Minimum Spanning Tree

https://cs.pomona.edu/classes/cs140/

Outline

Topics and Learning Objectives

• Discuss spanning tree and minimum spanning trees (MSTs)

• Introduce Prim's algorithms for MSTs

• Prove correctness of Prim's MST Algorithm

Exercise

• MST exercise questions 1 and 2

Extra Resources

- Introduction to Algorithms, 3rd, chapter 23
- Algorithms Illuminated Part 3, Chapter 15

Minimum Spanning Tree

*Given a graph, connect all points together as **cheaply** as possible.*

Why are we talking about this?

- It is a fundamental graph problem,

- It has several greedy-based solutions,

- And it has many applications:
    - Clustering
    - Networking
    - Many more

Greedy Solution

- Otakar Borůvka in 1926
- Vojtěch Jarník in 1930
    - Rediscovered by Robert Prim in 1957
    - Rediscovered by Edsger Dijkstra in 1959
- Joseph Kruskal in 1956

Blazingly fast algorithm for what you get as output:

- Can run in O(m lg n)
- Remember: it takes O(n + m) just to read the graph!
- There are an exponential number of possible spanning trees

Minimum Spanning Tree

Input: a weighted, <u>undirected</u> graph G = (V, E)

- A similar problem can be constructed for directed graphs, and it is then called the optimal branching problem

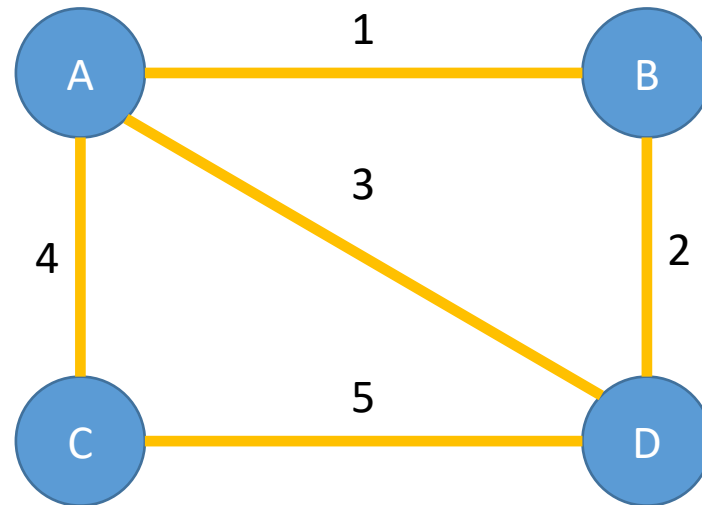- Each edge e has a cost c_e

- **Costs can be negative**

Output: the minimum cost tree T that spans all vertices

- Calculate cost as the sum of all edge costs

- What does it mean to **span** a graph?

- The tree T is just a subset of E

Spanning Tree Properties

1. The <u>spanning tree</u> T does not have any cycles

2. The subgraph (V, T) is <u>connected</u>

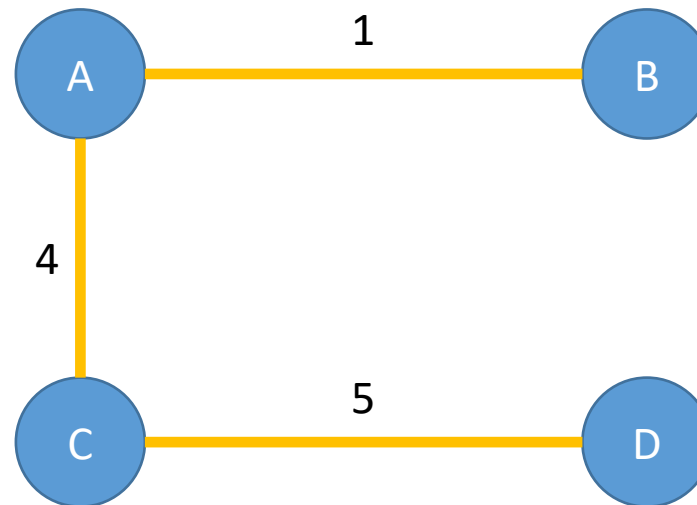What is a spanning tree for this graph?

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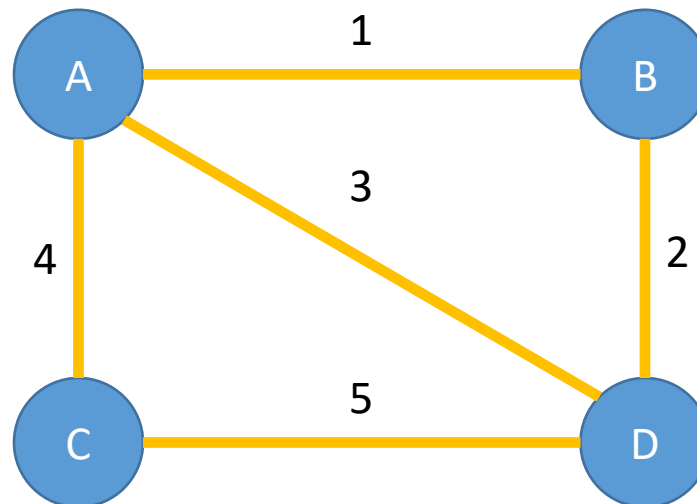This is **<u>not</u>** the **<u>minimum</u>** spanning tree

Our MST Problem Assumptions

1. The input graph is connected
   - This is easy to check. How?
   - Otherwise we're looking at the minimum spanning forest problem

2. Edge costs are distinct
   - All mentioned algorithms are correct with ties, but
   - It makes our correctness proof much easier if we assume no ties

Prim's Algorithm (aka Jarník's or Dijkstra's)

- A greedy algorithm that finds an MST for a weighted, undirected graph.

- It finds a subset of the edges that forms a tree that includes every vertex, where the total weight of all the edges in the tree is minimized.

What is a good criteria for finding the minimum spanning tree?

What is the minimum spanning tree for this graph?

Prim's Algorithm

- A greedy algorithm that finds an MST for a weighted, undirected graph.

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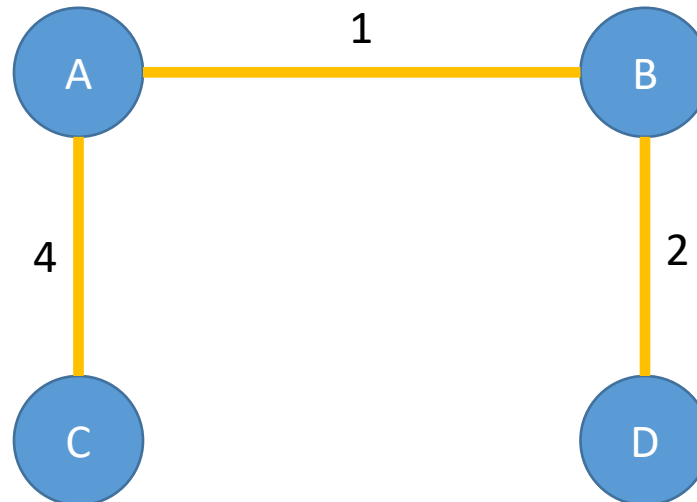
What is the minimum spanning tree for this graph?

```
FUNCTION Prims(G, start_vertex)
    found = {start_vertex}
    mst = {}
    mst_cost = 0

    WHILE found.size != G.vertices.size

        min_weight, min_edge = INFINITY, NONE
        FOR v IN found
            FOR vOther, weight IN G.edges[v]
                IF vOther NOT IN found
                    IF weight < min_weight
                        min_weight = weight
                        min_edge = (v, vOther)

        found.add(min_edge[1])
        mst.add(min_edge)
        mst_cost = mst_cost + min_weight

    RETURN mst, mst_cost
```

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How does this compare with Dijkstra's Algorithm?

Each iteration:

Extend MST in cheapest manner possible

Proof of Prim's

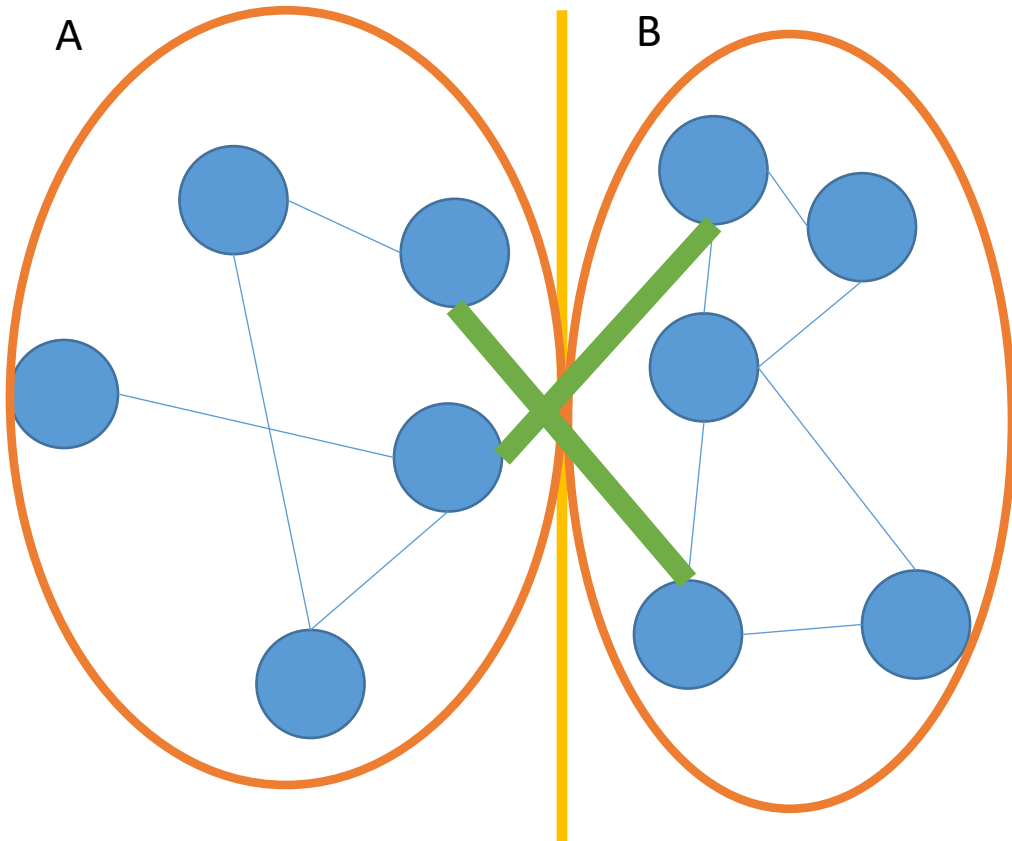Theorem: *Prim's algorithm always computes the (or a) MST when given a connected graph.*

Need to prove two things:

1. That Prim's algorithm creates a spanning tree T*

2. And that T* is the **minimum** spanning tree

We need to define a few things before we conduct the proof

Graph Cuts

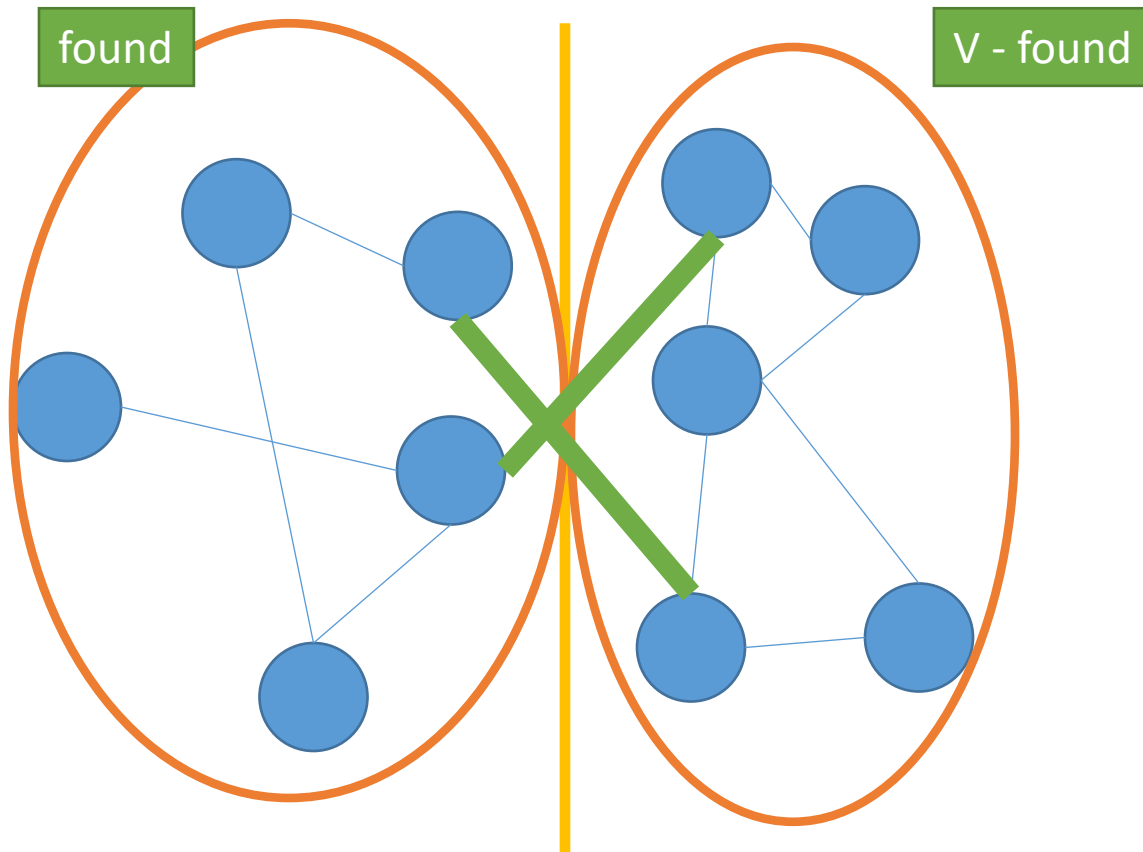*A cut of any graph G = (V, E) is a partition of V into two non-empty groups*

For a graph with n vertices, how many possible cuts are there?

a. $O(n)$
b. $O(n^2)$
c. $O(2^n)$
d. $O(n^n)$

Graph Cuts

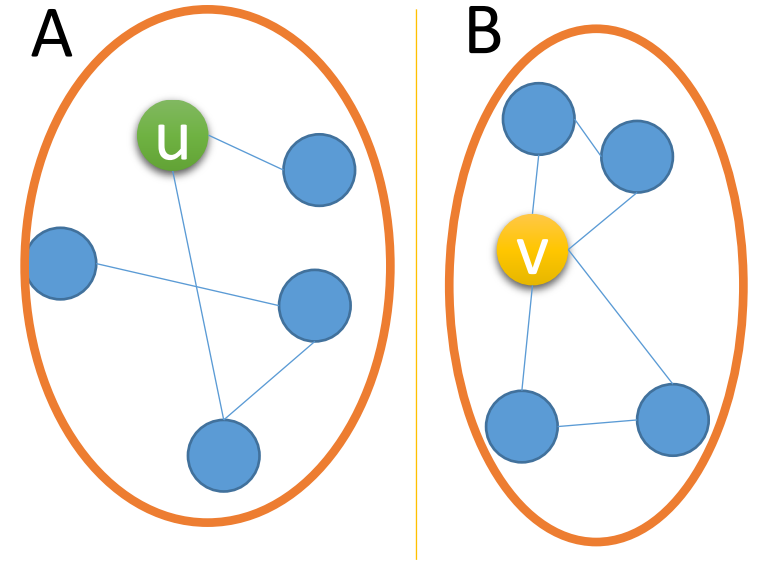*A cut of any graph G = (V, E) is a partition of V into two non-empty groups*

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V - found

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a. $O(n)$
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Lemma 1: Empty Cuts

*Empty Cut Lemma: a graph is **not connected** if there exists a cut (A, B) with zero crossing edges.*
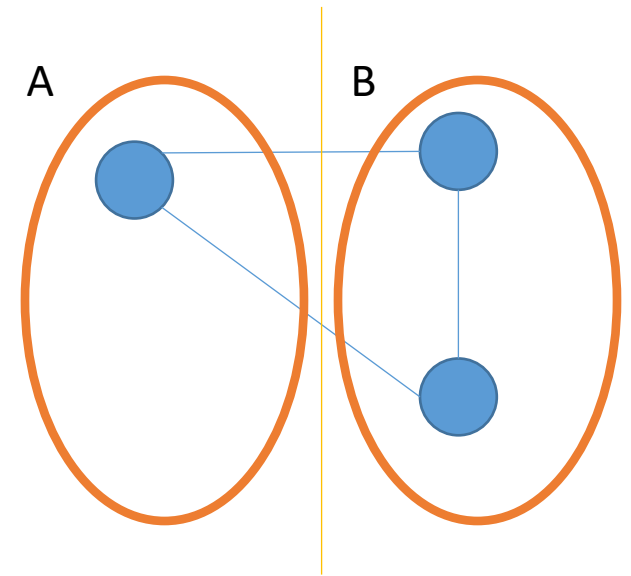
Proof A:

- Assume we have a cut with zero crossing edges
- Pick any u in A and v in B
- There is no path from u to v
- Thus the graph is not connected

Proof B:

- Assume the graph is not connected
- Suppose G has no path from u to v
- Put all vertices reachable from u into A
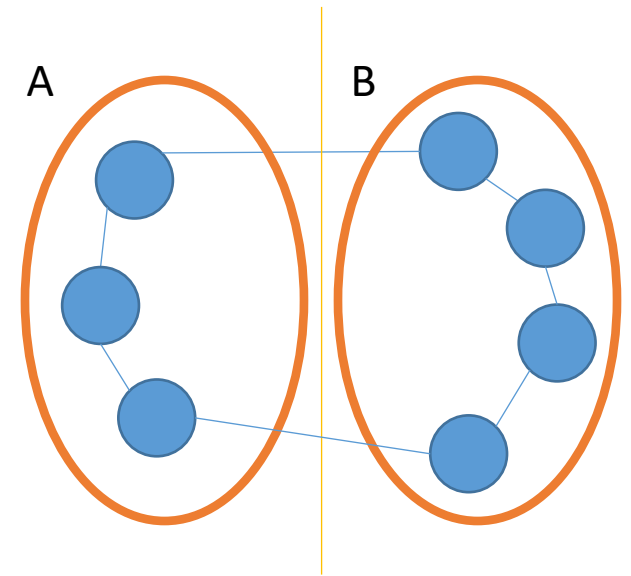- Put all other vertices in B
- Thus, no edges cross the cut

Lemma 2: Double-Crossings

double-crossing Lemma: suppose the cycle C has an edge crossing the cut (A, B). Then, there must be at least one more edge in C that crosses the cut.

A   B

Lemma 2: Double-Crossings

double-crossing Lemma: suppose the cycle C has an edge crossing the cut (A, B). Then, there must be at least one more edge in C that crosses the cut.

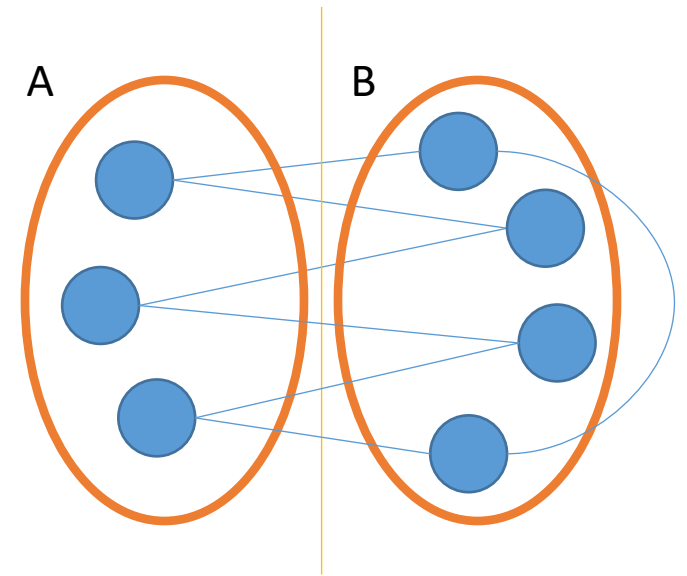# Lemma 2: Double-Crossings

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Lemma 2: Double-Crossings

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No Cycle Corollary: if e is the only edge crossing some cut (A, B), then it is not in any cycle.
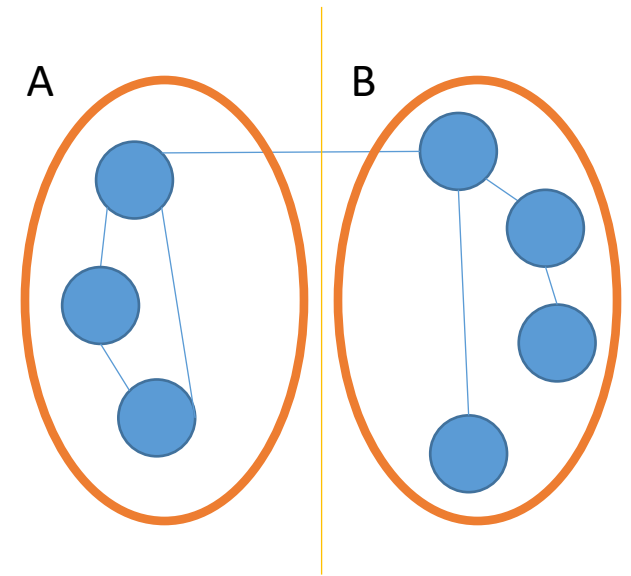
Proof of Prim's

Theorem: *Prim's algorithm always computes the (or a) MST when given a connected graph.*

Need to prove two things:

1. That Prim's algorithm creates a spanning tree T*

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We'll use graph cuts, the double-crossing lemma, and the no-cycle lemma in this proof.

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Theorem: *Prim's algorithm always computes the (or a) MST when given a connected graph.*

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1. That Prim's algorithm creates a spanning tree T*

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Claim 1: Prim's outputs a spanning tree

1. Prim's algorithm maintains the invariant that mst spans found

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FUNCTION Prims(G, start_vertex)
    found = {start_vertex}
    mst = {}
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        min_weight, min_edge = INFINITY, NONE
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                        min_weight = weight
                        min_edge = (v, vOther)
        found.add(min_edge[1])
        mst.add(min_edge)
        mst_cost = mst_cost + min_weight
    RETURN mst, mst_cost
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Claim 1: Prim's outputs a spanning tree

1. Prim's algorithm maintains the invariant that T spans X

Simplified Pseudocode for Prim's Algorithm

```
X = {s}    // list of found nodes
T = empty  // edges that belong to MST


while X is not V:
      let e = (u, v) be the cheapest edge of E
             with u in X and v not in X
      add e to T
      add v to X
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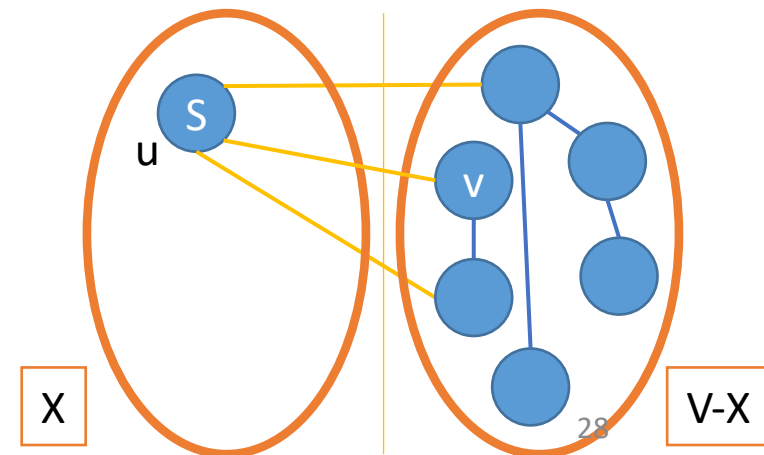
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1. Prim's algorithm maintains the invariant that T spans X

Assume the graph is <u>connected</u>.

X

V-X

28

Simplified Pseudocode for Prim's Algorithm

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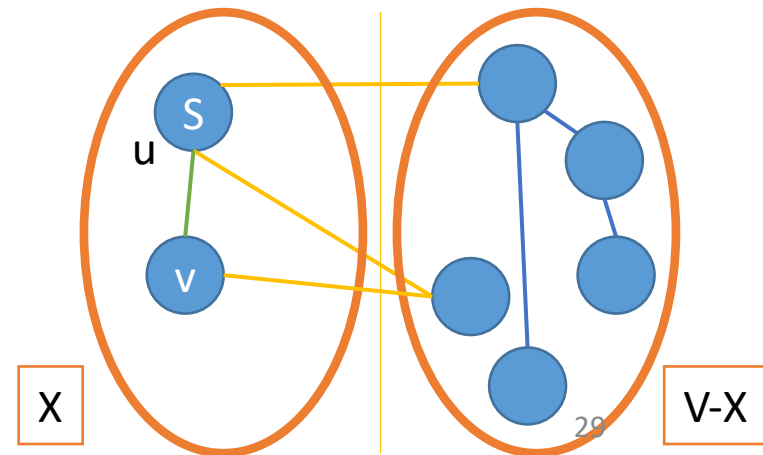T = empty // edges that belong to MST

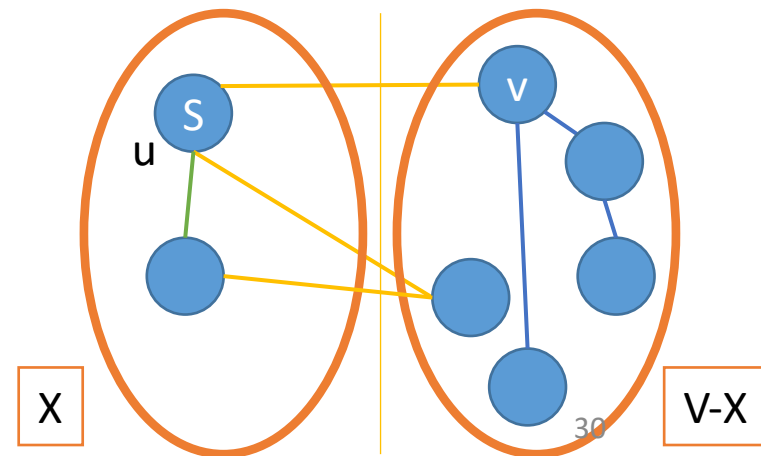> 1. Prim's algorithm maintains the invariant that T spans X

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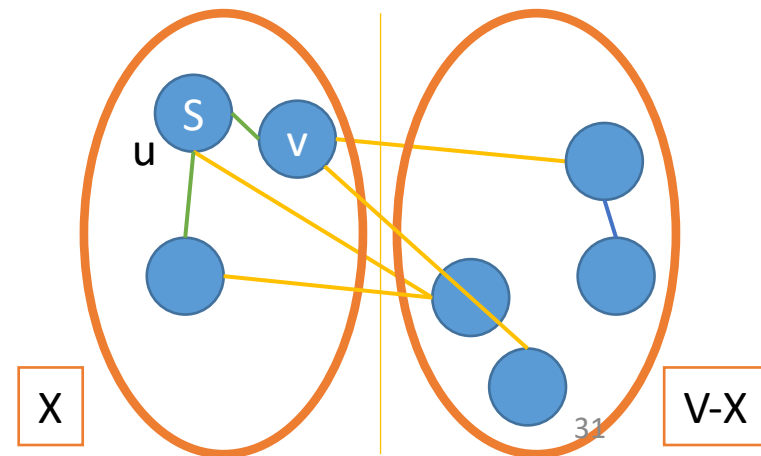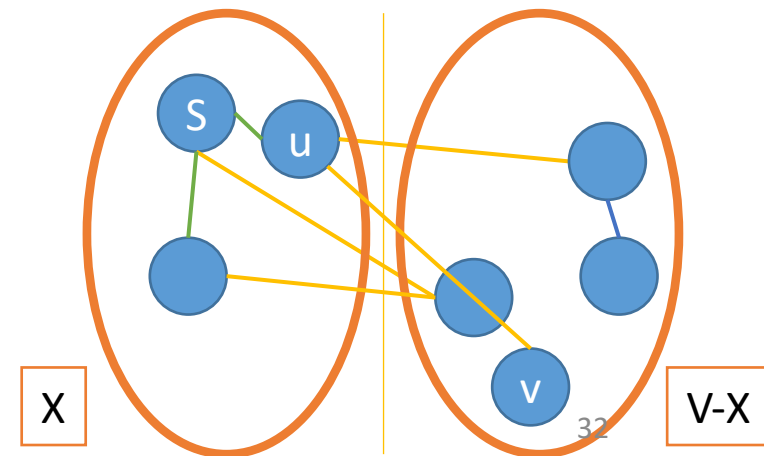
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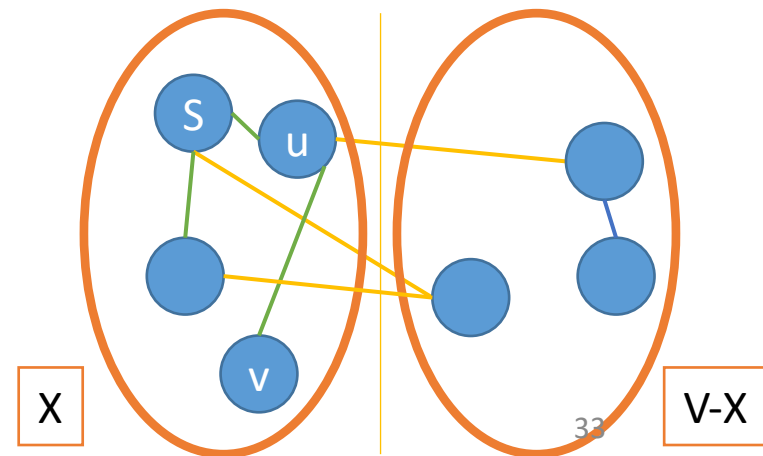
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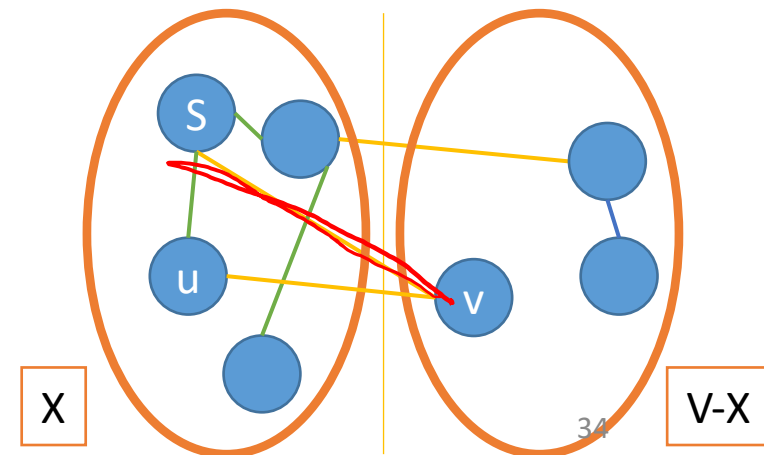
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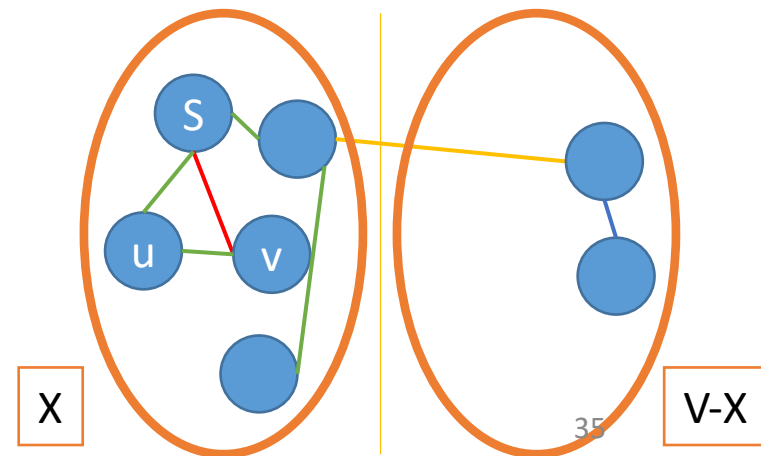> 1. Prim's algorithm maintains the invariant that T spans X

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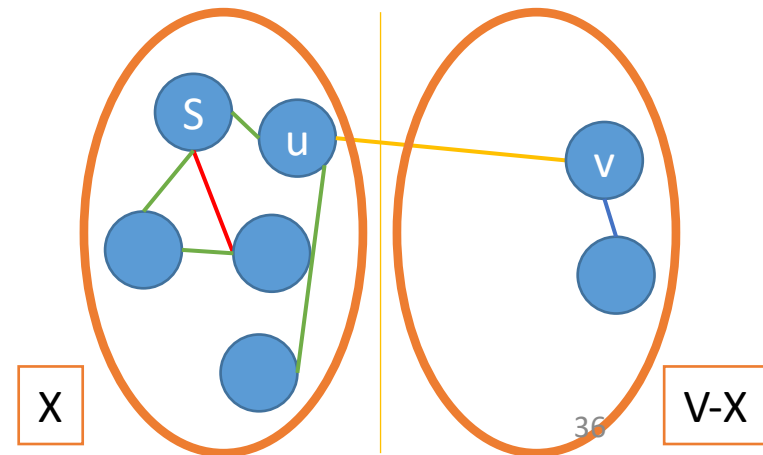
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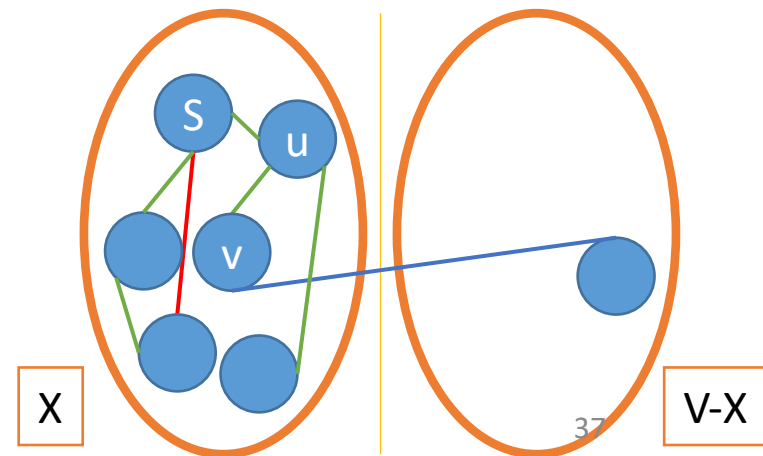> 1. Prim's algorithm maintains the invariant that
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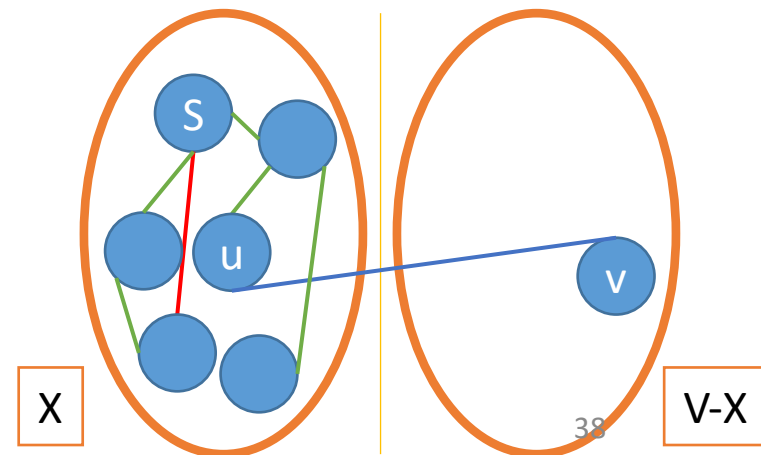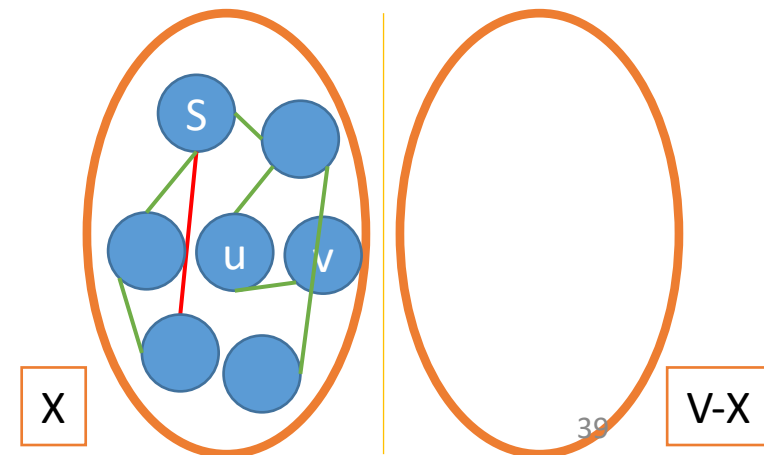
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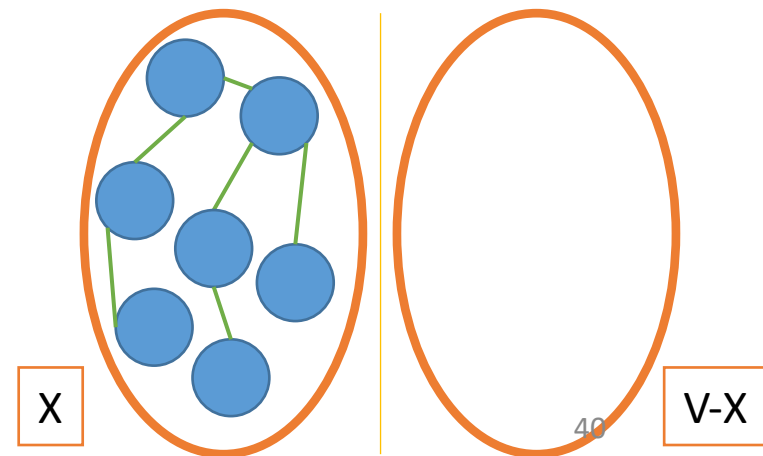
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X   V-X

Claim 1: Prim's outputs a spanning tree

1. Prim's algorithm maintains the invariant that T spans X
2. The algorithm is guaranteed to terminate with $X = V$

Simplified Pseudocode for Prim's Algorithm

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X = {s}    // list of found nodes
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while X is not V:
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    add e to T
    add v to X
```

> 2. The algorithm is guaranteed to terminate with X = V

> If the algorithm does not terminate, then by the Empty cut Lemma the input graph must be disconnected.

Claim 1: Prim's outputs a spanning tree

1. Prim's algorithm maintains the invariant that T spans X
2. The algorithm is guaranteed to terminate with $X = V$
3. The set of edges, T, does not contain any cycles

Simplified Pseudocode for Prim's Algorithm

```
X = {s}     // list of found nodes
T = empty   // edges that belong to MST
```

> 3. The set of edges, T, does not contain any cycles

```
while X is not V:
        let e = (u, v) be the cheapest edge of E
            with u in X and v not in X
        add e to T
        add v to X
```

> By the <u>No cycle corollary</u>, the addition of e cannot create a cycle (it is the only edge to cross the cut).

Claim 1: Prim's outputs a spanning tree

1. Prim's algorithm maintains the invariant that T spans X

2. The algorithm is guaranteed to terminate with X = V
   - Could anything go wrong here?
   - Under what circumstances cannot we not find an edge to cross the cut (X, V - X)?
   - By the <u>Empty cut Lemma</u> the input graph must be disconnected
   - However, we stated that only connected graphs would be used as inputs

3. The algorithm is guaranteed to create a tree (no cycles)
   - Consider any iteration and our sets X and T
   - Suppose we add an edge e to T
   - The edge e must be the first edge to cross (X, V - X) being added to T
   - By the <u>No cycle corollary</u>, the addition of e cannot create a cycle (only edge to cross the cut)

Claim 1: <mark>Prim's outputs a spa</mark>e

1. Prim's algorithm maintains the invariant

2. The algorithm is guaranteed to te   ✓
   - Could anythi   ng here?
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   - By the Em   sconnected
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   - The edge e must be the f   (X, V - X) being added to T
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Claim 2: Prim's outputs the Minimum ST

Before we can prove that the output is an MST, we need another helper definition

- Consider an edge e of G

- Suppose you can find a cut (A, B) such that e is the cheapest edge of G that crosses (A, B)

- *Cut Property: e belongs to the MST of G*

- ***Assume that this is true! We'll prove it later***
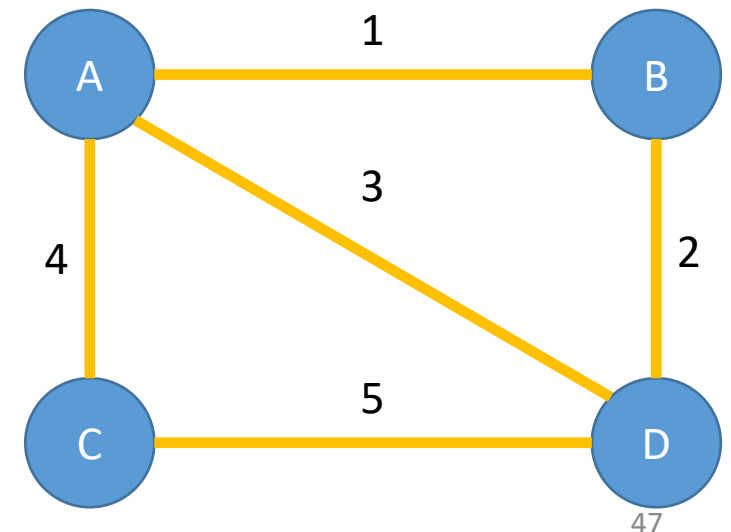
Claim 2: Prim's outputs the MST

- Claim: the <u>Cut Property</u> implies that Prim's algorithm outputs the MST

Simplified Pseudocode for Prim's Algorithm

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X = {s}    // list of found nodes
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```

> Claim: the <u>Cut Property</u> implies that Prim's algorithm outputs the MST

```
while X is not V:
    let e = (u, v) be the  cheapest  edge of G
            with u in X and v not in X
    add e to T
    add v to X
```

> Cut Property: if e is the cheapest edge that crosses the cut $(X, V - X)$ then it must be in the MST.

Claim 2: Prim's outputs the MST

- Claim: the <u>Cut Property</u> implies that Prim's algorithm outputs the MST
- Key point: every edge e in T is explicitly chosen via the cut property

At any given iteration:

- The tree T is a subset of the MST
- After termination, we are guaranteed that T is a spanning tree
- Given the cut property, we are also now guaranteed that T is minimal spanning tree

Claim 2: <mark>Prim's outputs the M</mark>

- Claim: the <u>Cut Property</u> implies that ... ...m outputs the MST
- Key point: every edge e in T is e ... ... via the cut property

At any given ...

- The tree T is ...
- After terminatio ... ... that T is a spanning tree
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Proof of Prim's

Theorem: *Prim's algorithm always computes the (or a) MST when given a connected graph.*

Need to prove two things:

1. That Prim's algorithm creates a spanning tree T*

2. And that T* is the **minimum** spanning tree ✳
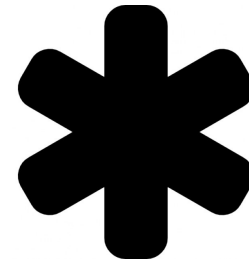
* Need to prove the cut property!

Proof of the Cut Property

Assume distinct edge costs

- Here is where our assumption of distinct edge costs is useful.

Cut Property: if e is the cheapest edge that crosses the cut (X, V − X) then it **must** be in the MST

We are going to prove this using exchange argument contradiction

Proof of the Cut Property

Claim: Suppose there is an edge e that is the cheapest one to cross a cut (X, V-X), but e is not in the MST T*

• What are we going to exchange?

Idea: exchange e with another edge in T* to make the cost of T* even cheaper (which would result in a contradiction)

What edge in T* can we swap with e?

Proof of the Cut Property

- The edge e is the cheapest to cross (X, V-X)

- MST T* must contain some other edge that crosses (X, V-X), otherwise T* would be disconnected.

- Let's call this other edge f

- Let's try to exchange e and f to get a spanning tree that is cheaper than T*
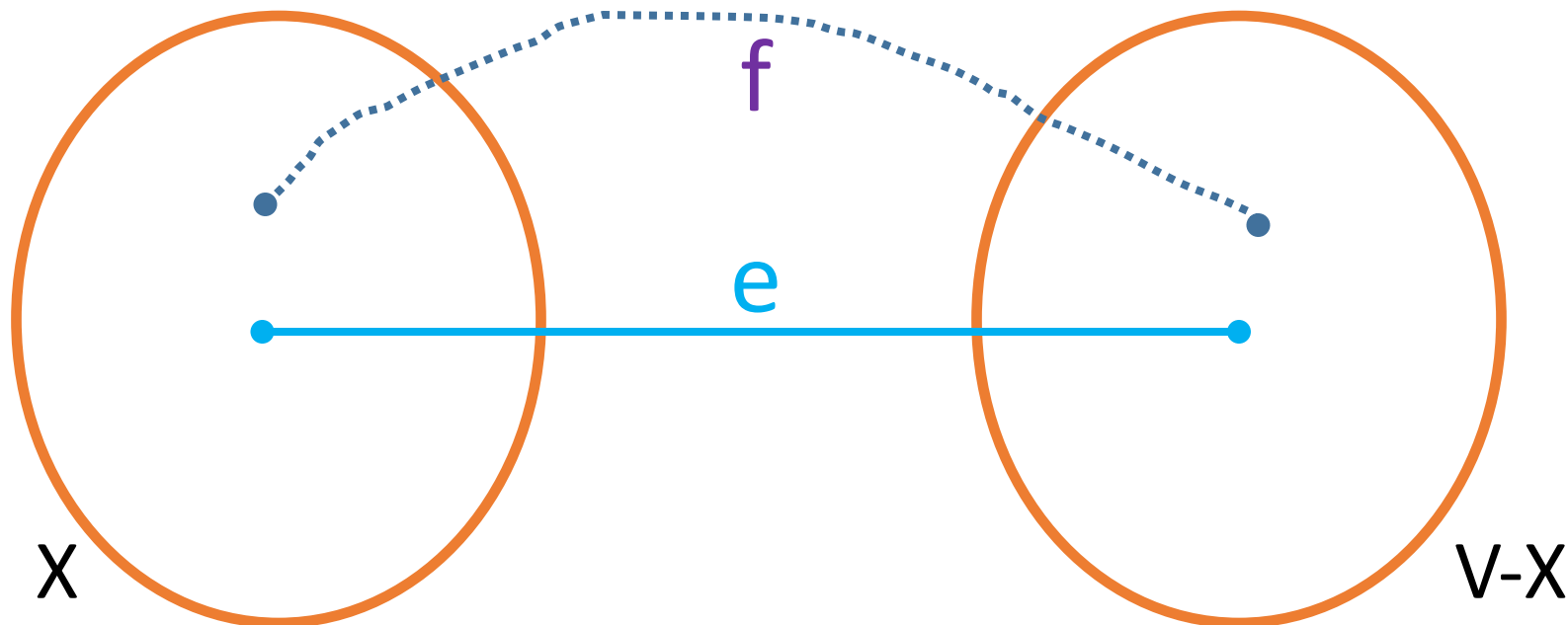
Proof of the Cut Property

Is T* ∪ {e} − {f} a spanning tree of G?
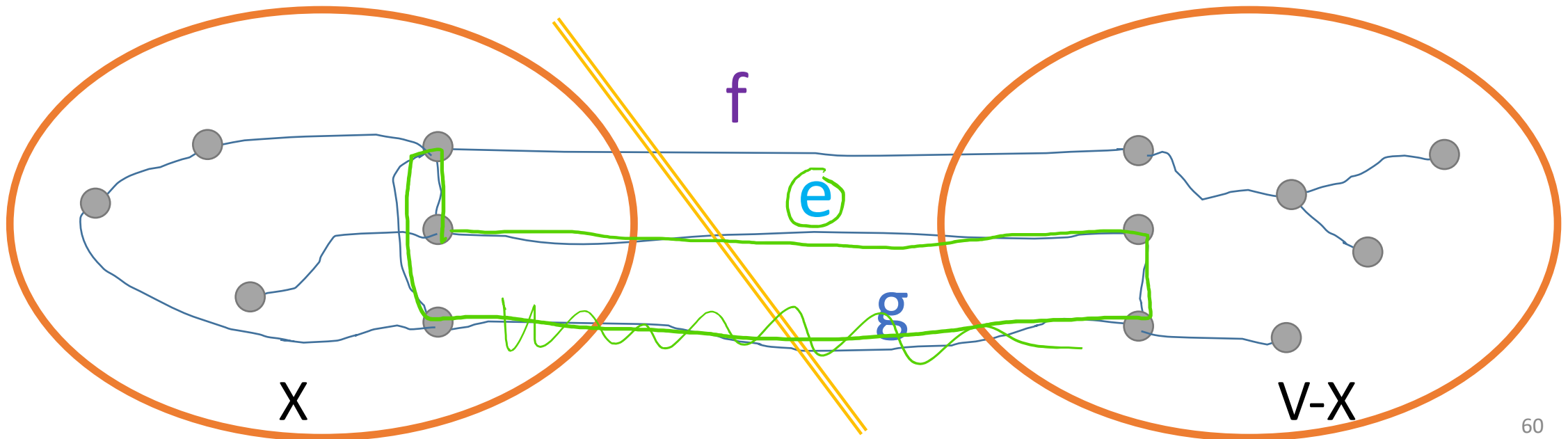
Yes	No	Only if e is the cheapest edge	Maybe

Proof of the Cut Property

Is T* U {e} − {f} a spanning tree of G?

Yes	No	Only if e is the cheapest edge	Maybe

Proof of the Cut Property

Is T* U {e} − {f} a spanning tree of G?

| Yes | No | Only if e is the cheapest edge | Maybe |

f

e

g

X

V-X

Proof of the Cut Property

Is T* U {e} − {f} a spanning tree of G?

| Yes | No | Only if e is the cheapest edge | Maybe |

f

e

g

X

V-X

60

Proof of the Cut Property

Hope: that we can always find a suitable edge e' so that exchanging edges yields a valid spanning tree

Solid green lines are those that are currently part of T*
Rainbow lines are other edges
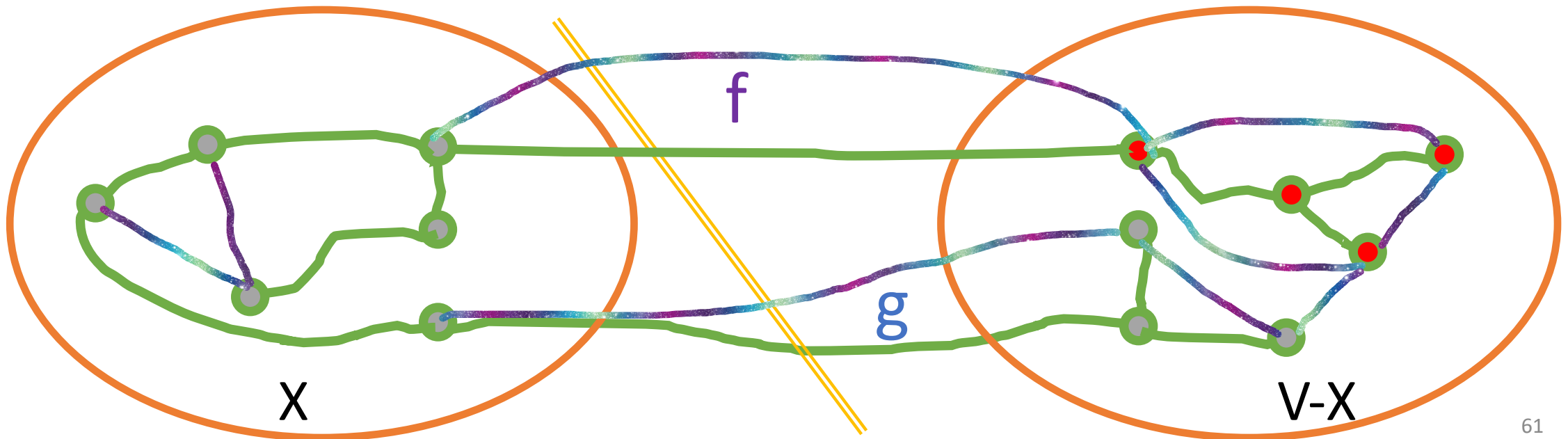
Proof of the Cut Property

Hope: that we can always find a suitable edge e' so that exchanging edges yields a valid spanning tree

Solid green lines are those that are currently part of T*
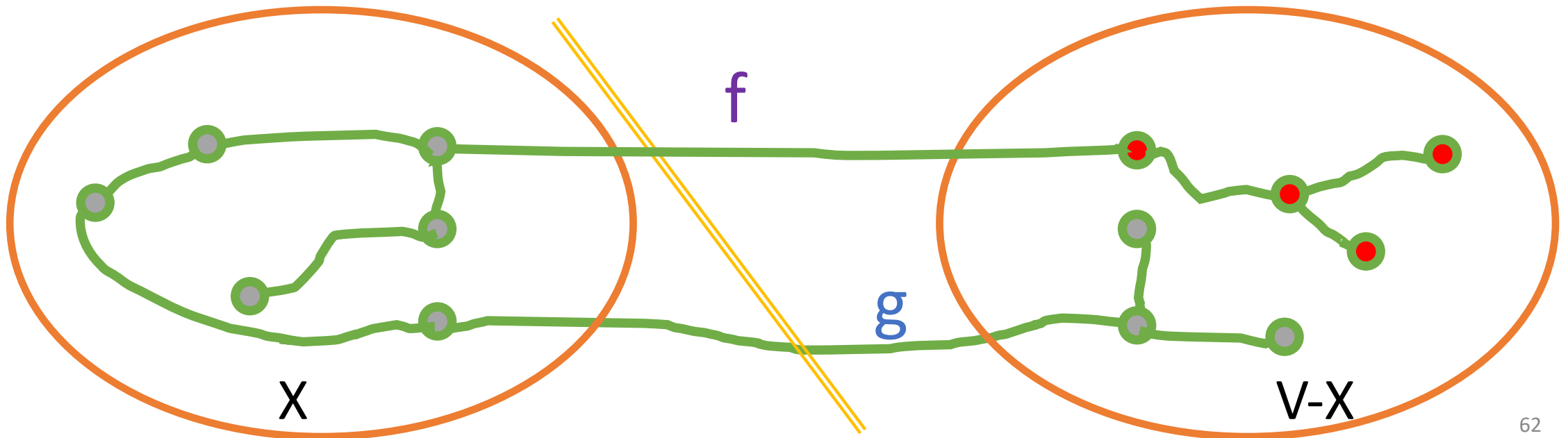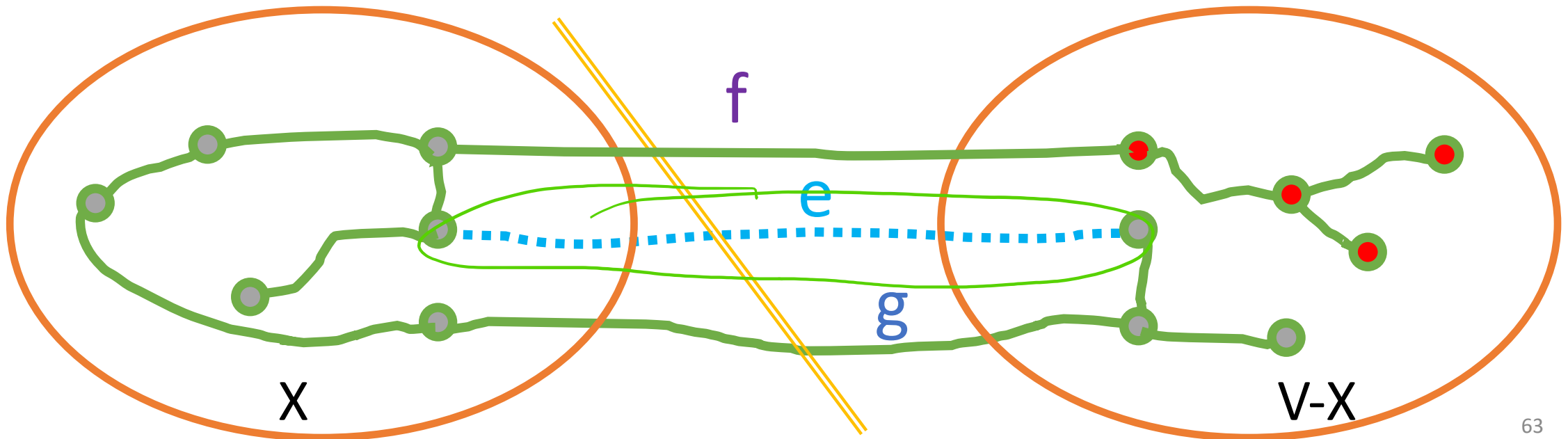
Proof of the Cut Property

Add the edge e.

What does adding e do?   A tree will always have n-1 edges   It creates a cycle that crosses the cut!
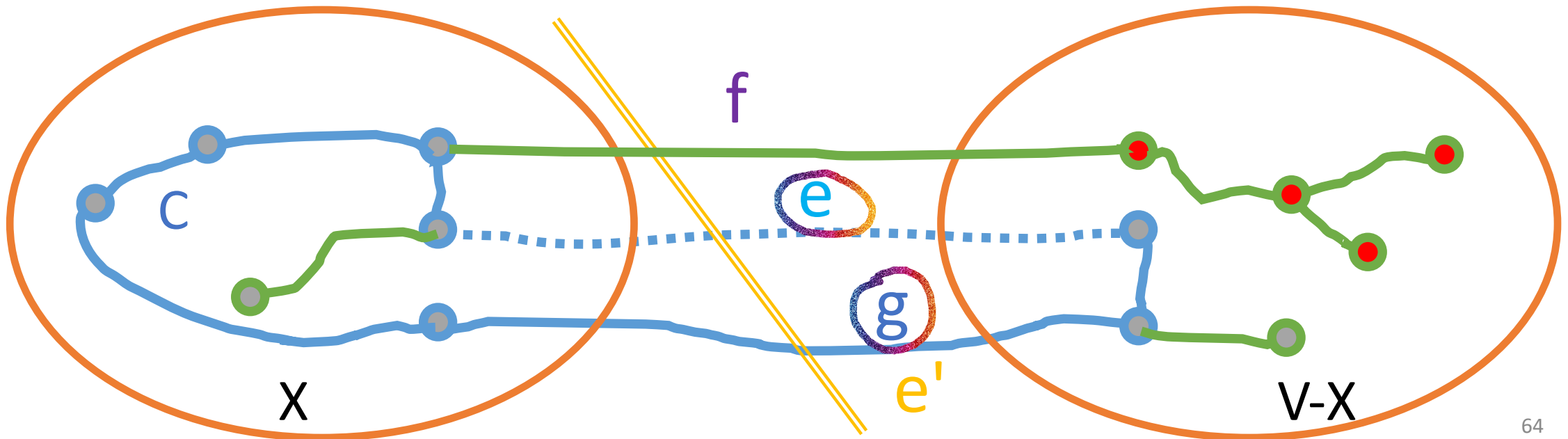
Which one of these edges can we exchange with e?

Solid green lines are those that are currently part of T*

Proof of the Cut Property

- Let C be the cycle created in T* by adding the edge e

- Find all edges that cross (X, V-X)

- By the double-crossing Lemma, there must be an edge e' that crosses (X, V-X)

Proof of the Cut Property
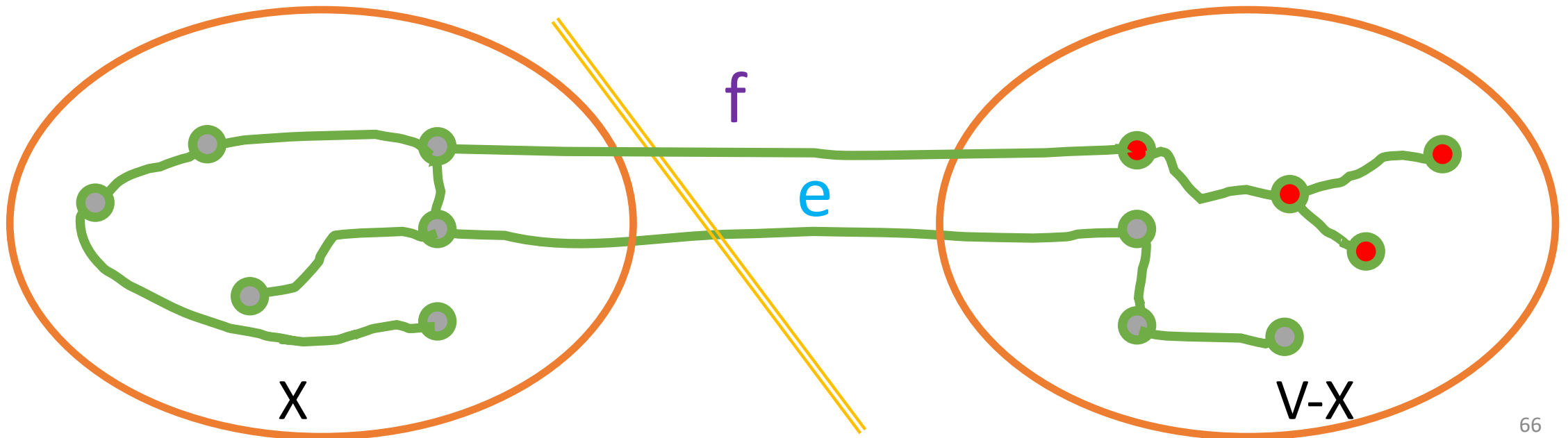
- Let T = T* U {e} − {e'}   Exchange

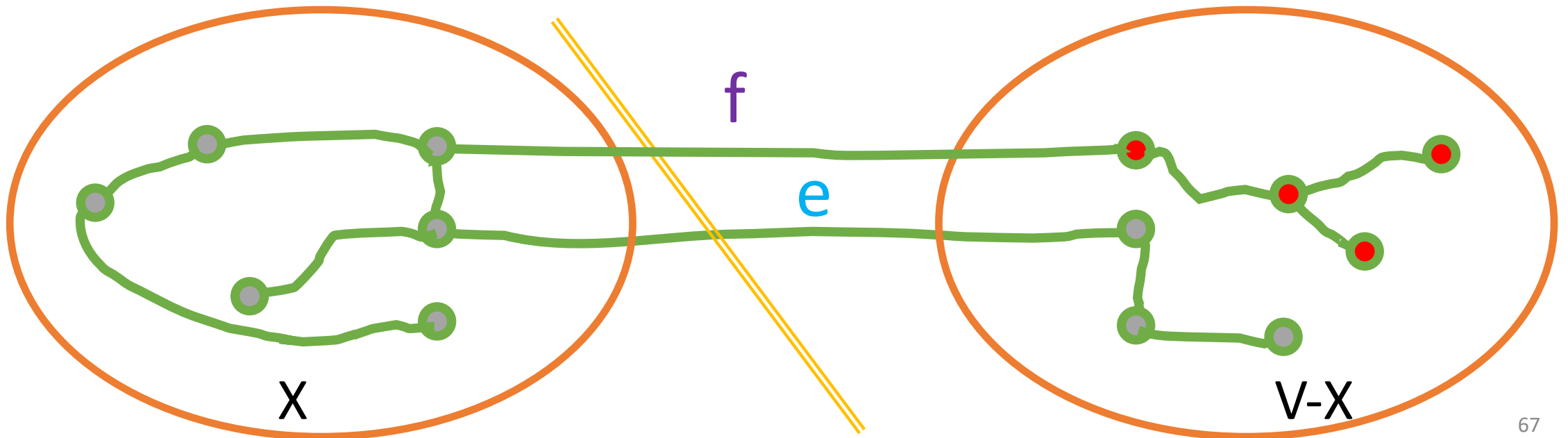The exchange argument was easier for greedy scheduling since every exchange resulted in a valid schedule

f

e

X

e'

V-X

Proof of the Cut Property

- Let T = T* ∪ {e} − {e'}   Exchange

f

e

X

V-X
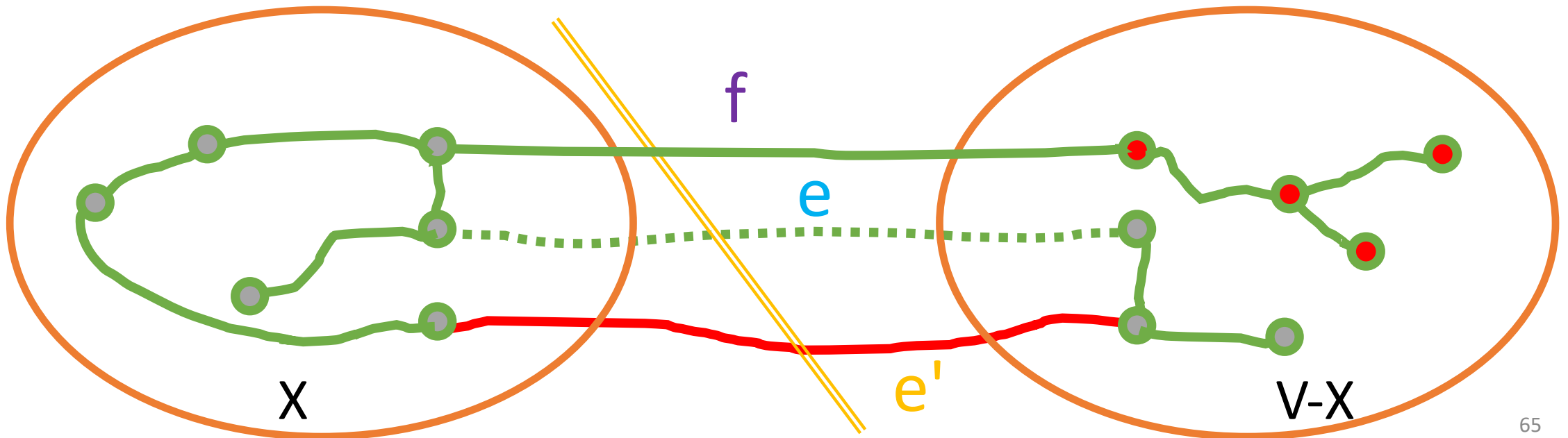
Proof of the Cut Property

- Let T = T^* ∪ {e} − {e'} Exchange
- T is also a spanning tree
- Since c_e < $c_{e'}$ T is a cheaper spanning tree than T^* (CONTRADICTION)
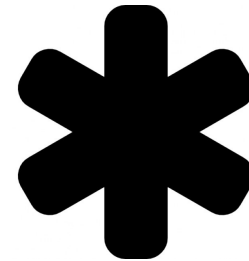

f

e

X

V-X

Proof of Prim's

Theorem: *Prim's algorithm always computes the (or a) MST when given a connected graph.*

Need to prove two things:

1. That Prim's algorithm creates a spanning tree T*
2. And that T* is the **minimum** spanning tree

* Need to prove the cut property!

What is the running time of Prim's?

Can we do better than O(mn)?

Can easily get to O(m lg n) using a heap (or faster with a Fibonacci Heap)

```
X = {s}    // list of found nodes
T = empty  // edges that belong to MST

while X is not V:
    let e = (u, v) be the cheapest edge of E
        with u in X and v not in X
    add e to T
    add v to X
```

O(n) for this while loop

O(m) to find cheapest edge that crosses the cut (X, V-X)