Kruskal’s MST Algorithm

https://cs.pomona.edu/classes/cs140/

Outline

Topics and Learning Objectives

- Introduce Kruskal’s algorithms for MSTs
- Discuss disjoint sets

Exercise

- Kruska’s exercise

Extra Resources

- Introduction to Algorithms, 3rd, chapter 23
- Algorithms Illuminated Part 3, Chapter 15

Trick Question for the Day

Which is asymptotically bigger?

$$O(m \lg n) \ or \ O(m \lg m)$$

Minimum-Spanning-Tree Overview

Input: an undirected graph where each edge has an associated cost

Output: a minimum-spanning-tree

1. Connects the entire graph as a tree, but
2. Has a minimal cost

Assumptions:

1. The input graph is connected
2. The edges costs are distinct (only necessary/useful for our proof)

Cut Property: if e is the cheapest edge crossing a cut, then it must be in the MST

Kruskal’s

A greedy algorithm for finding the minimum spanning tree

Why are we learning another one?

- Kruskal’s will motivate a new data structure: Union-Find (disjoint-set)
- It will also let us talk a bit about clustering

Can you think of another greedy algorithm for solving MST?

Kruskal’s Minimum Spanning Tree Algorithm

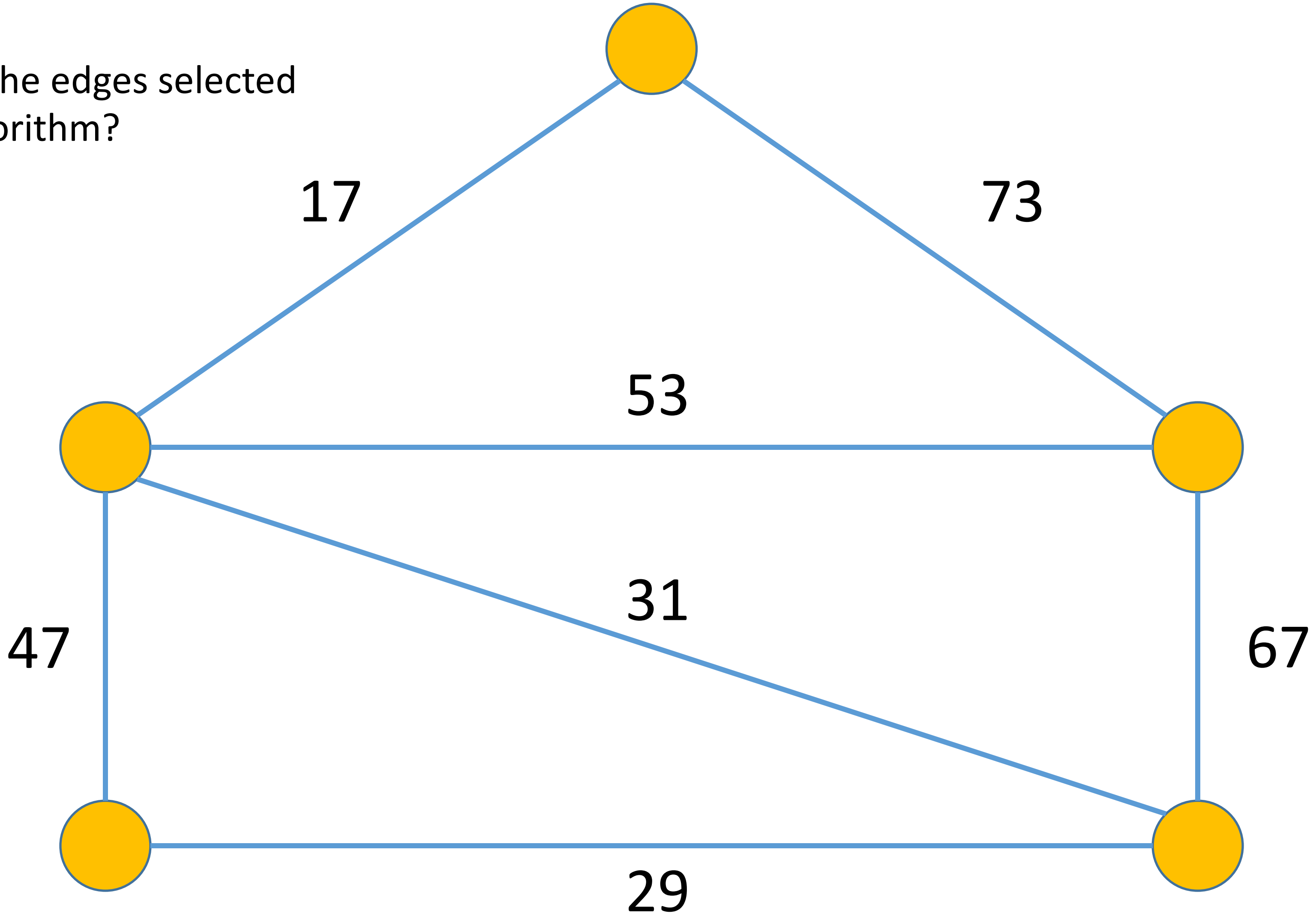
```
Sort E by edge cost
T = empty

For e in E:
    if T U {(u, v)} has no cycles
        add e to T
```

Edge based

Exercise question 1.

1. In what order are the edges selected using Kruskal's Algorithm?



Proof of Kruskal's Algorithm

Theorem: Kruskal's algorithm is correct (computes the MST)

Let T* = the output of Kruskal's algorithm

Graph/Cut/Tree Lemmas and Properties

- Empty Cut Lemma: a graph is not connected if there exists a cut (A, B) with zero crossing edges
- Double Crossing Lemma: suppose the cycle C has an edge crossing the cut (A, B), then there must be at least one more edge in C that crosses the cut
- No Cycle Corollary: if e is the only edge crossing some cut (A, B), then it is not in any cycle
- Cut Property: if e is the cheapest edge that crosses the cut (A, B) then it must be in the MST

Proof of Kruskal's Algorithm

Theorem: Kruskal's algorithm is correct (computes the MST)

Let T* = the output of Kruskal's algorithm

Does Kruskal's output a spanning tree (what are the properties)?

- No cycles
- Connected

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Proof of Kruskal’s Algorithm

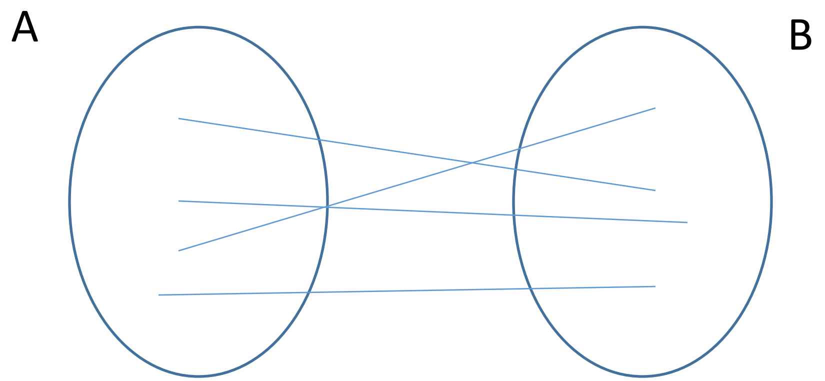
Theorem: Kruskal’s algorithm is correct (computes the MST)

Let T* = the output of Kruskal’s algorithm

Does Kruskal’s output a spanning tree (what are the properties)?

- No cycles (this is given by the definition of the algorithm)
- Connected

Proof of Kruskal's Algorithm



Proof of Connectivity

- Given the Empty Cut Lemma, we only need to show that Kruskal's produces a tree T* that crosses every cut.

- Fix a cut (A,B)
- Since G is connected, at least one of its edges crosses (A,B)
- Kruskal's algorithm considers each edge once
- Let's fast-forward to the first time that it encounters an edge crossing (A,B)
- Claim: this 1st edge is guaranteed to be in T*
- Given the No Cycle Corollary the claim is true
- It is also the minimum edge to cross the cut (sorted edges)

Proof of Kruskal's Algorithm

For the second part of the proof, we need to prove that T* is minimal

- We just finished proving that Kruskal's outputs some spanning tree T*

Claim: every edge is justified by the Cut Property

- Remember that satisfying the Cut Property implies that we have an MST
- This was very explicit in Prim's Algorithm

Prim's Minimum Spanning Tree Algorithm

```
X = {s}
T = empty

while X is not V:
    let e = (u, v) be the cheapest edge of G
        with u in X and v not in X
    add e to T
    add v to X
```

Proof of Kruskal's Algorithm

Proving that we can use the <u>Cut Property</u>

- Consider each iteration where edge (u, v) is added to T*
- Since T* U {(u, v)} has no cycle, T* currently has no u->v path
- Thus, there must be a cut (A, B) separating u and v. For example:
  - All findable from u in A
  - All findable form v in B
  - All other vertices can be partitioned arbitrarily
- Hence, (u, v) is the first crossing cut for (A, B)
- Additionally, it must be the cheapest such cut since we sorted the edges
- Finally, the edge (u, v) is justified by the <u>Cut Property</u>

Proof of Kruskal's Algorithm

What have we done?

We proved that Kruskal's outputs a spanning tree

- No cycles by definition
- Connectivity by the Empty Cut Lemma

We then proved that Kruskal's outputs the minimum spanning tree

- The Cut Property implies that we are left with the MST
- We showed that Kruskal's uses the Cut Property because the edges are sorted

Implementation of Kruskal’s

Kruskal’s Minimum Spanning Tree Algorithm

```
Sort E by edge cost
T = empty

For e in E:
    if T U {(u, v)} has no cycles
        add e to T
```

$O(m \lg m)$

$O(m)$

How would you detect if adding (u,v) creates creates a cycle?

Kruskal's Minimum Spanning Tree Algorithm

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Sort E by edge cost
T = empty

For e in E:
    if T U {(u, v)} has no cycles
        add e to T
```

$O(m \lg m)$

$O(m)$

Naïvely $O(n+m)$

O(m lg m) + O(m) * O(n+m)

O(mn+m^2)

Kruskal’s Minimum Spanning Tree Algorithm

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Sort E by edge cost
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For e in E:
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```

What can we change (should we change) to do better?

Kruskal's Minimum Spanning Tree Algorithm

```
Sort E by edge cost
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For e in E:
    if T U {(u, v)} has no cycles
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```
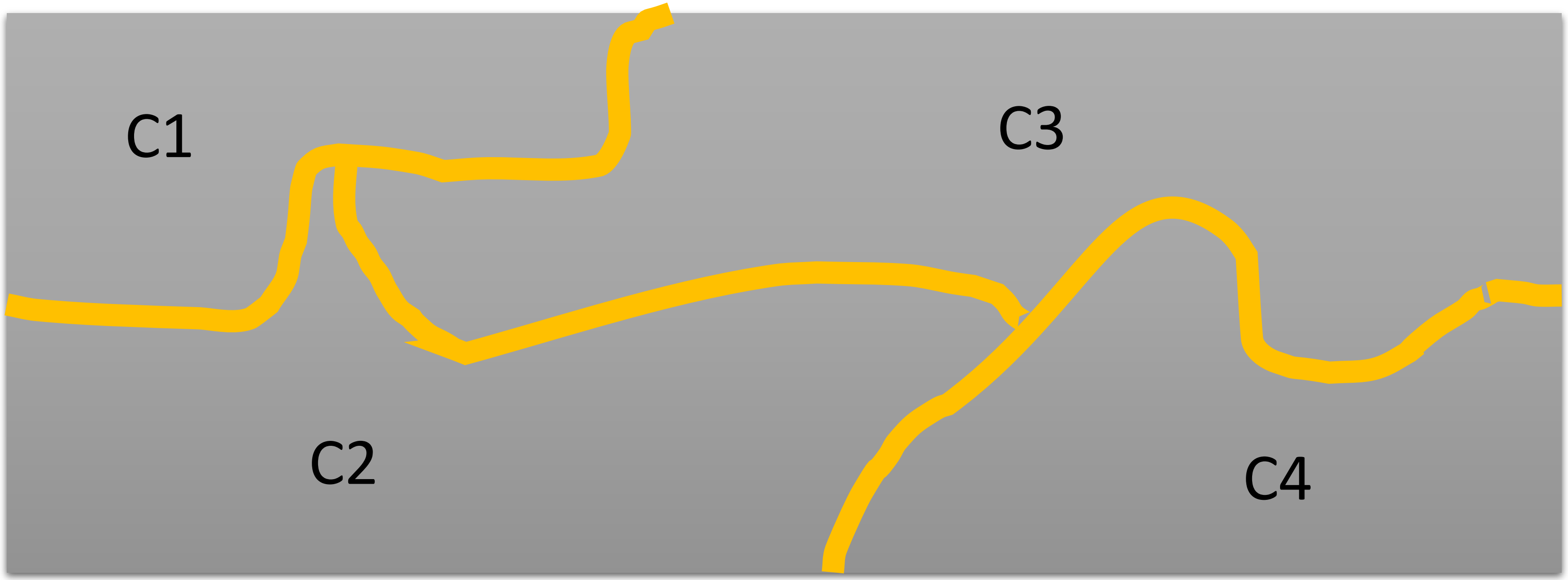

What can we change (should we change) to do better?

The Union-Find Data Structure

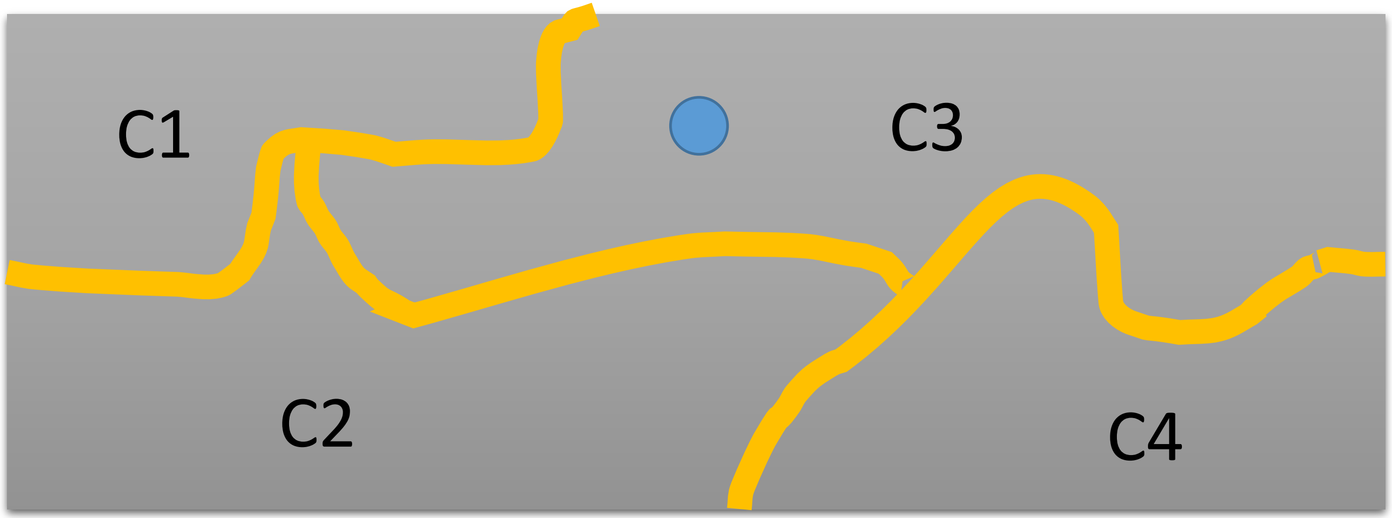
- Also known as the disjoint-set data structure
- Used to maintain a partition of objects



Union-Find

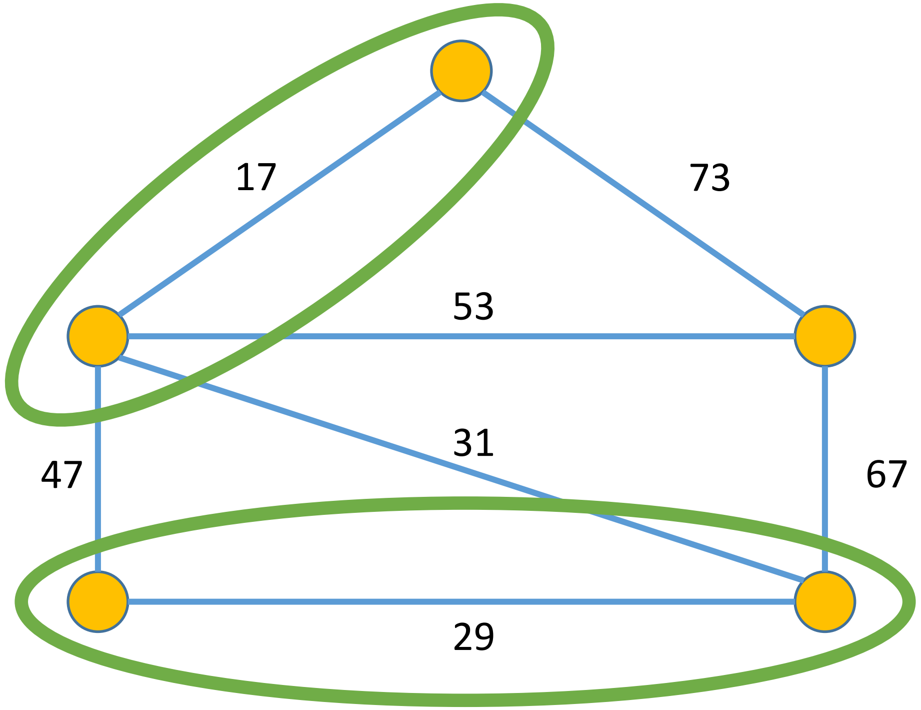
Operations:

- Find(x): return the name of the group to which x belongs
- Union(Ci, Cj): merge the two partitions into a single partition



How does this help us with Kruskal's?

- What do we store in the data structure?
- What makes a group/partition?



Motivation

- Speed up the way in which we check for cycles.
- How would you implement the Union-Find data structure?
- Augment each vertex to include another piece of information: the name of its leader
  - Or use a separate data structure (what kind? → what operations matter?)

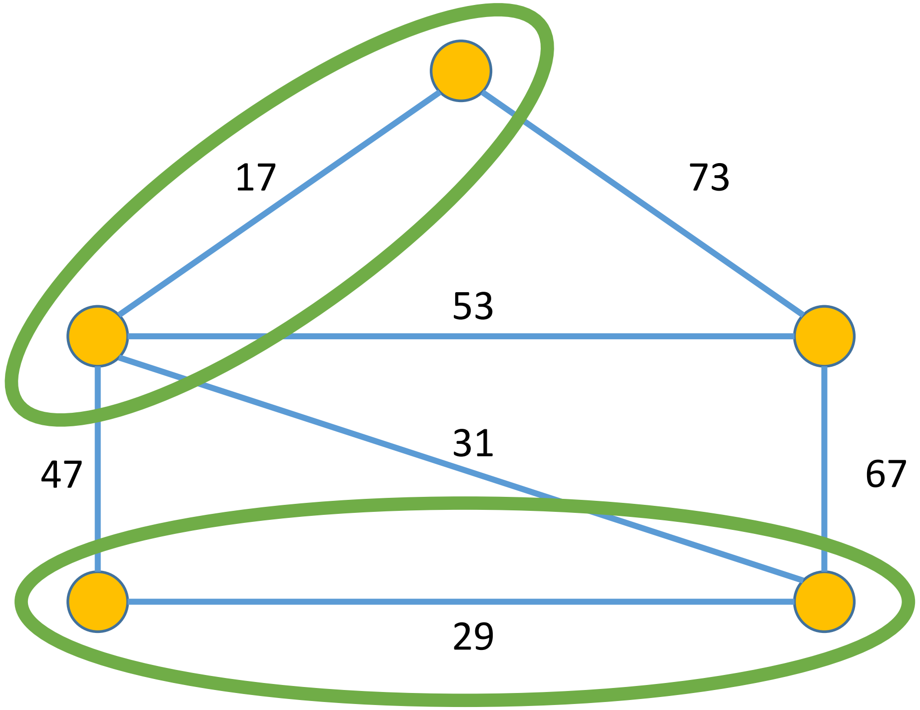
- <u>Invariant: each vertex knows its leader</u>

- How long does it take to check for a cycle now?

Checking for cycles

- Given an edge (u, v), we can check if u and v are in the same partition in constant time O(1).

$$Find(u) == Find(v)?$$

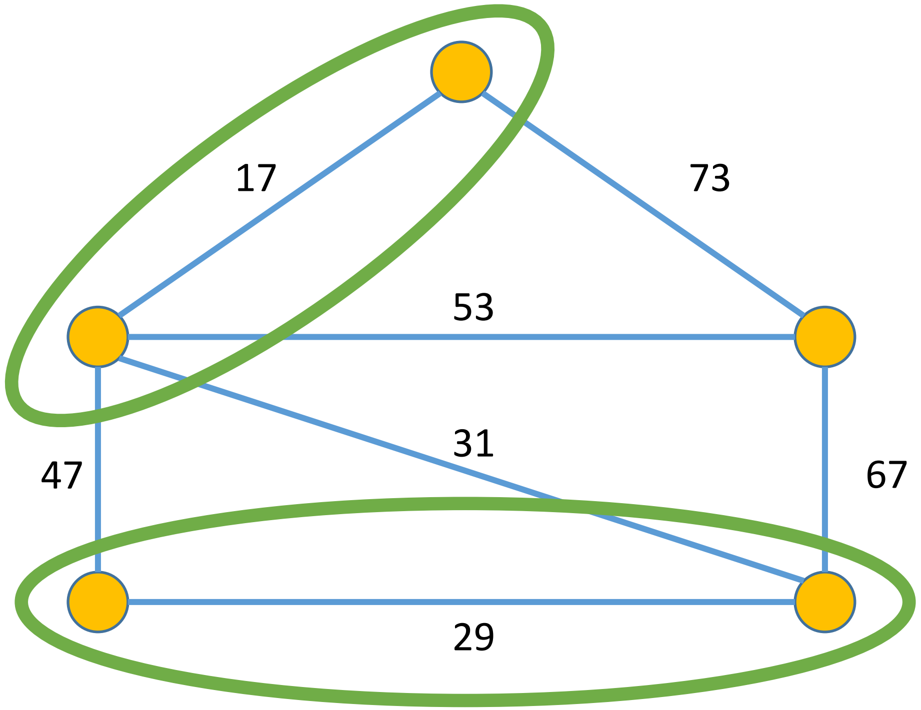


What happens during the next iteration?

What's the catch?

Maintaining the Invariant

- <u>Invariant: each vertex knows its leader</u>



What is the maximum number of vertex leaders that must be fixed after a union?

Exercise Question 2

Union example.

Union-Find Data Structure

- Put every element in its own partition
  - Every element has its own leader

- Join partitions by copying the leader of the larger partition elements to all elements of the smaller partition

- You can use an array or hash table to keep track of leaders

- No other information/memory is needed

Kruskal's Minimum Spanning Tree Algorithm

```
Sort E by edge cost
T = empty

For e in E:
    if T U {(u, v)} has no cycles
        add e to T
        union
```

$O(m \lg m)$

$O(m)$

What do we have as a running time now?

What happened?

```
Sort E by edge cost
T = empty

For e in E:
    if T U {(u, v)} has no cycles
        add e to T
        union
```

$O(m \lg m)$

$O(m)$

O(n+m) → O(1) (checking leaders)

We don't do this every iteration

O(1) → O(n) (updating leaders)

Maximum number of leader updates?

How many times can we update the leader of a single vertex?

- We only update the leader of a vertex if we merge it with a bigger partition.
- How many times can we update a vertex's leader?
    - (Or: How many times can we double the size of a partition?)

This is our global view of something happening inside the loop.

Kruskal's Minimum Spanning Tree Algorithm

```
Sort E by edge cost
T = empty

For e in E:
    if T U {(u, v)} has no cycles
        add e to T
        union
```

$O(m \lg m)$

$O(m)$

$O(1)$ just for the cycle check

$O(n \lg n)$ for Union (not per iteration)

Technically this is O(n lg n + m lg m)

O(m lg m)	+	O(n lg n)	+	O(m) * O(1)	=	O(m lg m)
Sort		Union		Loop		Total

Cutting Edge

- Can we do better than O(m lg n)?
- Yes!
- **Average** O(m) using a randomized algorithm (1995)
- We do not actually know if a deterministic O(m) algorithm exists.
- We do have a deterministic algorithm that is O(m α(n))
- α is the inverse Ackermann function
- Which is slower than the Iterated logarithm: lg*
    - the number of times the logarithm function must be iteratively applied before the result is less than or equal to 1
- An optimal deterministic algorithm was developed in 2002
- But we do not know the exact asymptotic complexity
- Just that it is between O(m) and O(m α(n))

x	$\lg^* x$
$(-\infty, 1]$	0
$(1, 2]$	1
$(2, 4]$	2
$(4, 16]$	3
$(16, 65536]$	4
$(65536, 2^{65536}]$	5