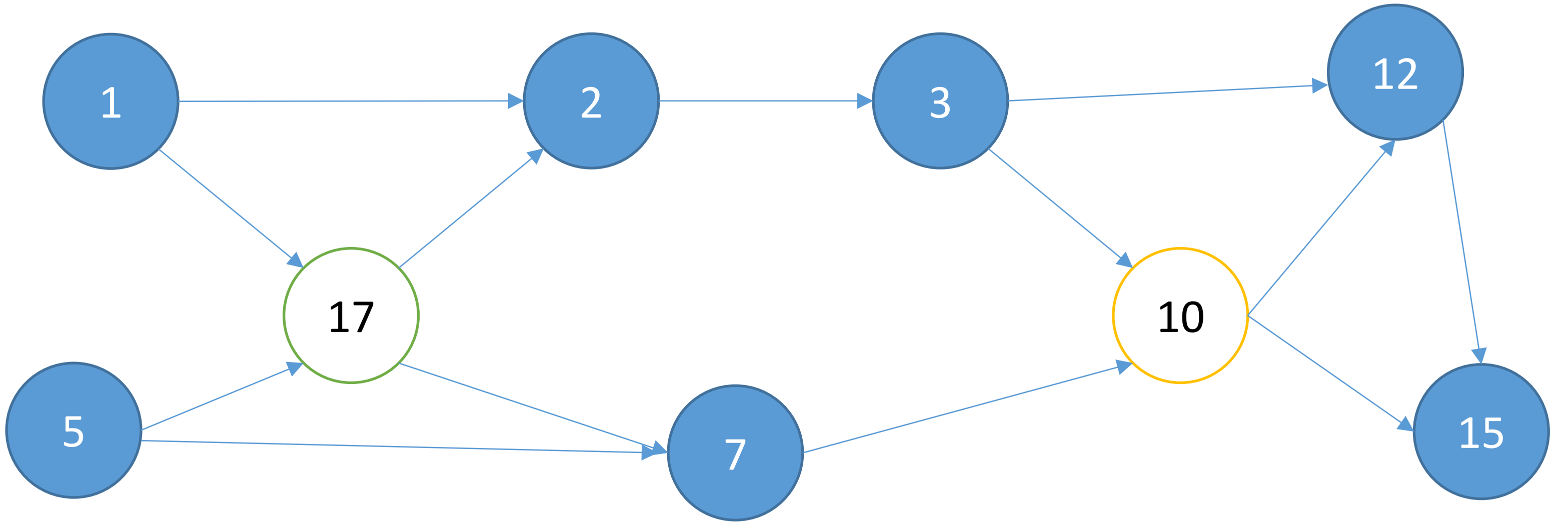


Example Substructure

Optimal Substructure Lemma:

- Fix a source vertex i, a destination vertex j, and a value for k
- Then let P be the shortest i → j path with **<u>internal</u>** nodes from $V^{(k)}$

i = 17
j = 10



Example Substructure

Optimal Substructure Lemma:

- Fix a source vertex i, a destination vertex j, and a value for k
- Then let P be the shortest i → j path with **internal** nodes from $V^{(k)}$

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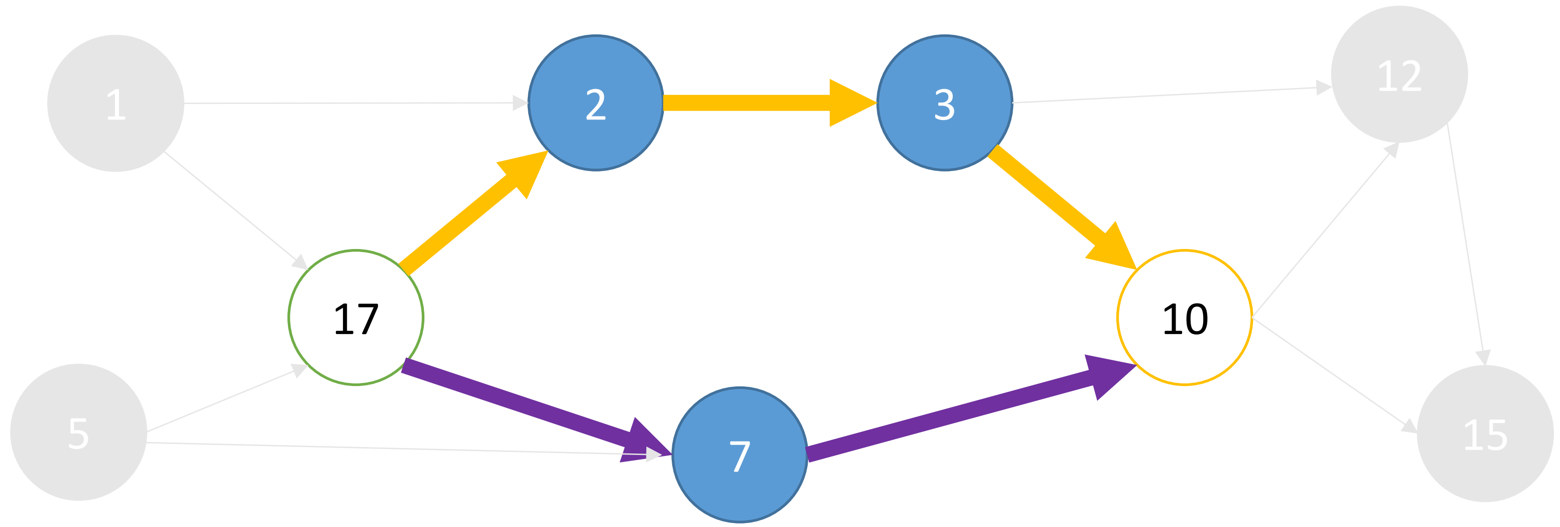
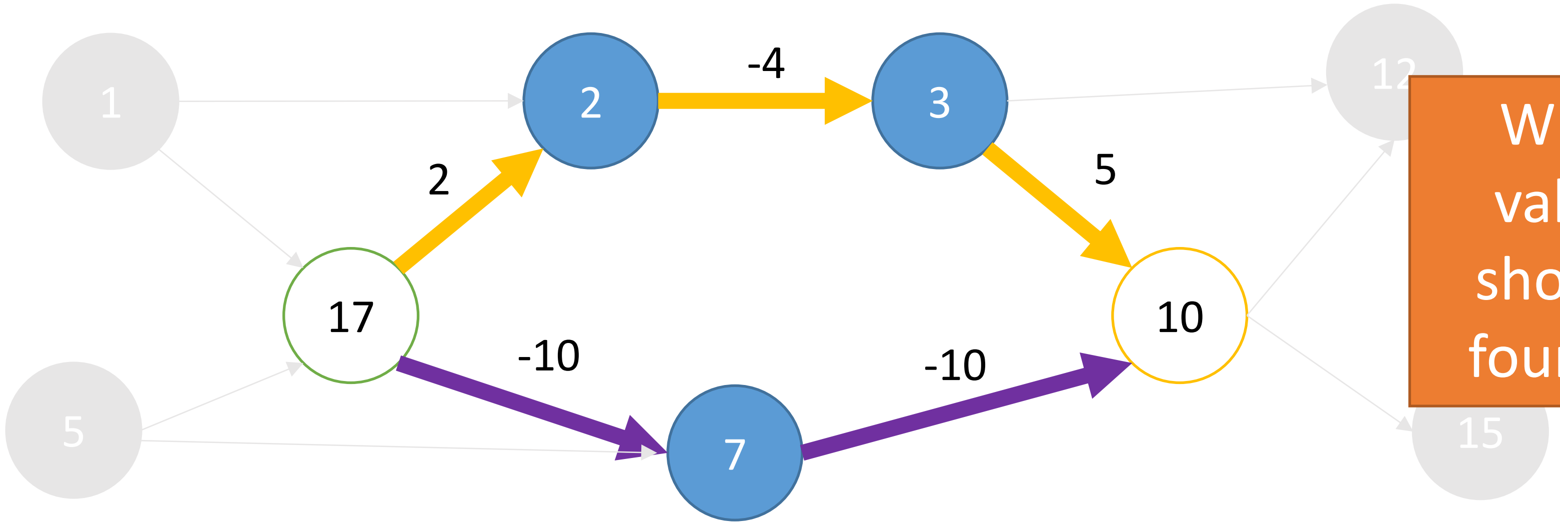


Example Substructure

Optimal Substructure Lemma:

- Fix a source vertex i, a destination vertex j, and a value for k
- Then let P be the shortest i → j path with **internal** nodes from $V^{(k)}$

i = 17
j = 10



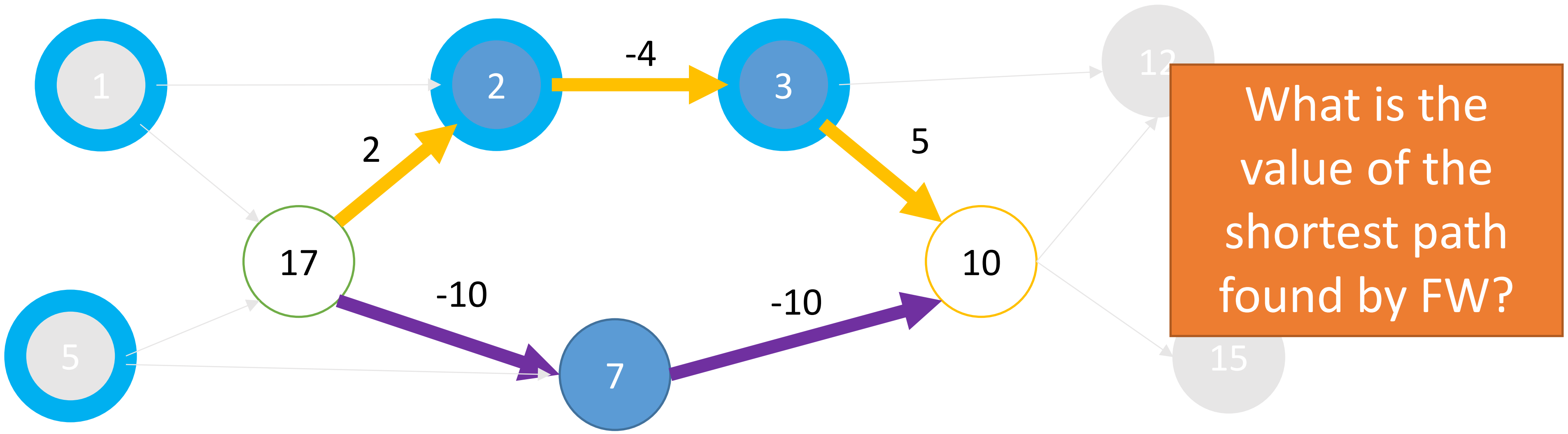
What is the value of the shortest path found by FW?

Example Substructure

Optimal Substructure Lemma:

- Fix a source vertex i, a destination vertex j, and a value for k
- Then let P be the shortest i → j path with **internal** nodes from $V^{(k)}$

i = 17
j = 10
k = 5

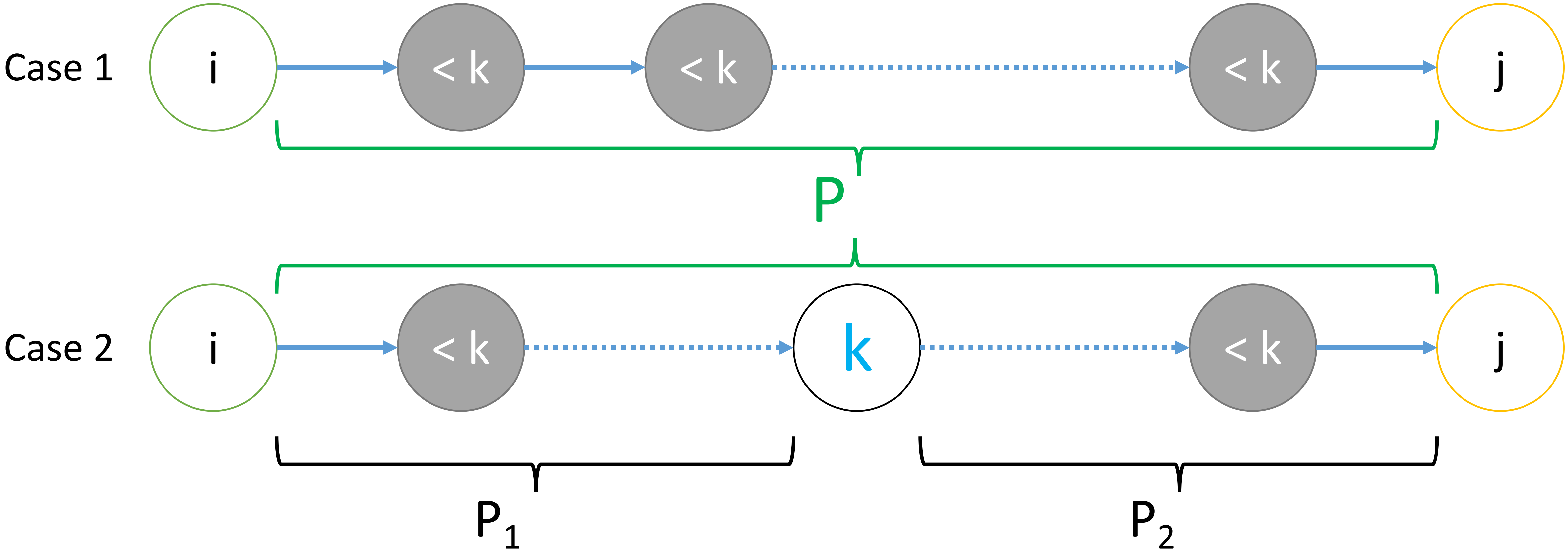


Optimal Substructure Lemma

Suppose that G has no negative cycles. Let P be the shortest (cycle-free) path $i \rightarrow j$, where all internal nodes come from $V^{(k)}$. Then:

- Case 1: if k is not internal to P, then P is also a shortest path $i \rightarrow j$ with all internal nodes from $V^{(k-1)}$.
- Case 2: if k is internal to P, then:
  - Let P_1 = the shortest $i \rightarrow k$ path with nodes from $V^{(k-1)}$, and
  - Let P_2 = the shortest $k \rightarrow j$ path with nodes from $V^{(k-1)}$
  - Effectively, k splits the path into two optimal subproblems

Picture of our cases



Floyd-Warshall Algorithm Base Cases

Let A = 3D array, where A[i, j, k] = the length of the shortest i → j path with all internal nodes from {1, 2, ..., k}

- Which index (i, j, or k) do you think represents our base case?

What is the value of A[i, j, 0] when…

- i = j? 0
- there is a direct edge from i to j c_{ij}
- there is no edge directly connecting i to j ∞

```
FUNCTION FloydWarshall(graph)
    # Base 1 indexing for vertices labeled 1 through n
    pathLengths = [n by n by (n + 1) array]

    # Base case
    FOR vFrom IN [1 ..= n]
        FOR vTo IN [1 ..= n]

            IF i == j
                length = 0

            ELSE IF graph.hasEdge(vFrom, vTo)
                length = graph.edges[vFrom][vTo].weight

            ELSE
                length = INFINITY

            pathLengths[vFrom][vTo][0] = length

    # Table building
    continued next slide…
```

```
FUNCTION FloydWarshall(graph)
    # Base 1 indexing for vertices labeled 1 through n
    pathLengths = [n by n by (n + 1) array]

    # Base case
    cut from previous slide…

    # Table building
    FOR k IN [1 ..= n]
        FOR vFrom IN [1 ..= n]
            FOR vTo IN [1 ..= n]

                # Case 1
                withoutK = pathLengths[vFrom][vTo][k - 1]

                # Case 2
                withKSubPathA = pathLengths[vfrom][k][k - 1]
                withKSubPathB = pathLengths[k][vTo][k - 1]

                pathLengths[vFrom][vTo][k] = min(
                    withoutK,
                    withKSubPathA + withKSubPathB
                )
```

Floyd-Warshall Algorithm

Running time?

- $O(n^3)$

Correctness?

- Substructure lemma

```
# Table building
FOR k IN [1 ..= n]
   FOR vFrom IN [1 ..= n]
      FOR vTo IN [1 ..= n]

         # Case 1
         withoutK = pathLengths[vFrom][vTo][k - 1]

         # Case 2
         withKSubPathA = pathLengths[vfrom][k][k - 1]
         withKSubPathB = pathLengths[k][vTo][k - 1]

         pathLengths[vFrom][vTo][k] = min(
            withoutK,
            withKSubPathA + withKSubPathB
         )
```

- Where are the final answers?
- How does it handle negative cycles?
- Reconstruction is similar to other dynamic programming problems.