Heaps

https://cs.pomona.edu/classes/cs140/

Outline

Topics and Learning Objectives

- Discuss data structure operations
- Cover heap sort
- Discuss heaps

Exercise

- Heap practice

Extra Resources

- Introduction to Algorithms, 3rd, chapter 6
- Algorithms Illuminated, Part 2: Chapter 10

Data Structures

Used in essentially every single programming task that you can think of

- What are some examples of data structures?
- What are some example programs?

What do they do?

- They organize data so that it can be effectively accessed.

- A data structure is not necessarily a method of laying out data in memory
- It is a way of logically thinking about your data.

The Heap Data Structure (not heap memory)

A container for objects that have key values
(Sometimes called a "Priority Queue")

Operations:

- Insert : O(lg n)
- Extract-Min (or max) : O(lg n)

- Heapify : O(n) for batched inserts
- Arbitrary Deletion : O(lg n)

- Good for continually getting a minimum (or maximum) value

Heap used to improve algorithm

Selection sort

- Continually look for the smallest element
- The element currently being considered is in blue
- The current smallest element is in red
- Sorted elements are in yellow

	8
	5
	2
	6
	9
	3
	1
	4
	0
	7

Heap used to improve algorithm

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Heap used to improve algorithm

Selection sort

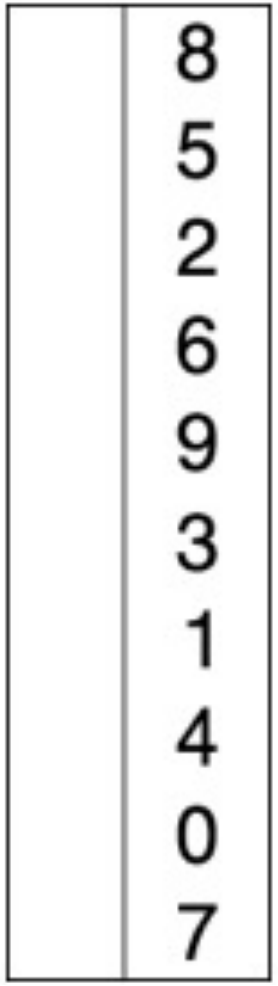
- Continually look for the smallest element

What is the runtime of selection sort?

How can we make it faster with a heap?

With a heap: $O(n^2)$ → $O(n \lg n)$

- Insert all elements into a heap: n
- Extract each element: n * lg n



Example : Event Manager

Uses a priority queue (synonym for Heap)

Example: simulation or game

- play sounds
- render animation We can probably delay this without much trouble.
- detect collisions
- register input Probably the most important to get correct. But does it need to be the highest priority?

Heap Implementation

No pointer following
Fewer heap-allocations
No pointer storing

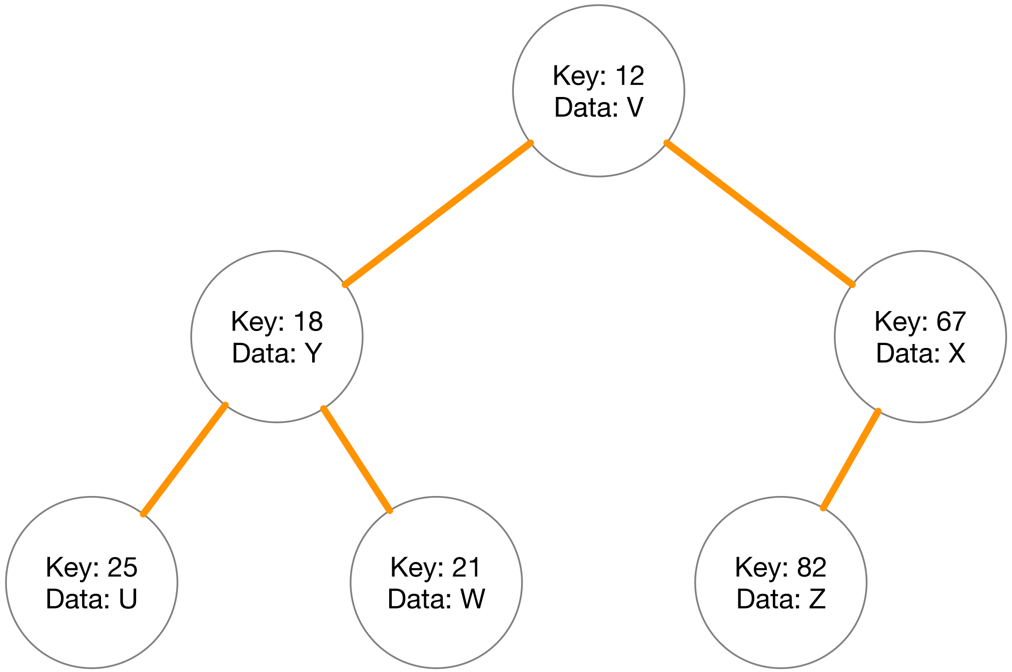
Conceptually you should think of a Heap as a binary tree

But it is implemented using an array (why?)

Heap Property: for any given node x,

1. key[x] ≤ key[x's left child], and
2. key[x] ≤ key[x's right child]

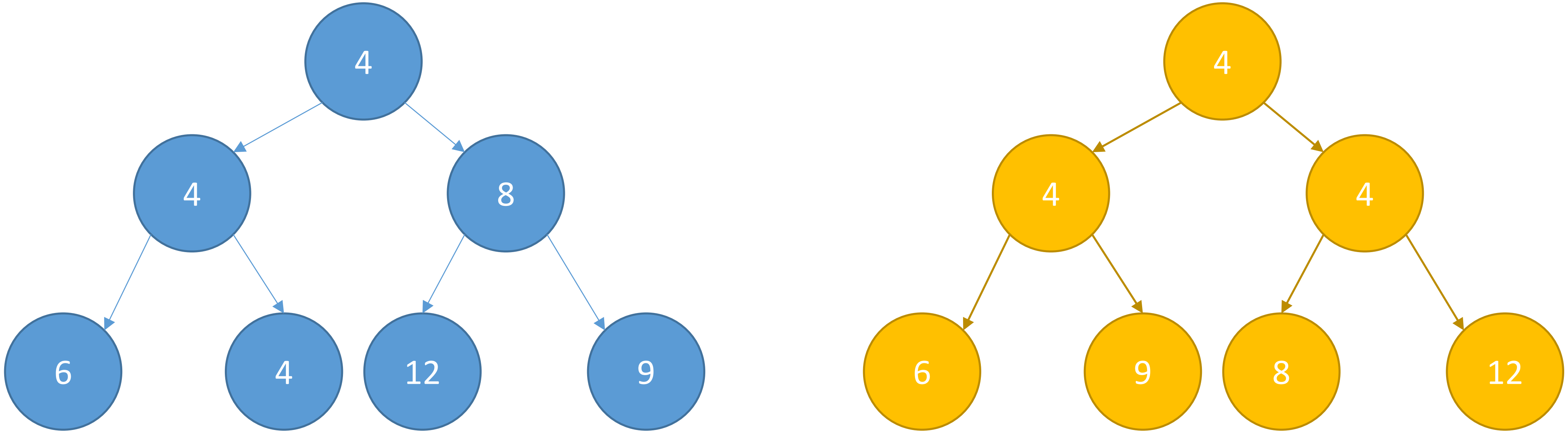
Where is the minimum key?



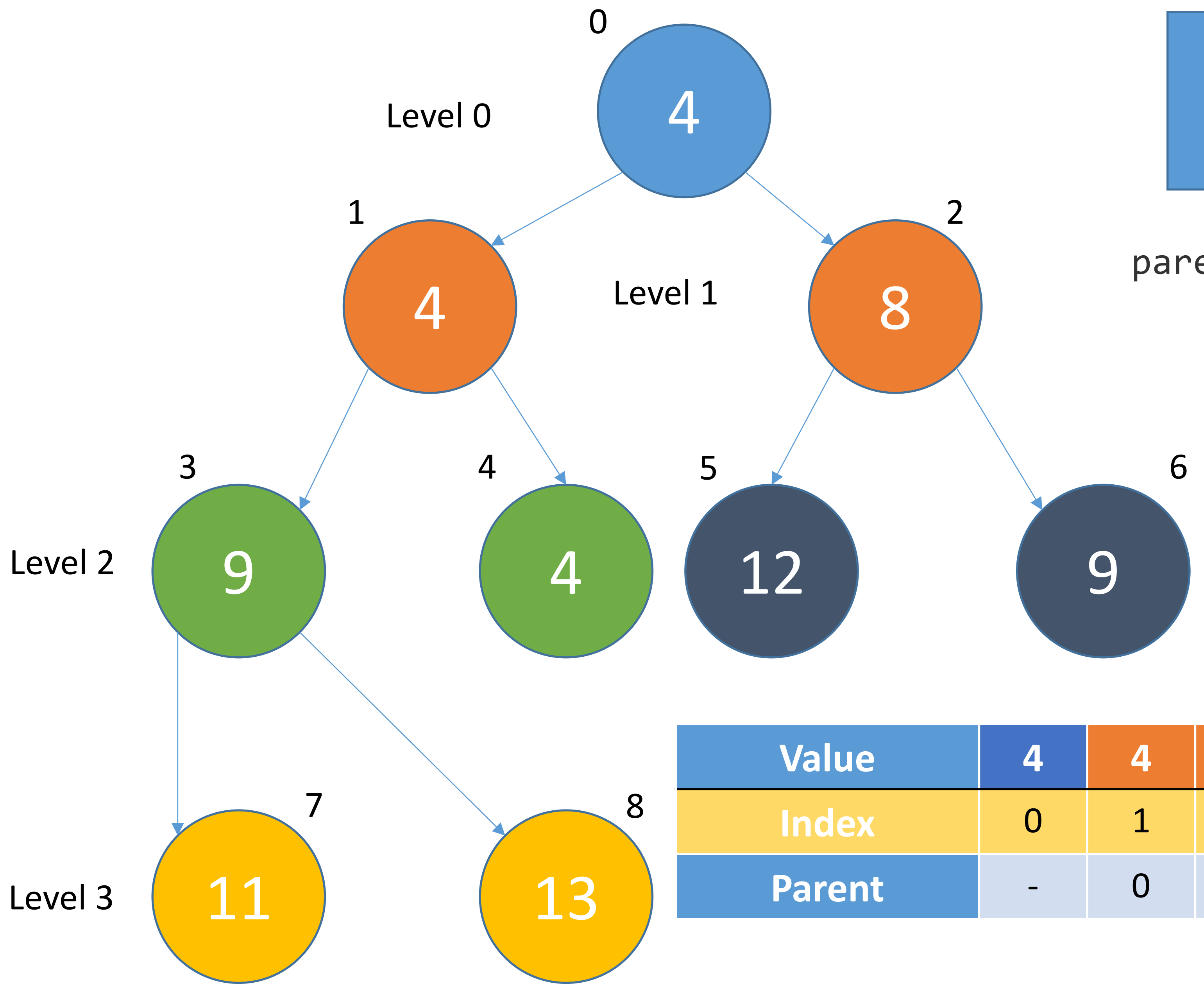
Heap Implementation

Note: Heaps are not unique

You can have multiple different configurations that hold the same data



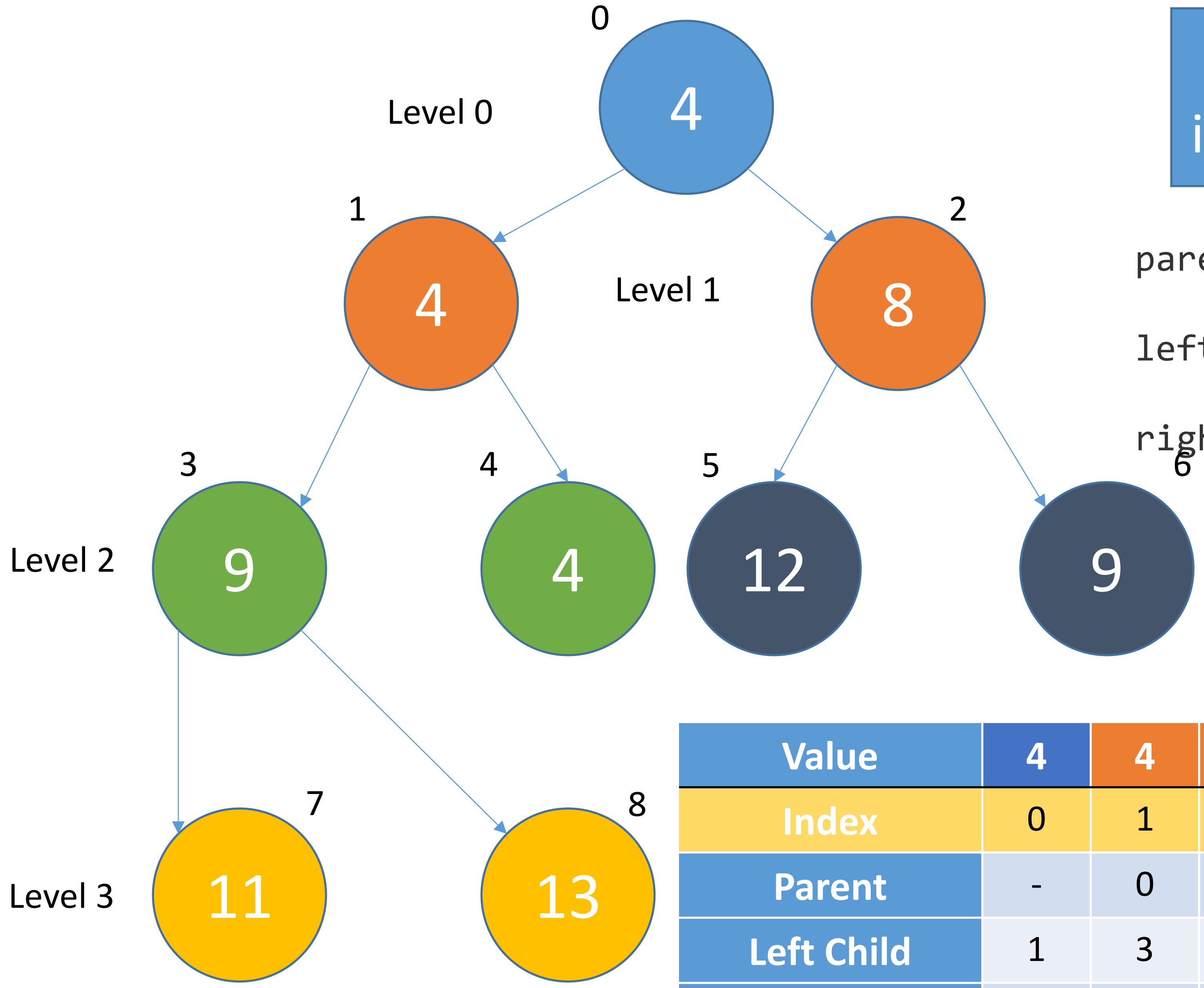
How do you calculate the index of a node's parent?




```
parent_index = (node_index - 1) // 2
```

Value	4	4	8	9	4	12	9	11	13
Index	0	1	2	3	4	5	6	7	8
Parent	-	0	0	1	1	2	2	3	3

How do you calculate the indices of a node's children?




```
parent_index = (node_index - 1) // 2

left_child_index = 2 * node_index + 1

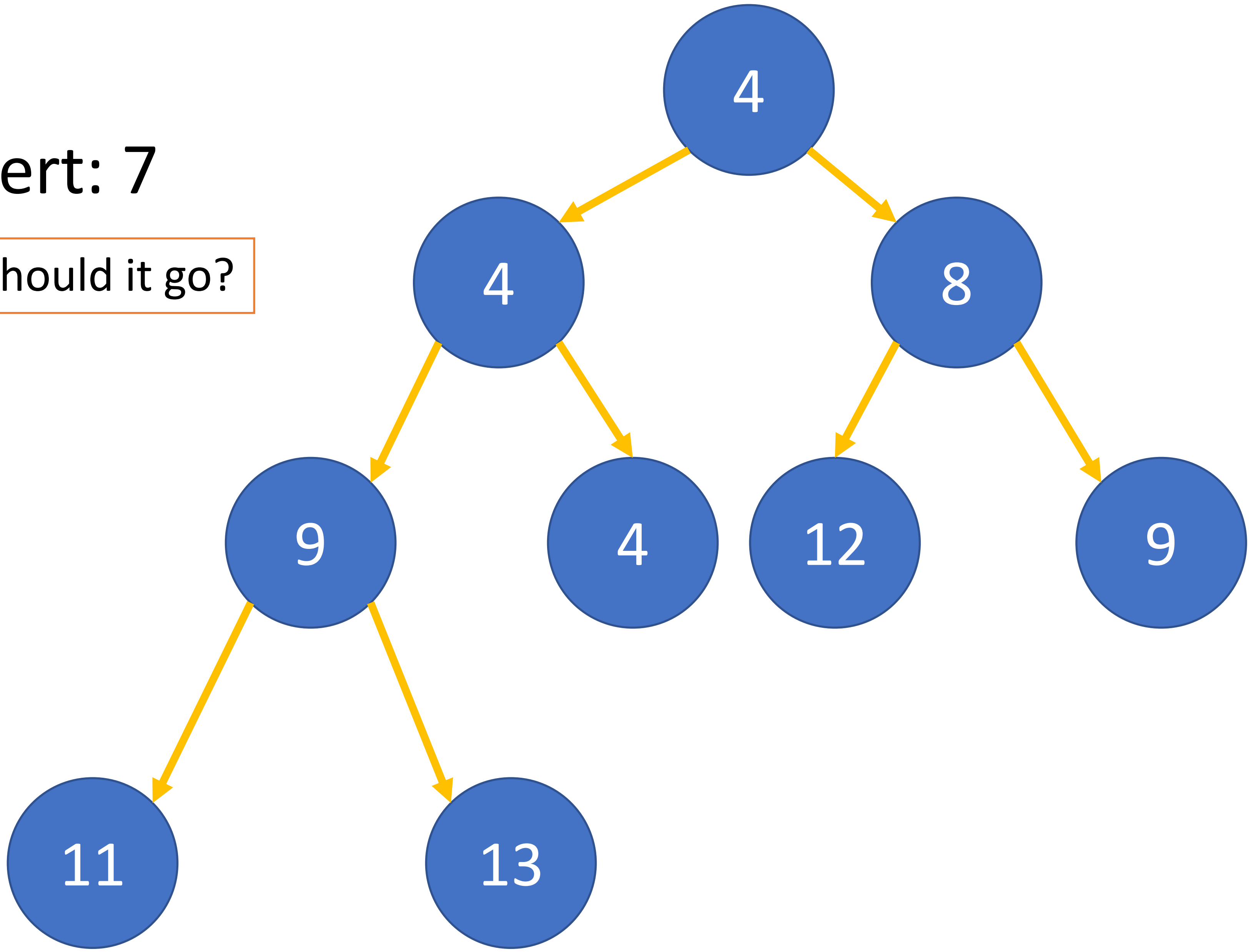
right_child_index = 2 * (node_index + 1)
```

Value	4	4	8	9	4	12	9	11	13
Index	0	1	2	3	4	5	6	7	8
Parent	-	0	0	1	1	2	2	3	3
Left Child	1	3	5	7	9	11	13	15	17
Right Child	2	4	6	8	10	12	14	16	18

Exercise

Insert: 7

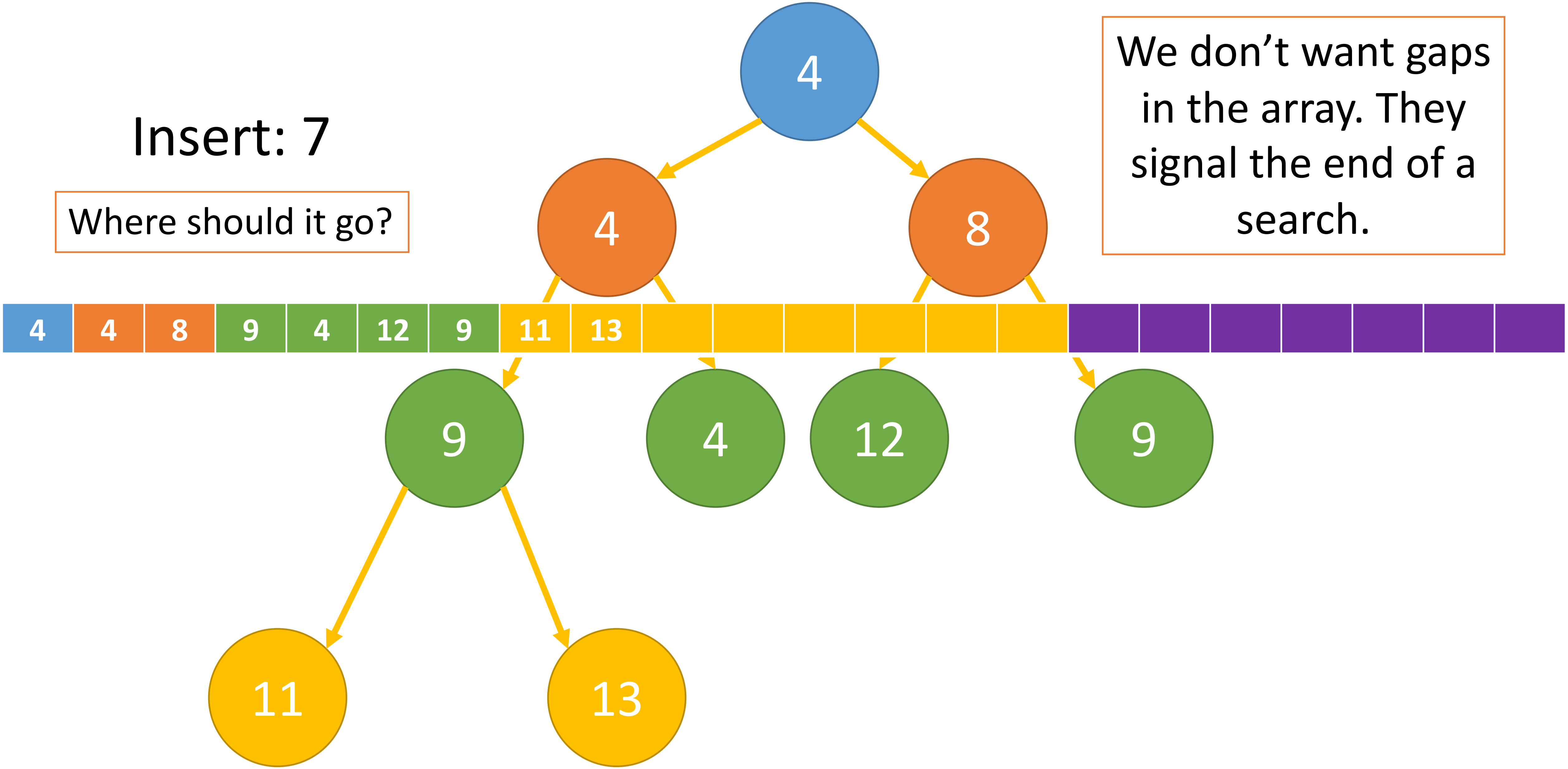
Where should it go?



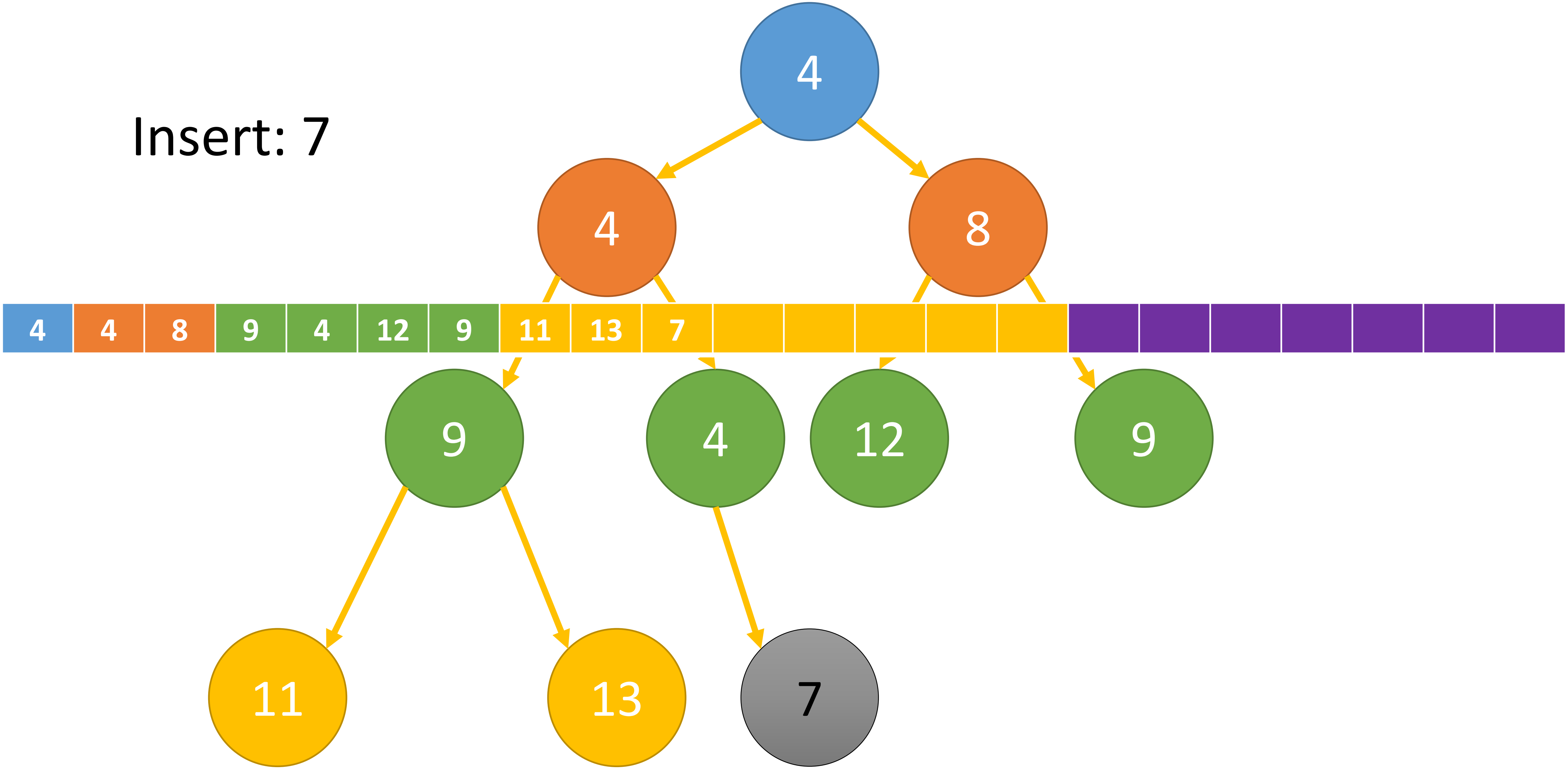
Insert: 7

Where should it go?

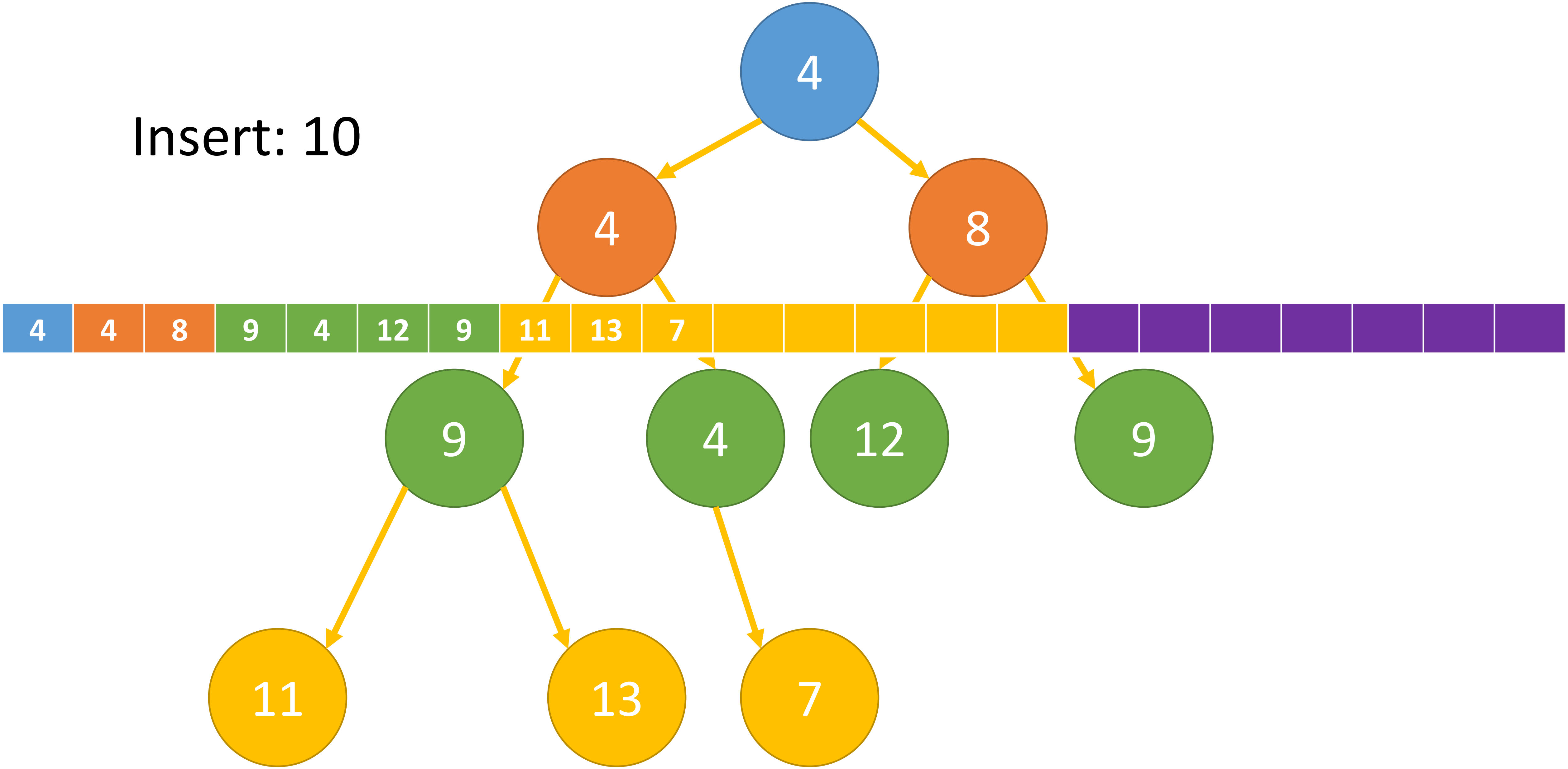
We don't want gaps in the array. They signal the end of a search.



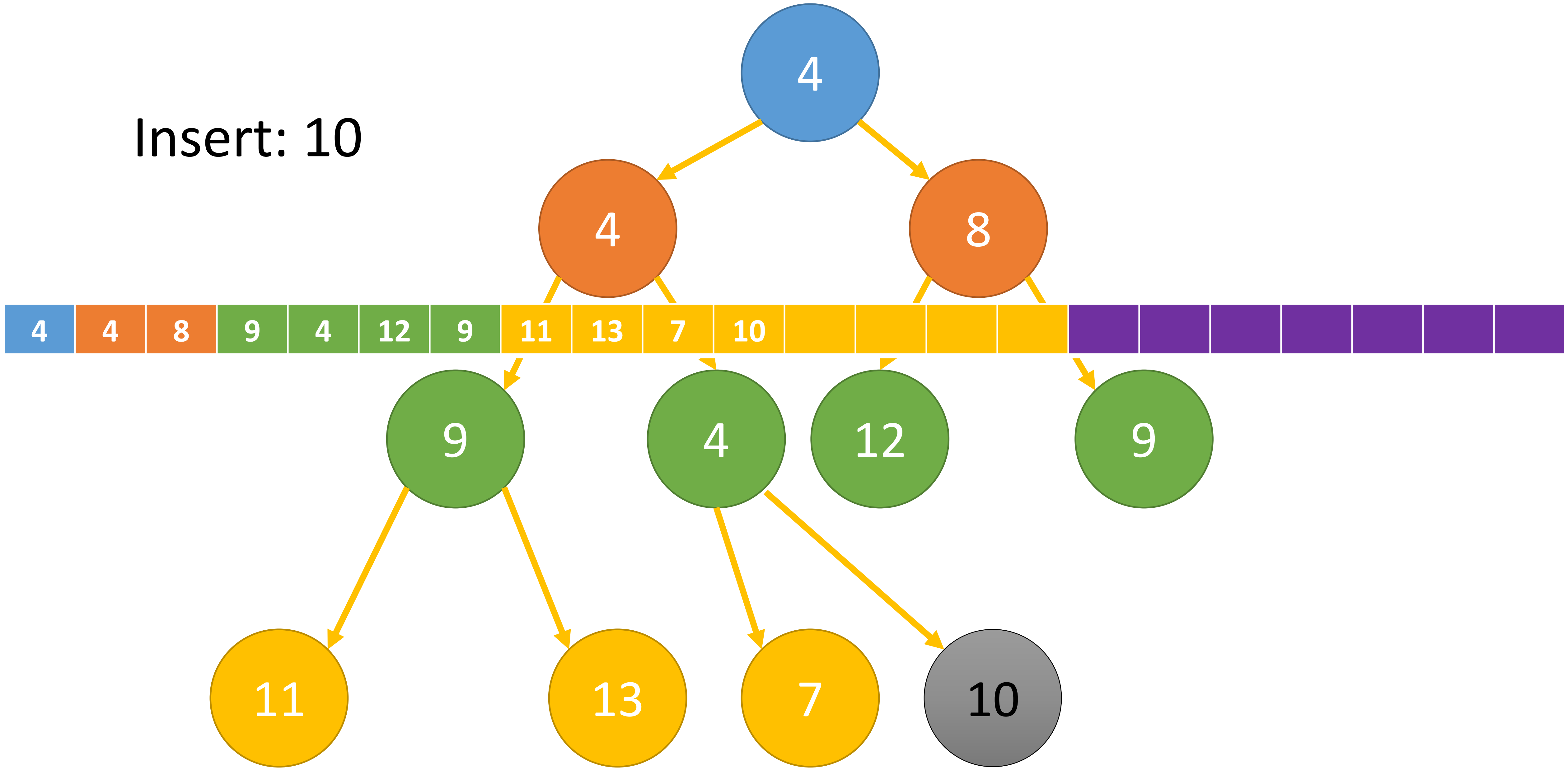
Insert: 7



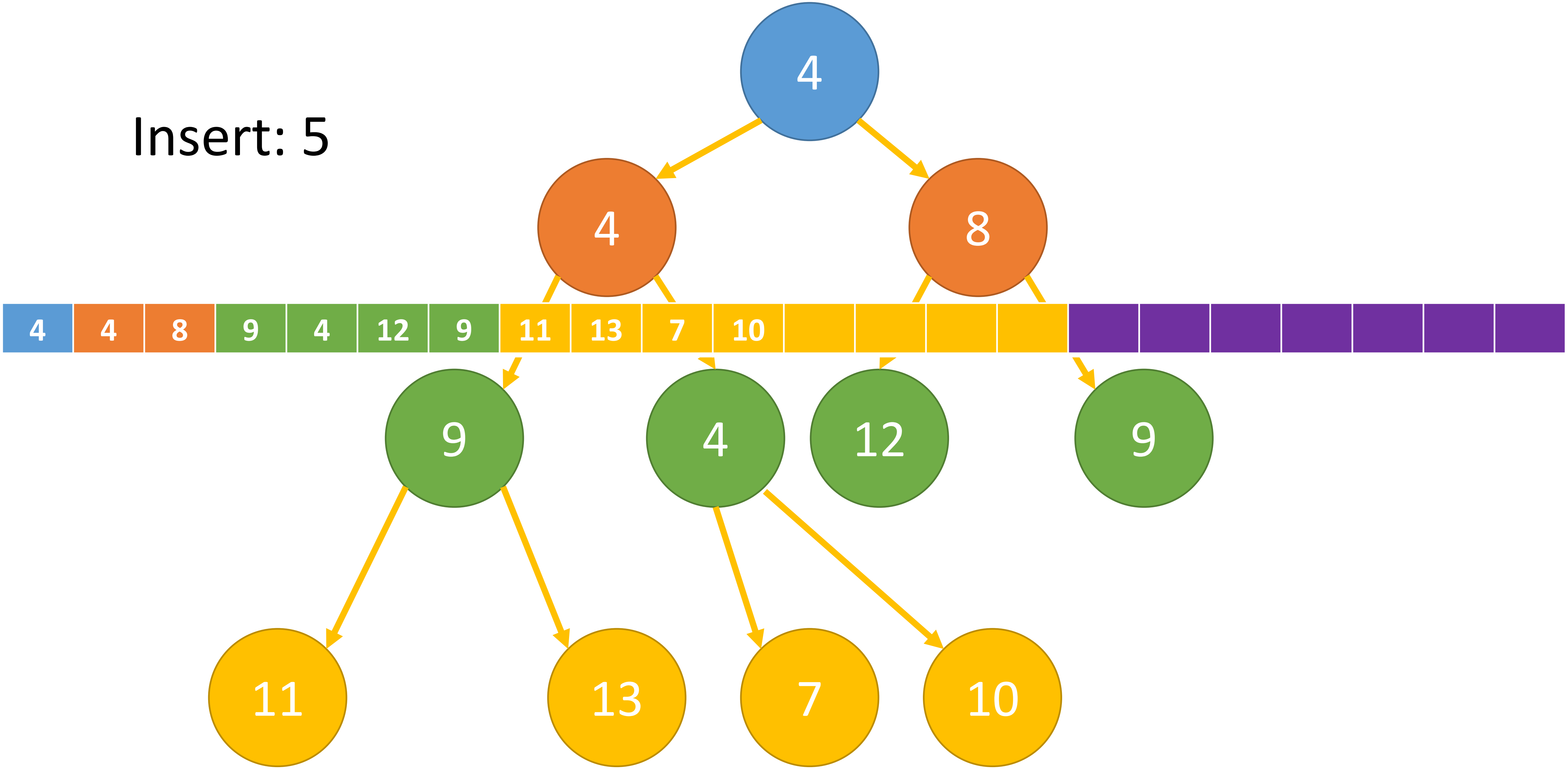
Insert: 10



Insert: 10



Insert: 5



Insert: 5

```
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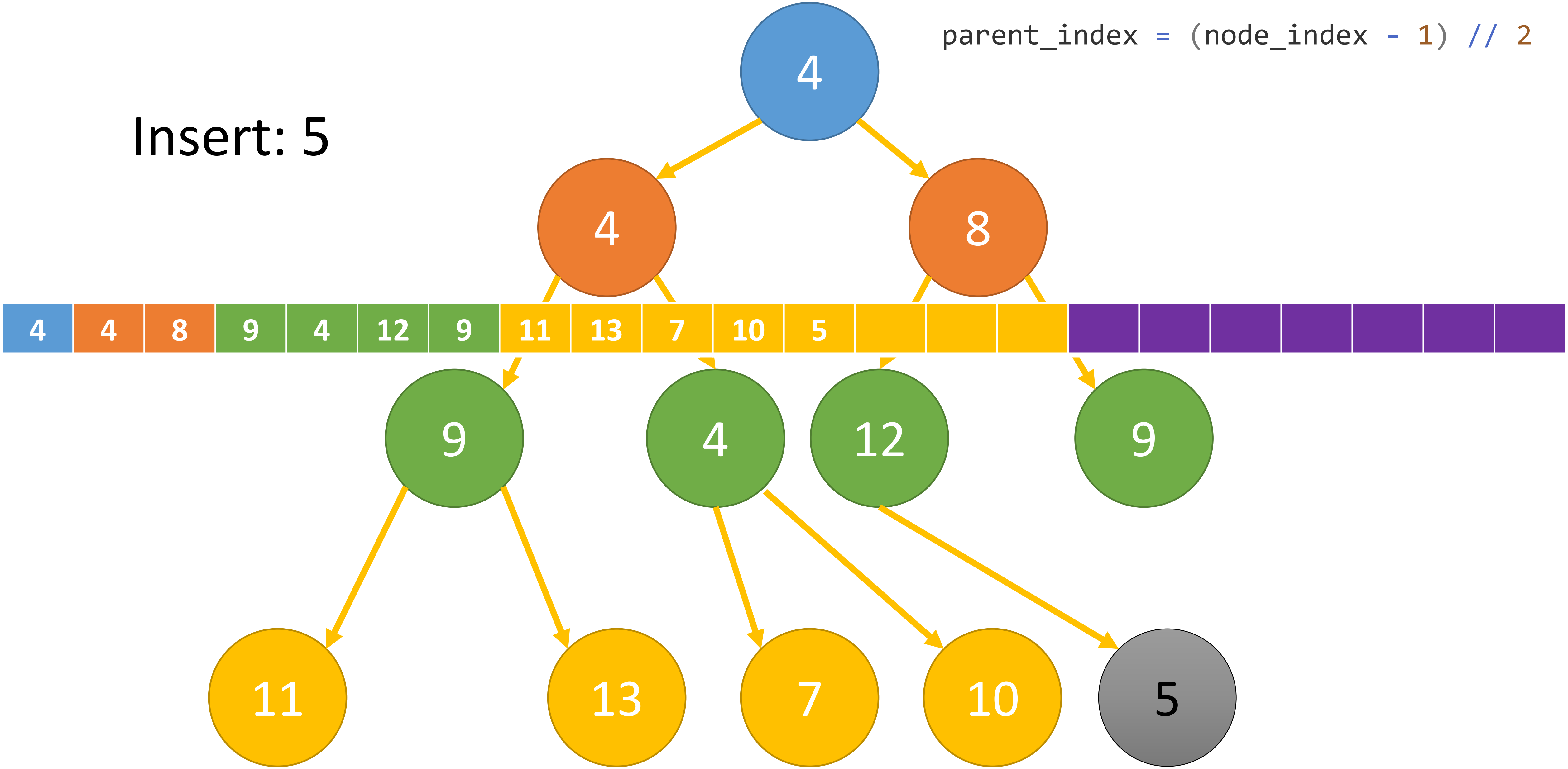
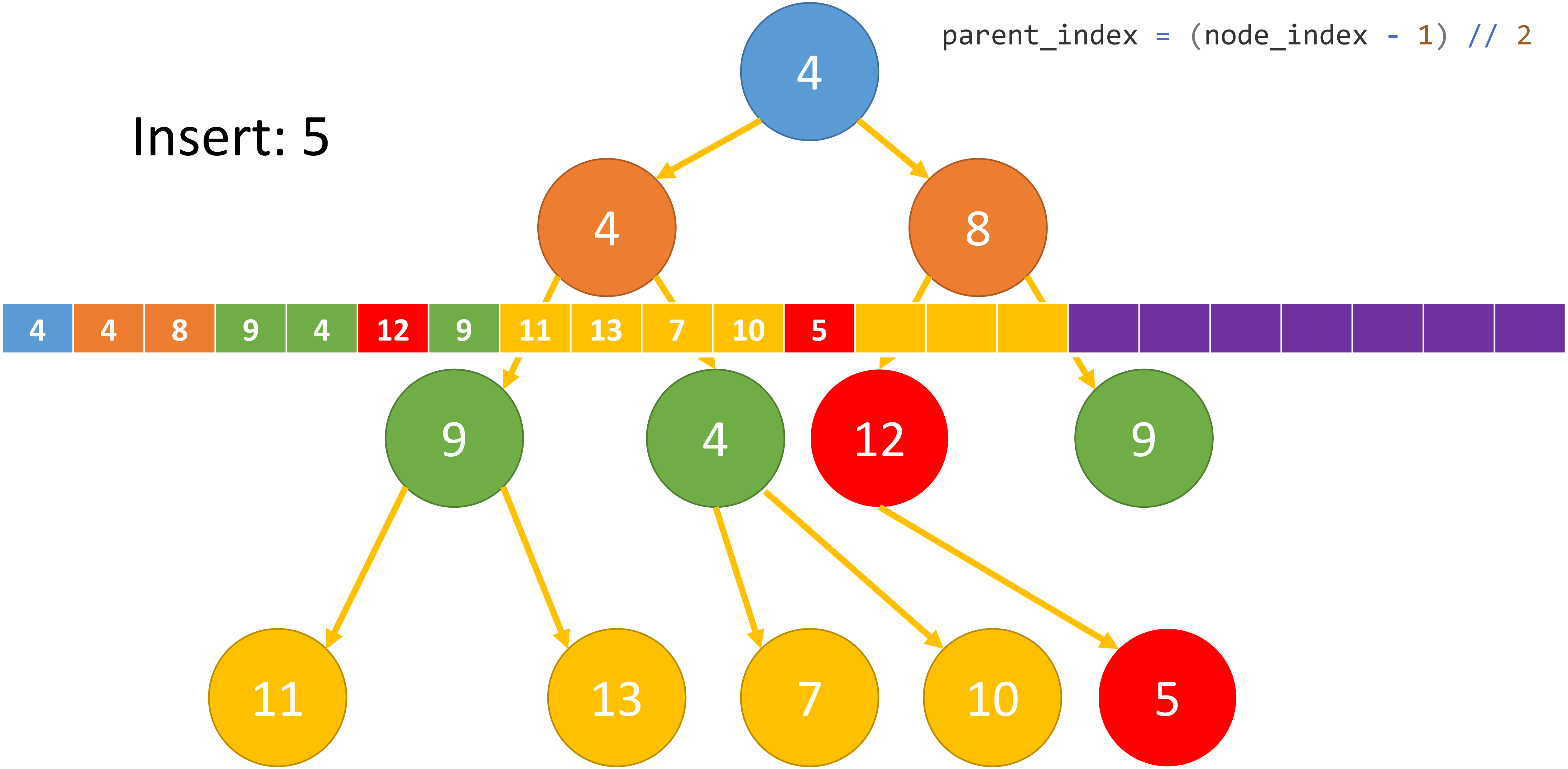


Insert: 5

`parent_index = (node_index - 1) // 2`



Insert: 5

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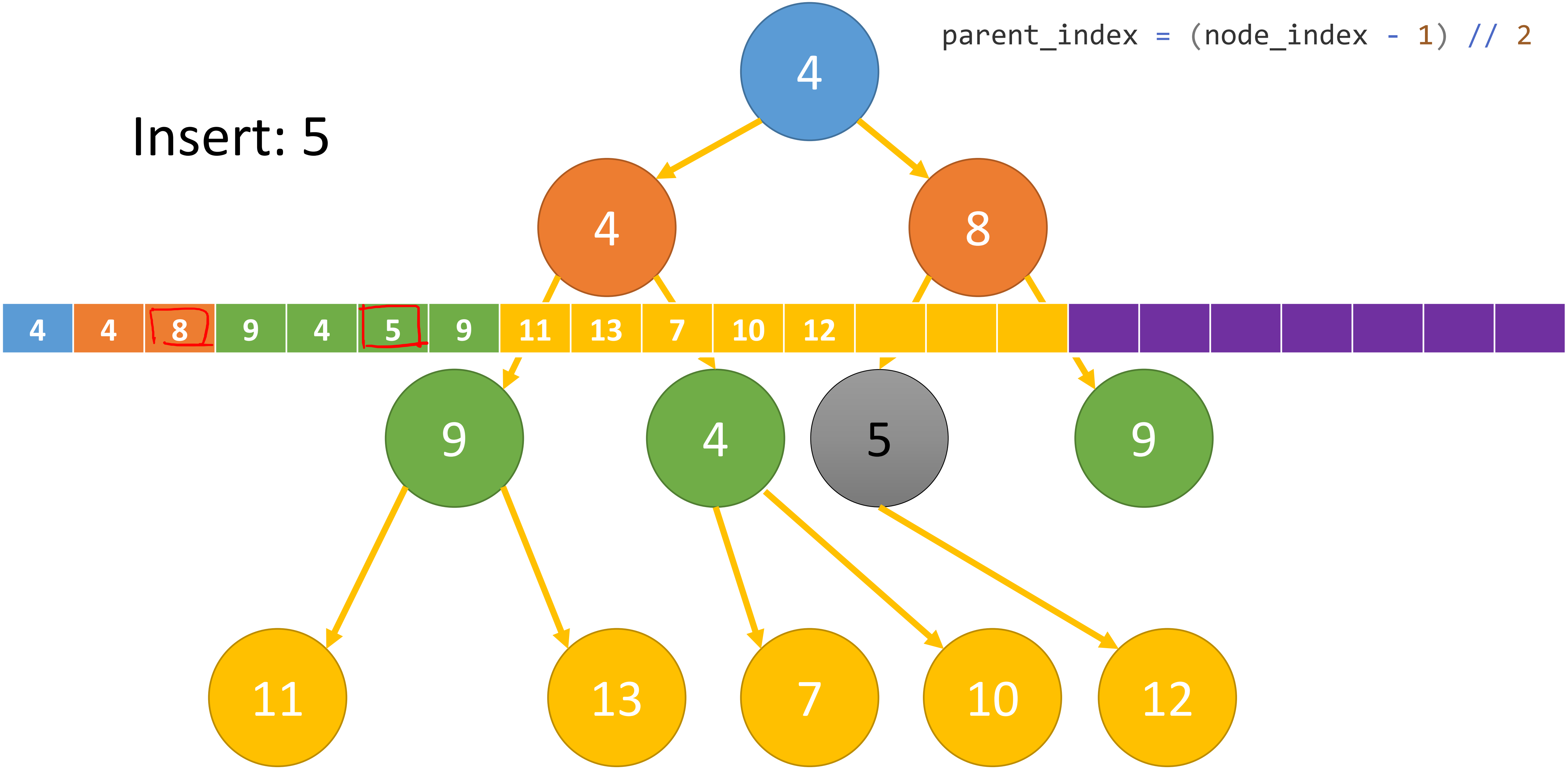

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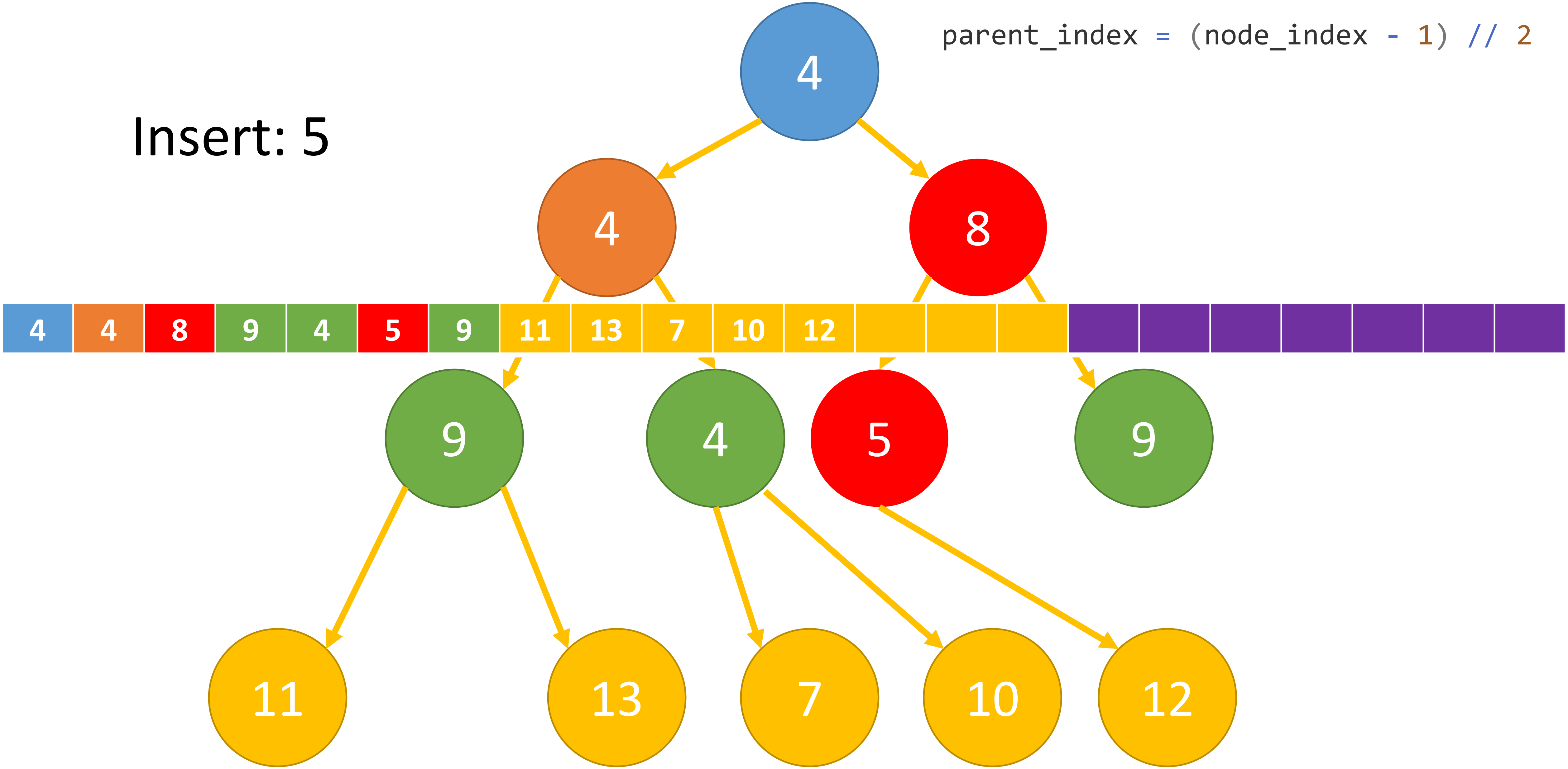
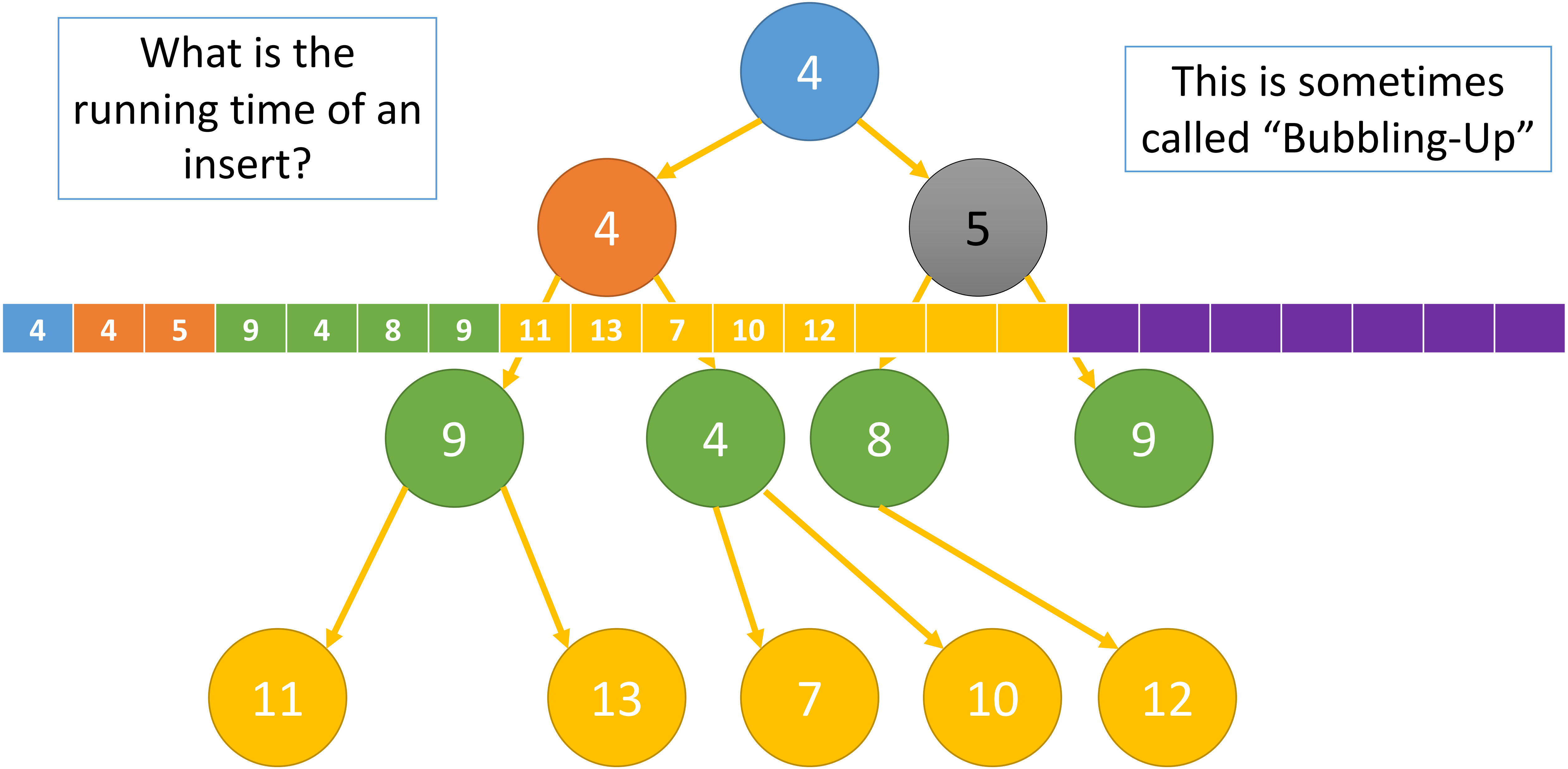


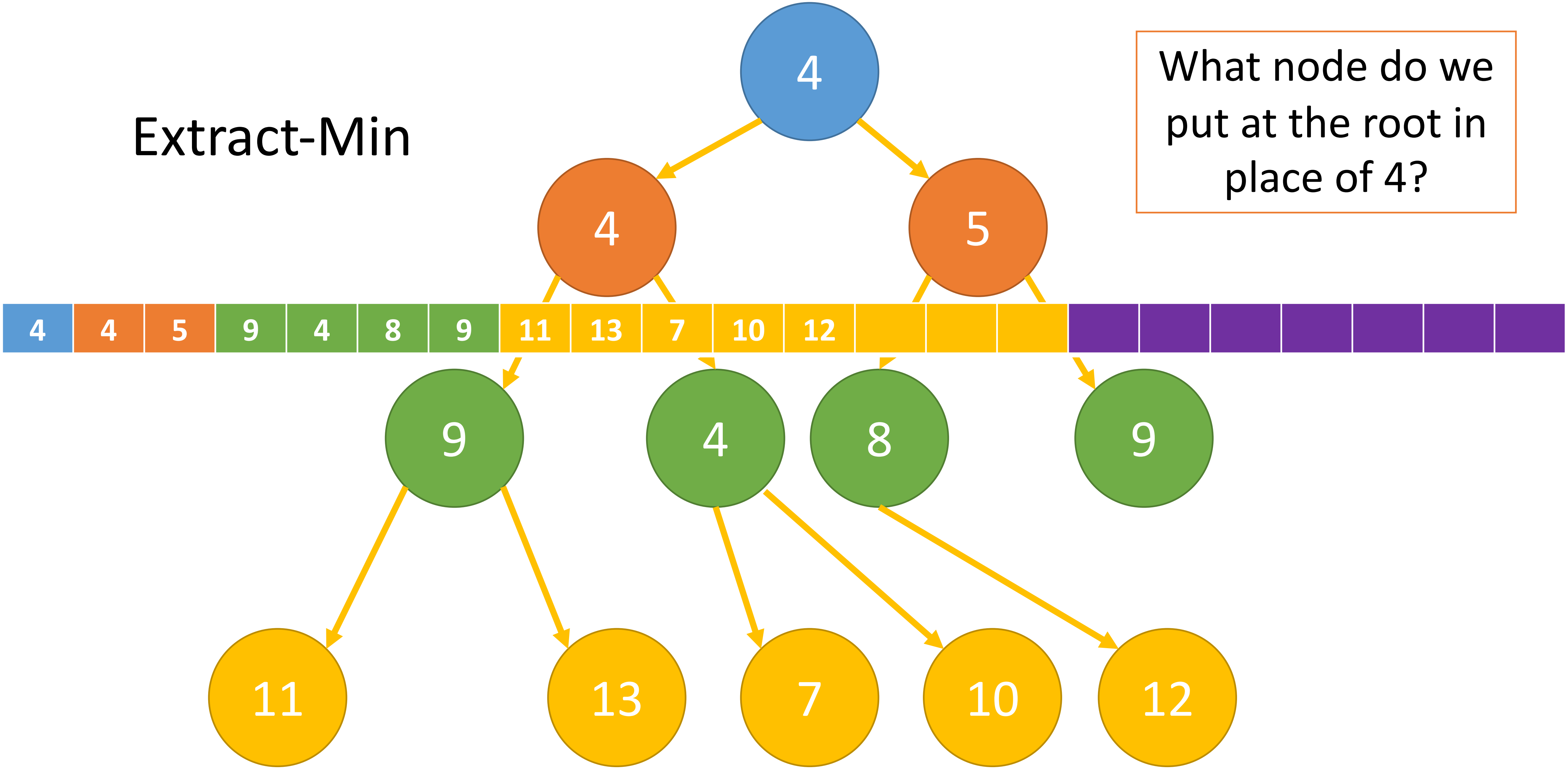
What is the running time of an insert?

This is sometimes called “Bubbling-Up”



Extract-Min

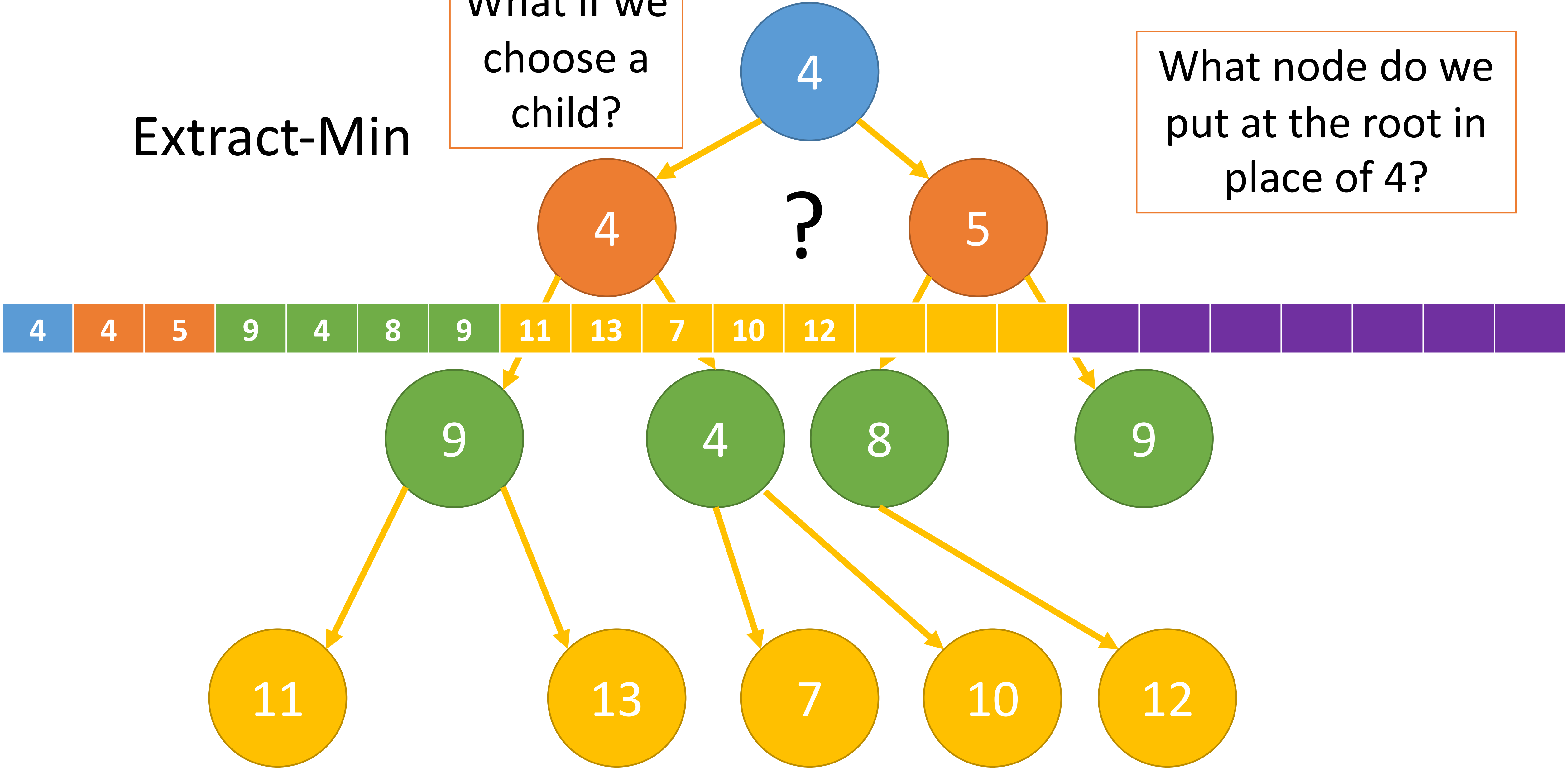
What node do we put at the root in place of 4?



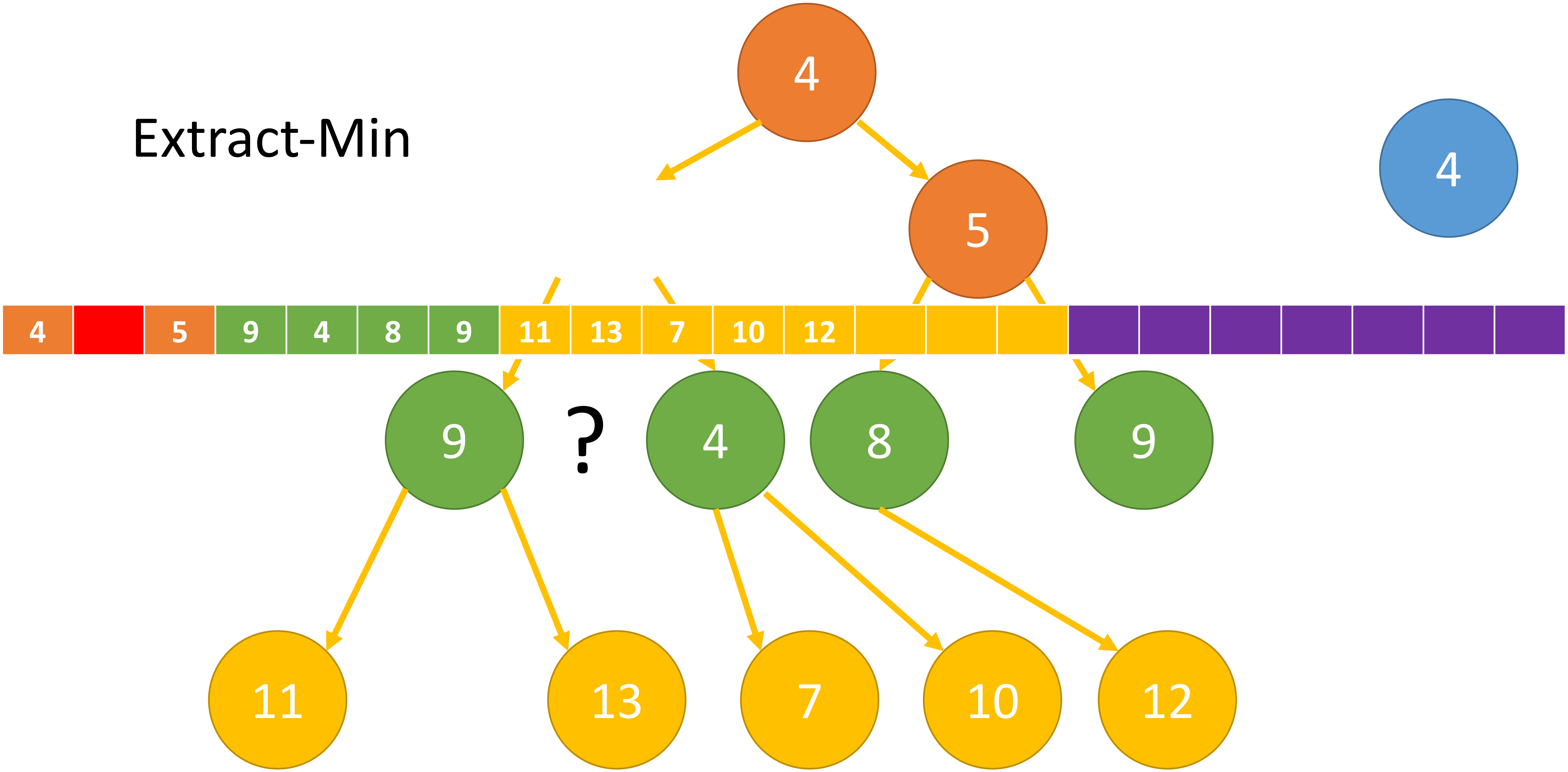
Extract-Min

What if we choose a child?

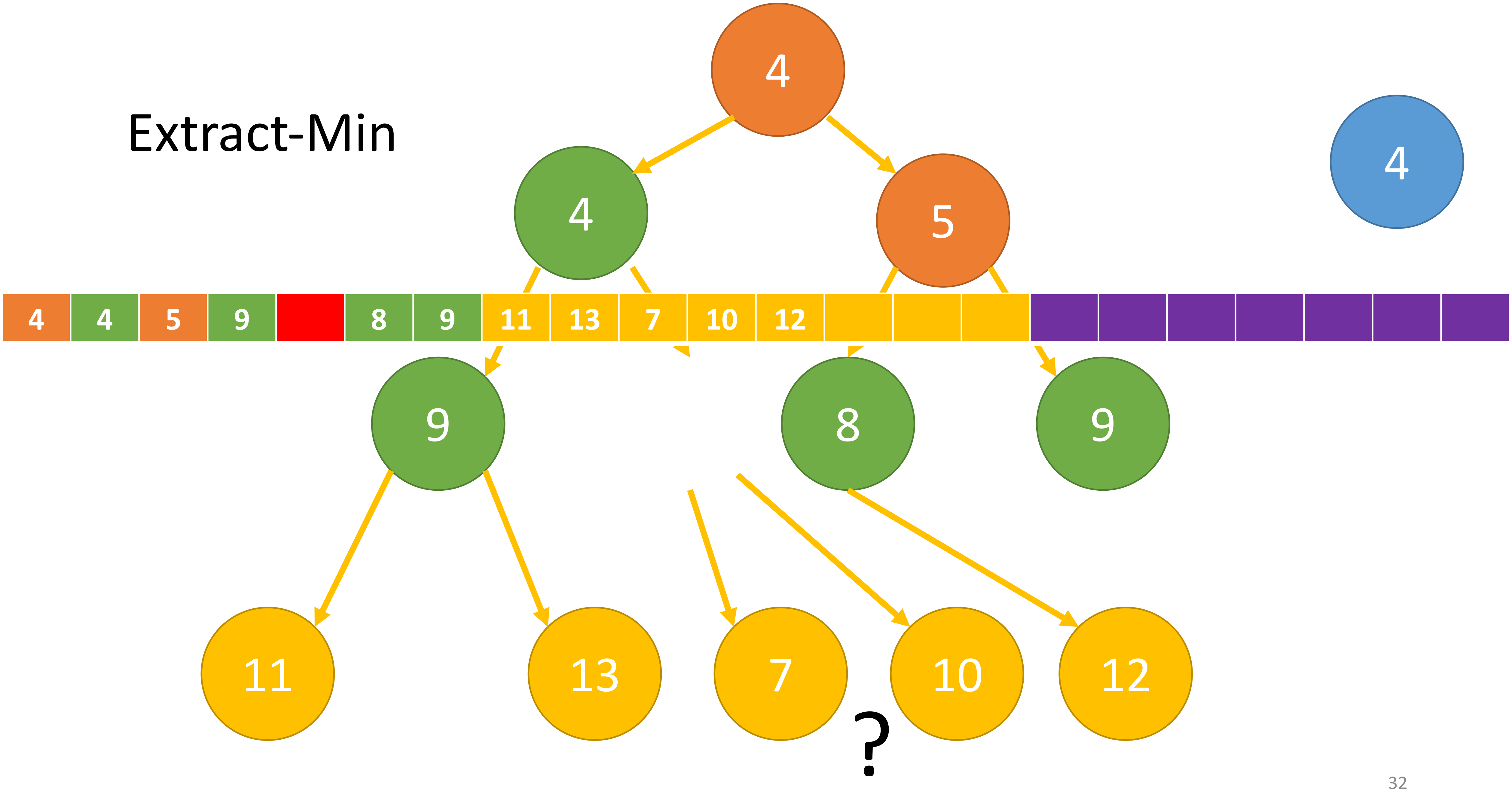
What node do we put at the root in place of 4?



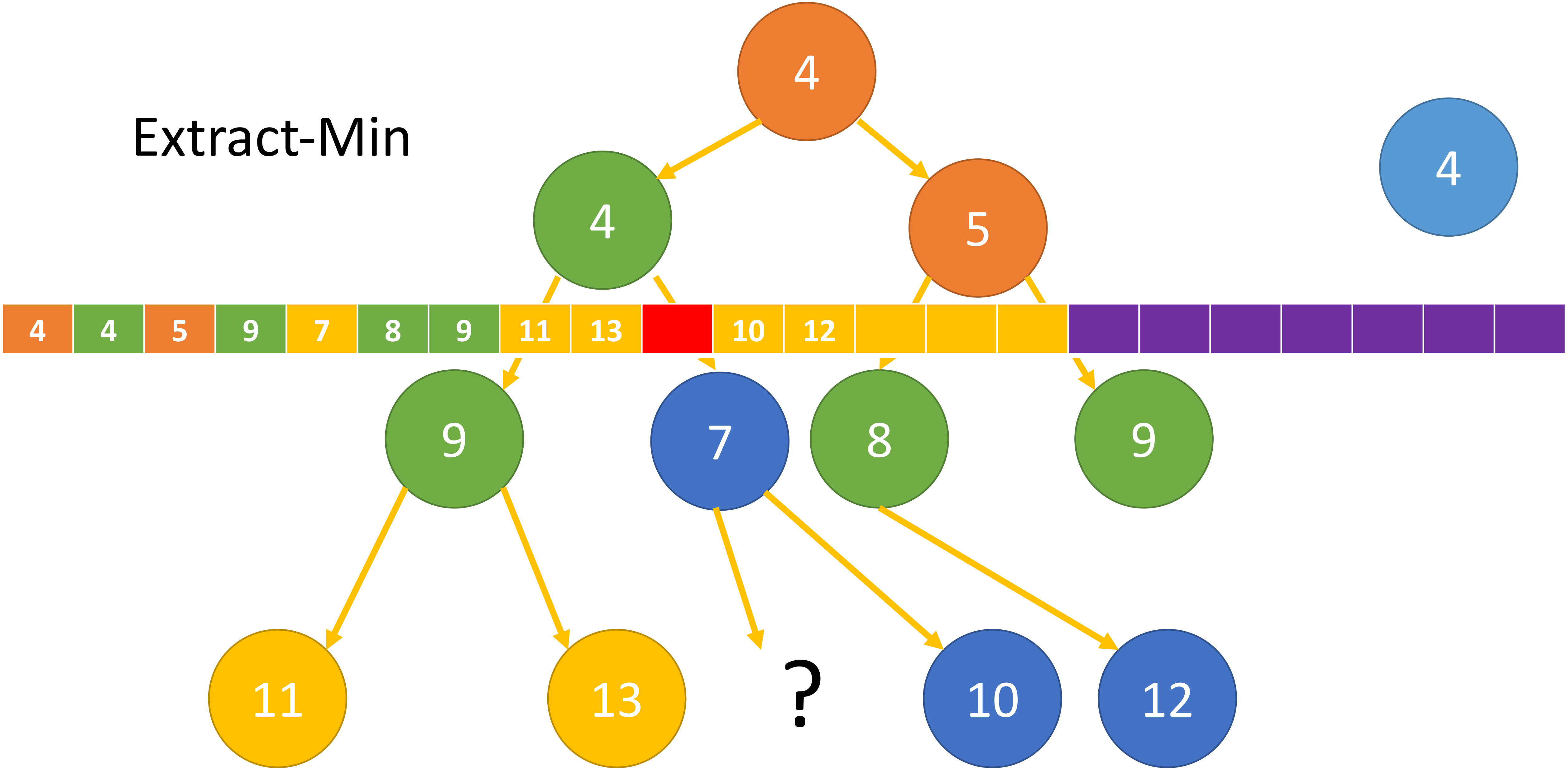
Extract-Min



Extract-Min



Extract-Min



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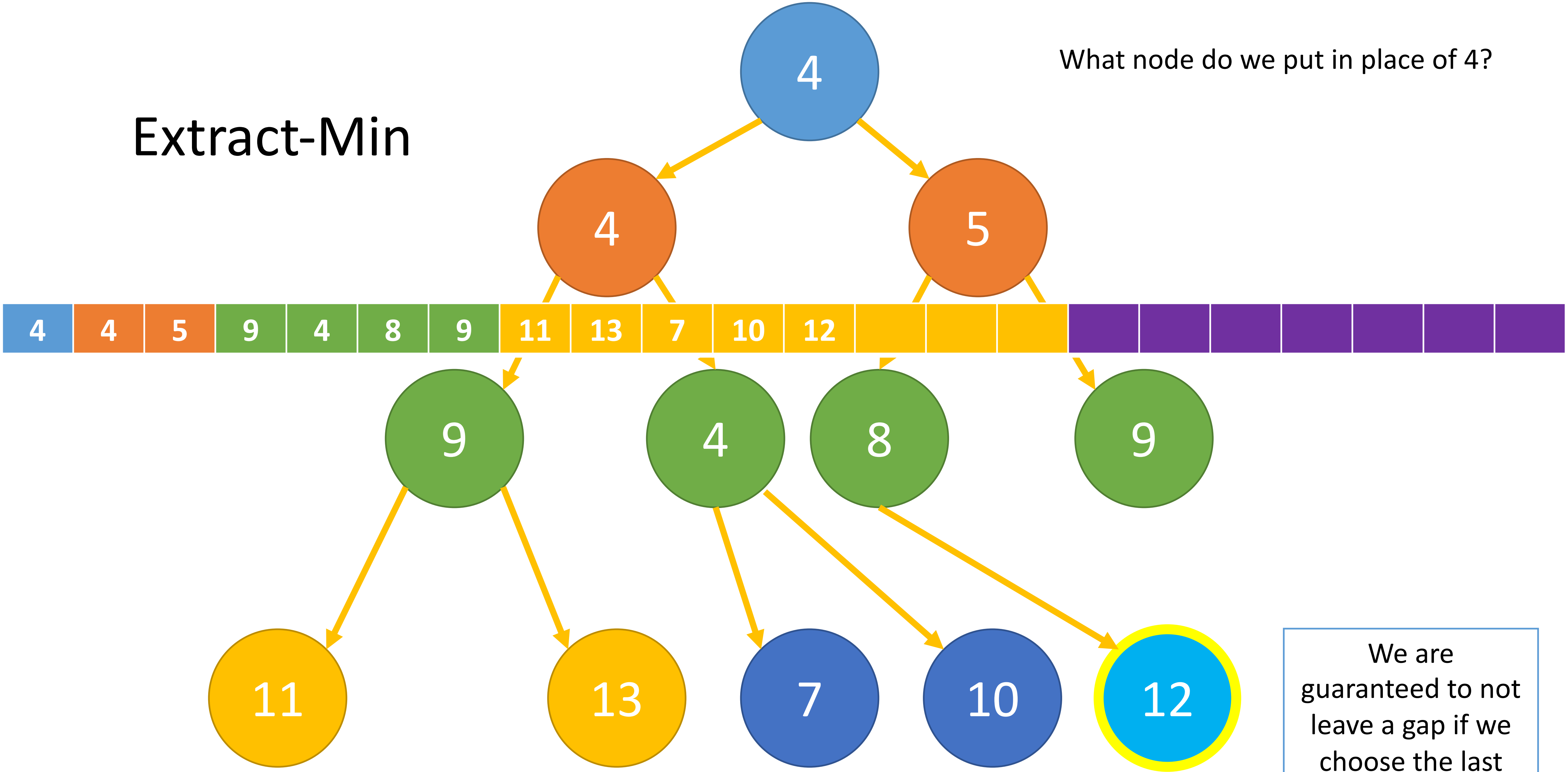
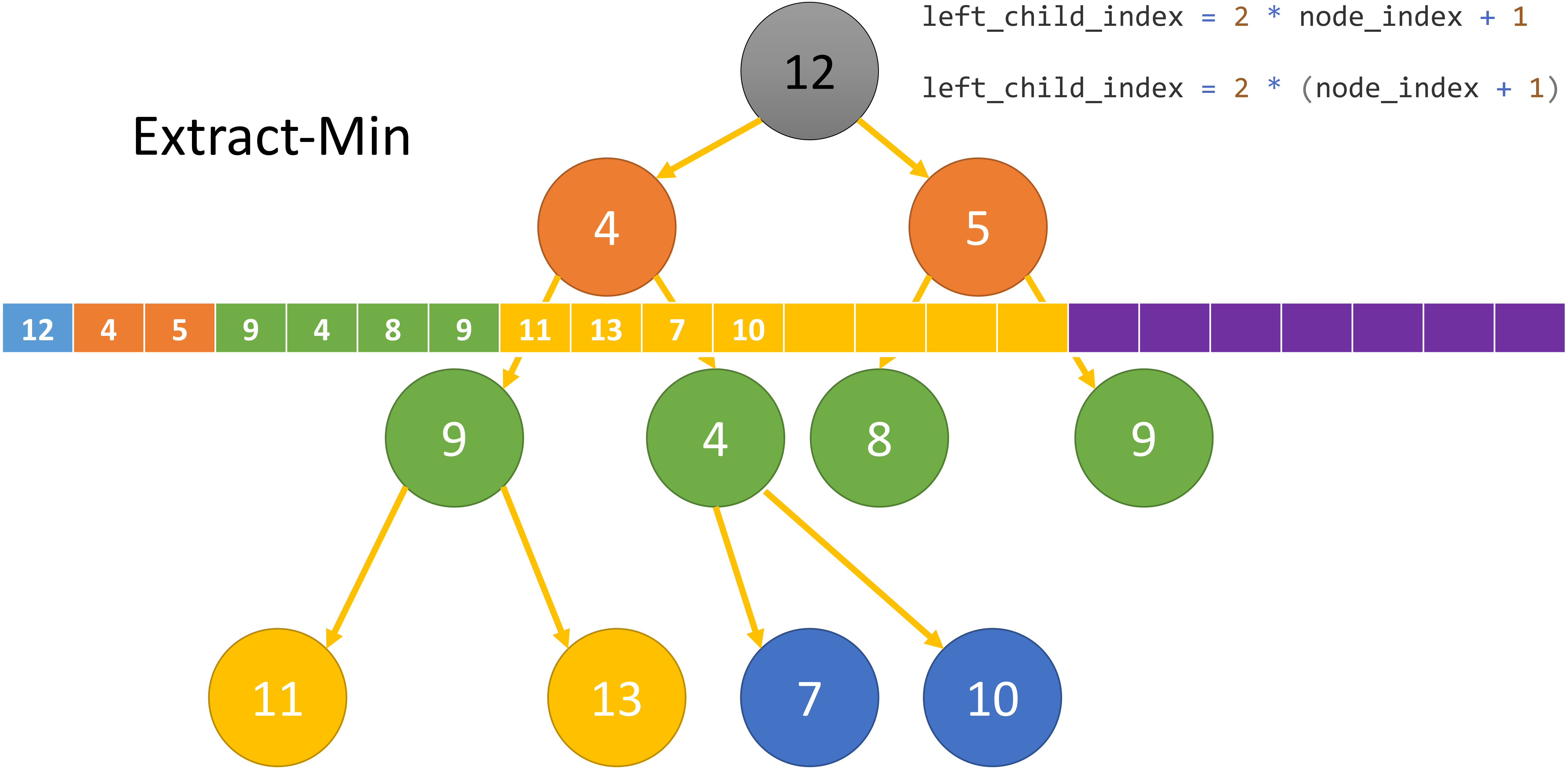


We are guaranteed to not leave a gap if we choose the last node.

Extract-Min

```
left_child_index = 2 * node_index + 1

left_child_index = 2 * (node_index + 1)
```

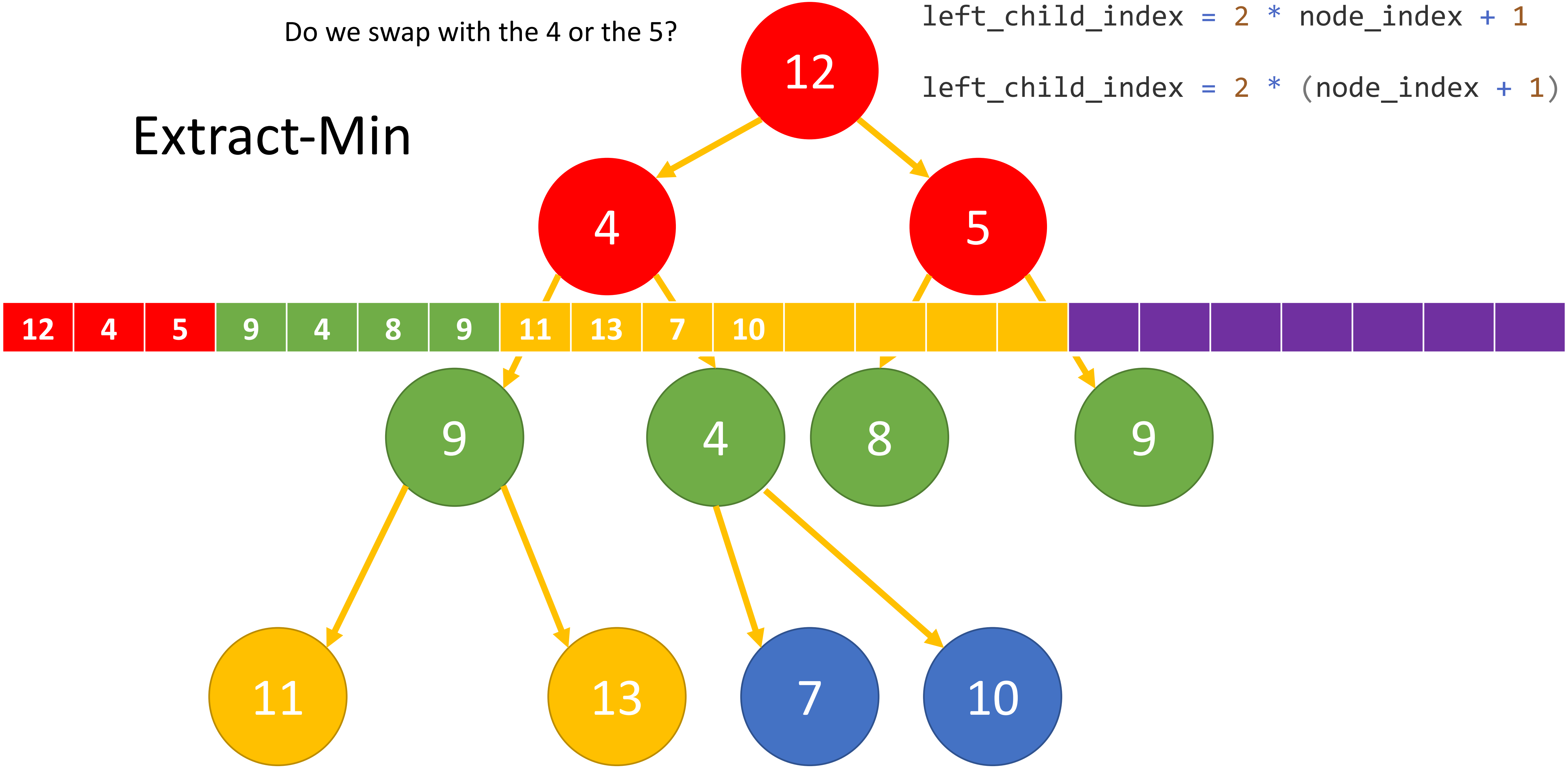


Do we swap with the 4 or the 5?

Extract-Min

```
left_child_index = 2 * node_index + 1

left_child_index = 2 * (node_index + 1)
```



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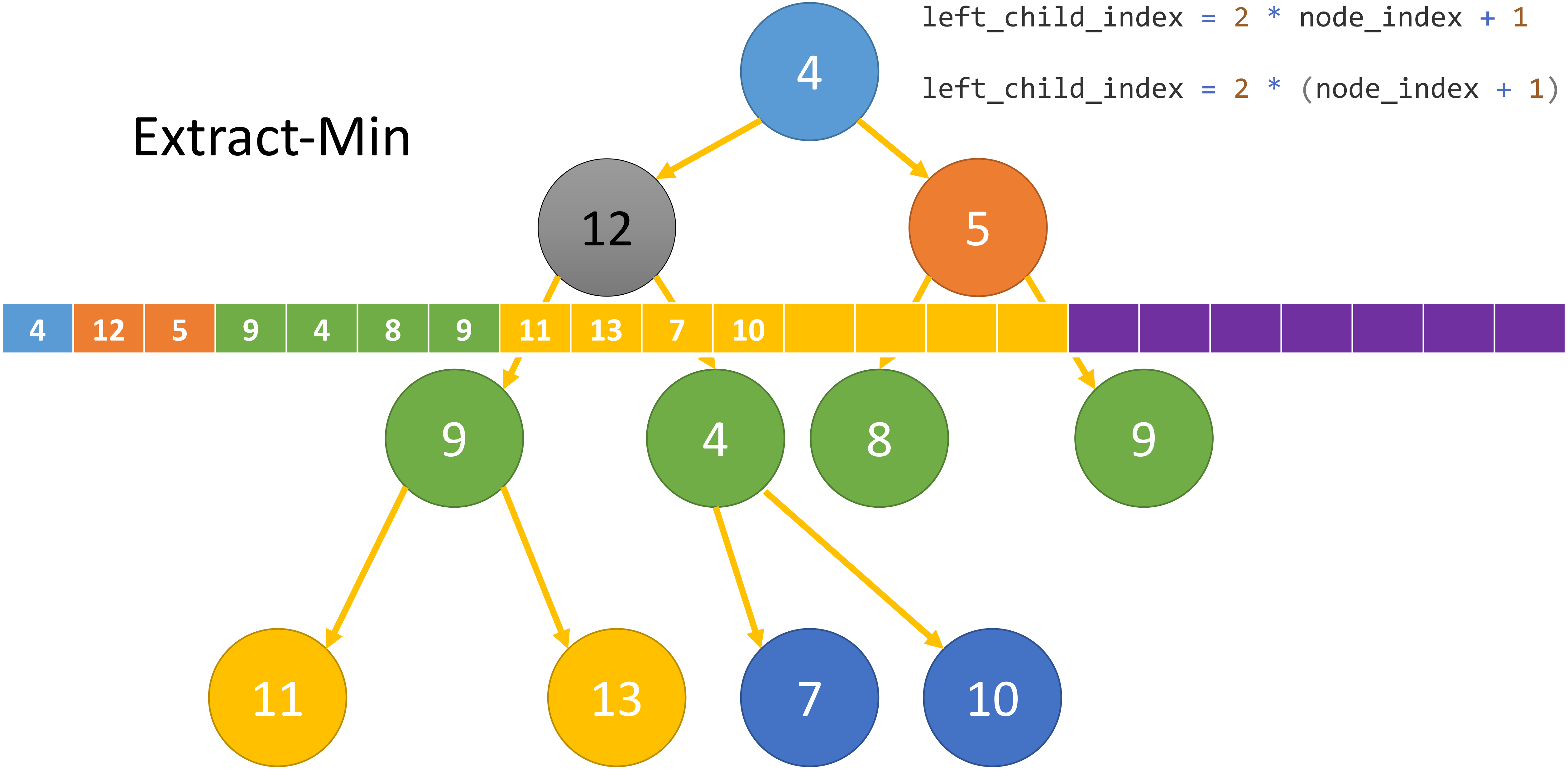

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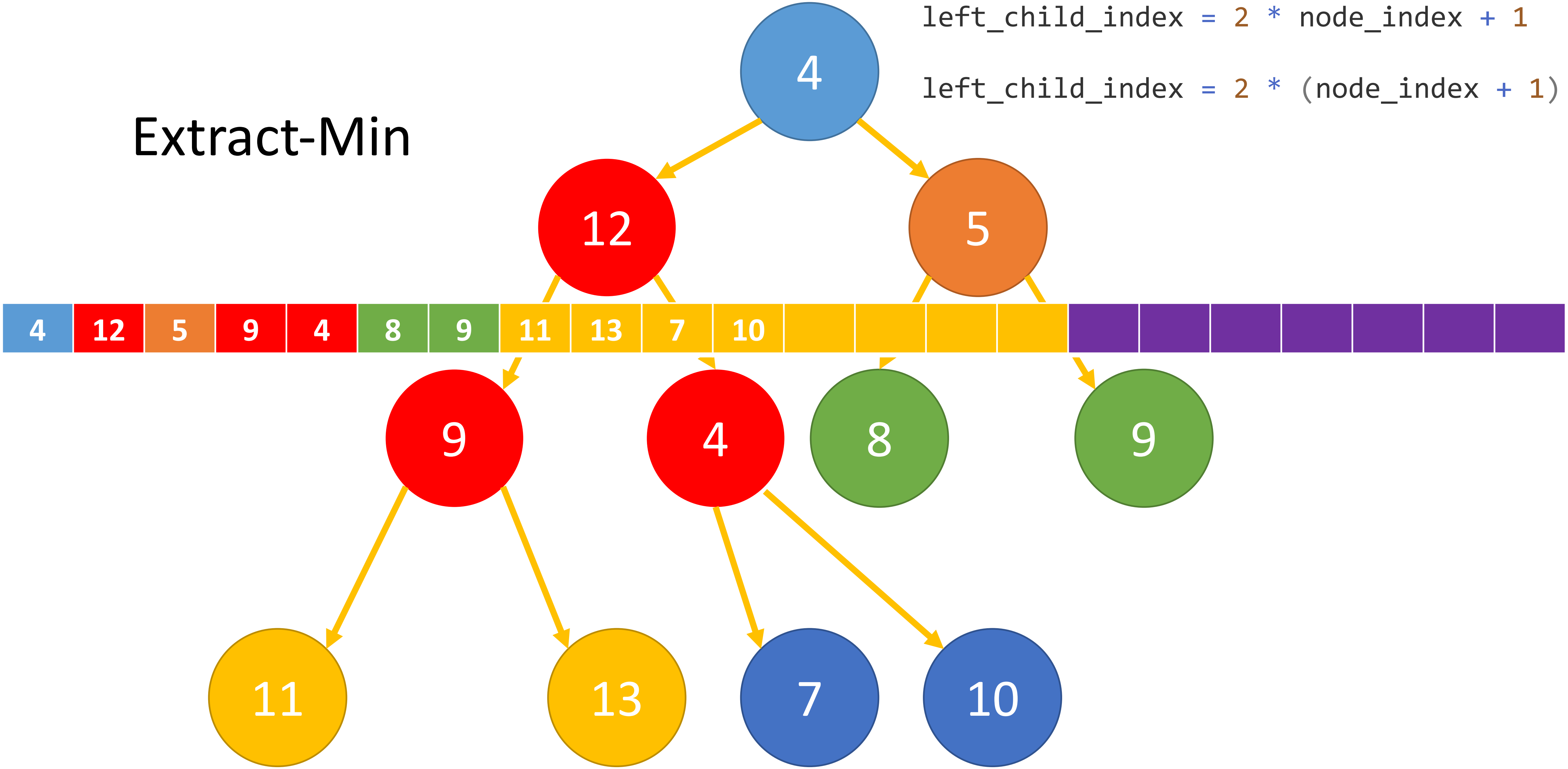

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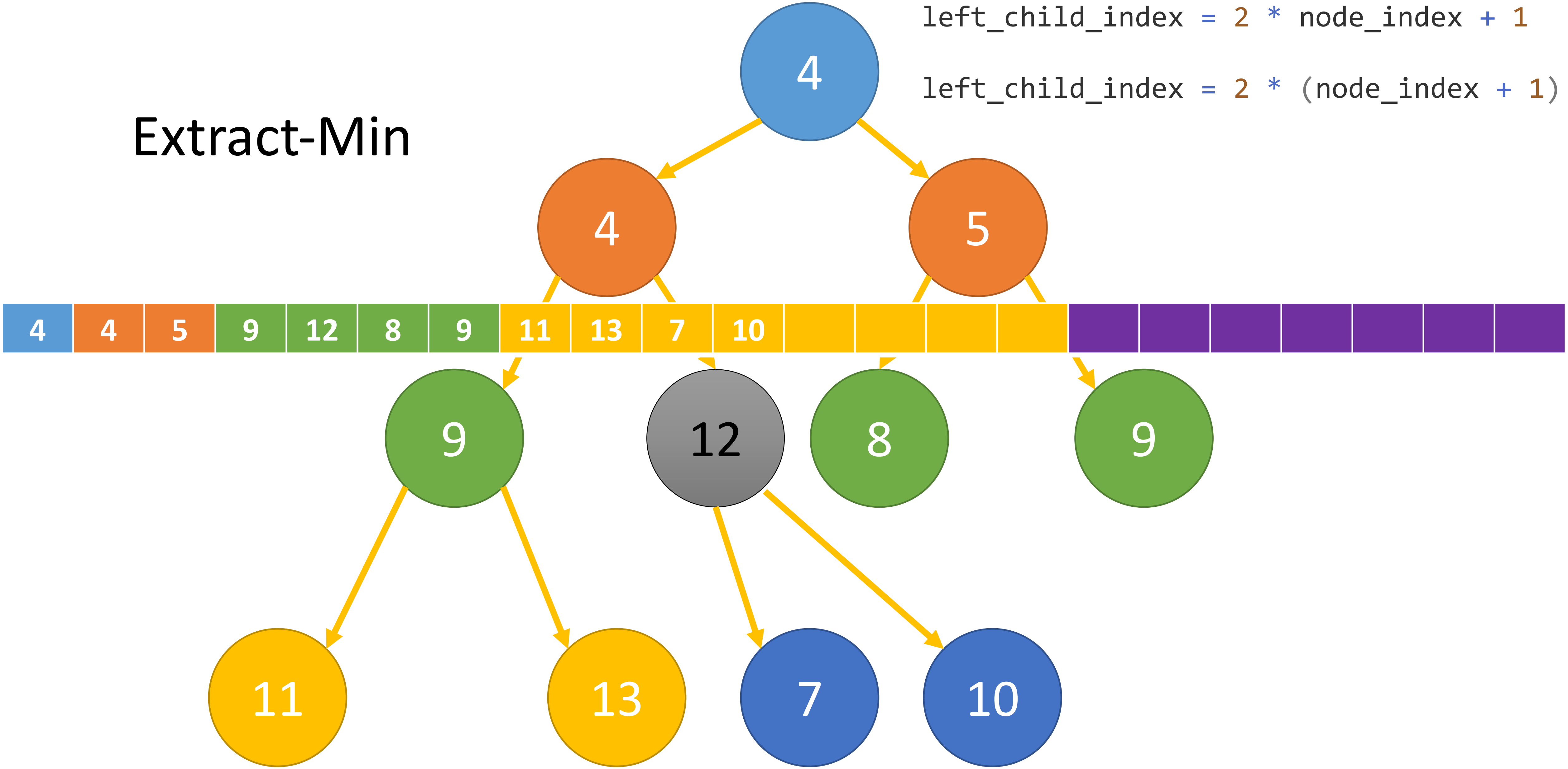

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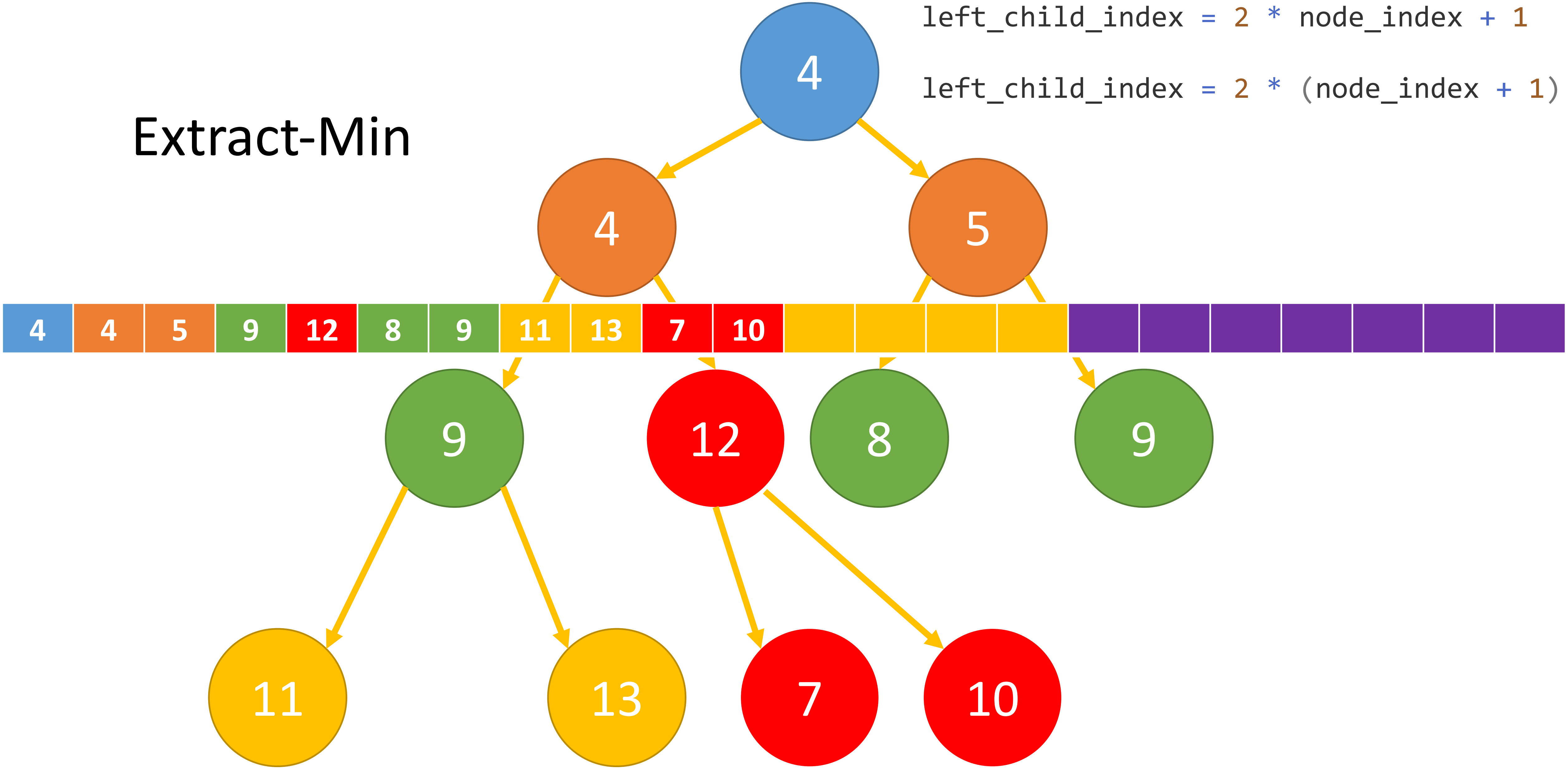

Extract-Min

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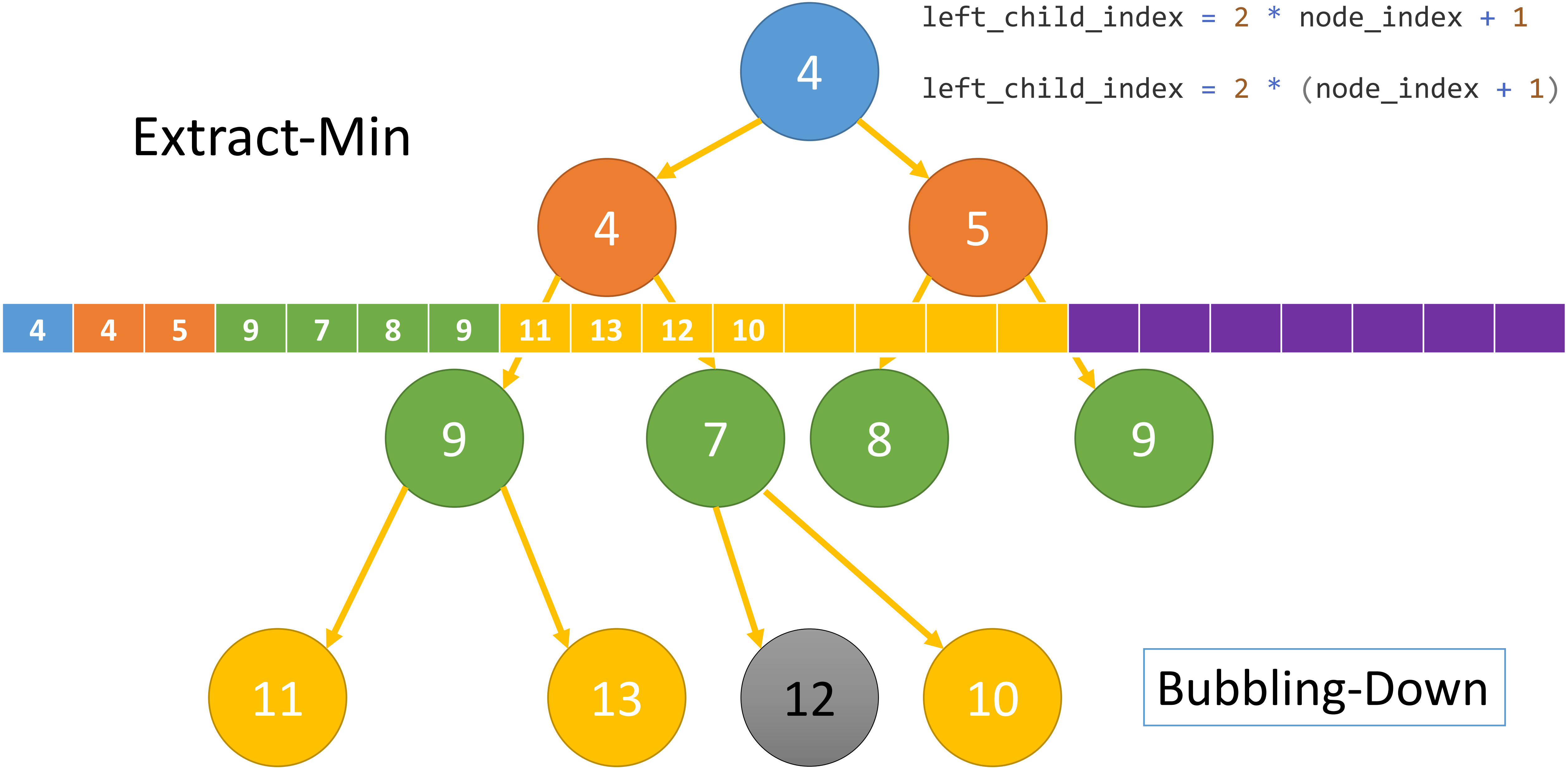

Bubbling-Down

```
FUNCTION Dijkstra(G, start_vertex)
    found = {}
    lengths = {v: INFINITY FOR v IN G.vertices}

    found.add(start_vertex)
    lengths[start_vertex] = 0

    WHILE found.length != G.vertices.length
        FOR v IN found
            FOR vOther, weight IN G.edges[v]
                IF vOther NOT IN found
                    vOther_length = lengths[v] + weight
                    IF vOther_length < min_length
                        min_length = vOther_length
                        vMin = vOther
        found.add(vMin)
        lengths[vMin] = min_length

    RETURN lengths
```

What is the running time?

How many times does the outer loop run?

O(n)

How many times do the inner two loops run?

O(m)

```
FUNCTION Dijkstra(G, start_vertex)
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        FOR v IN found
            FOR vOther, weight IN G.edges[v]
                IF vOther NOT IN found
                    vOther_length = lengths[v] + weight
                    IF vOther_length < min_length
                        min_length = vOther_length
                        vMin = vOther
        found.add(vMin)
        lengths[vMin] = min_length

    RETURN lengths
```

What is the running time?

We can bring this down to O(m lg m) with a simple change.

State of the art of Dijkstra's:
O(m + n lg n)
(uses Fibonacci heap)

```python
def dijkstras_heap(adjacency_list, start_vertex):
    """Dijkstra's Algorithm implemented with all vertices placed in a heap.

    This version of Dijkstra's Algorithm has a running time of O(m lg m).
    """

    n = len(adjacency_list)

    path_lengths = {v: inf for v in adjacency_list}
    predecessors = {v: None for v in adjacency_list}

    path_lengths[start_vertex] = 0
    predecessors[start_vertex] = None

    found = set()
    vertex_min_heap = [(path_lengths[start_vertex], start_vertex)]

    while len(found) != n:
        vfrom_length, vfrom = heappop(vertex_min_heap)
        found.add(vfrom)

        for vto, weight in adjacency_list[vfrom]:
            path_length = vfrom_length + weight
            if path_length < path_lengths[vto]:
                path_lengths[vto] = path_length
                predecessors[vto] = vfrom

                heappush(vertex_min_heap, (path_lengths[vto], vto))

    return path_lengths, predecessors
```

Not optimal but works very well in practice.

```python
while len(found) != n:
    vfrom_length, vfrom = heappop(vertex_min_heap)
    found.add(vfrom)

    for vto, weight in adjacency_list[vfrom]:
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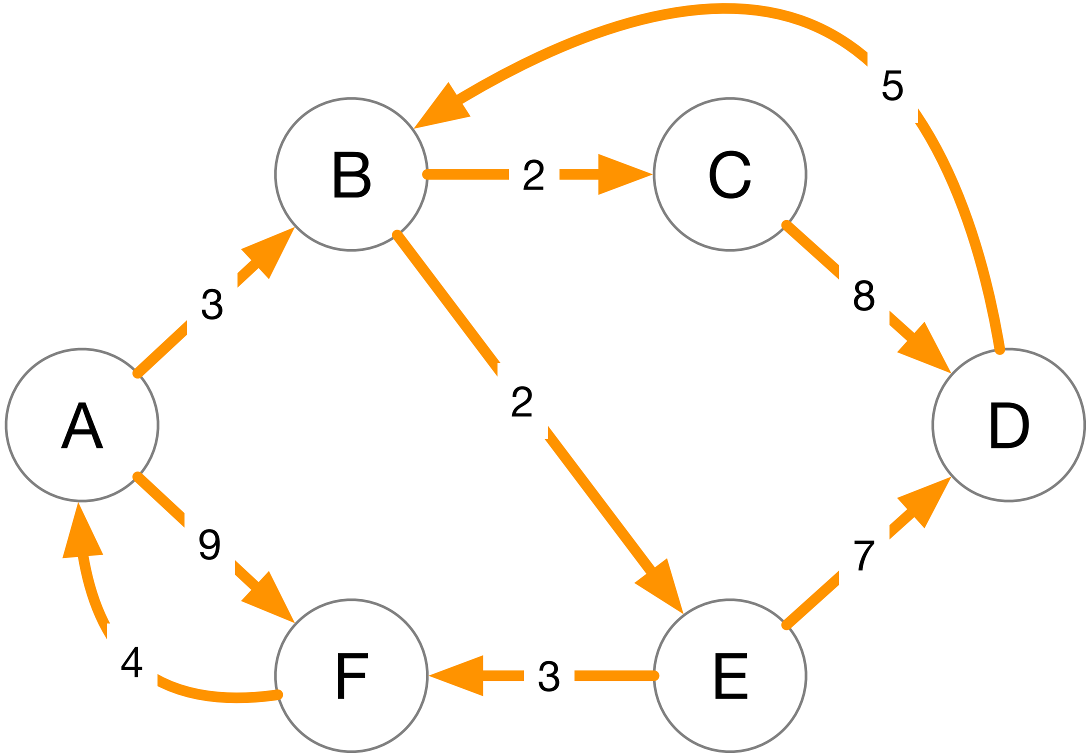

```python
def print_path(end_vertex, predecessors):

    path = [end_vertex]
    pred = predecessors[end_vertex]

    while pred is not None:

        path.append(pred)
        pred = predecessors[pred]

    print(" -> ".join([str(v) for v in reversed(path)]))
```

Dijkstra’s Algorithm Correctness

Theorem:

- Dijkstra’s algorithm will find the shortest path from the start vertex to every other vertex on any graph with non-negative weights.

Proof using a loop invariant. Loop predicate:

- At the start of each iteration of the while loop, the shortest path has been found for every vertex in the found set

Initialization

- Initially, the found set is empty. So, the invariant is trivially true.

Loop predicate/invariant: At the start of each iteration of the while loop, the shortest path has been found for every vertex in the found set

```
…
found = set()
…
while len(found) != n:
    vfrom_length, vfrom = heappop(vertex_min_heap)
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```

Maintenance (1)

Loop predicate/invariant: At the start of each iteration of the while loop, the shortest path has been found for every vertex in the found set

- Assume all previous iterations have produced the correct shortest path for all vertices in the found set.
- For purposes of a contradiction, assume that when a vertex u is added to the found set its path length is not optimal.
- At the time u is found we must have some path to u

```
while len(found) != n:
    vfrom_length, vfrom = heappop(vertex_min_heap)
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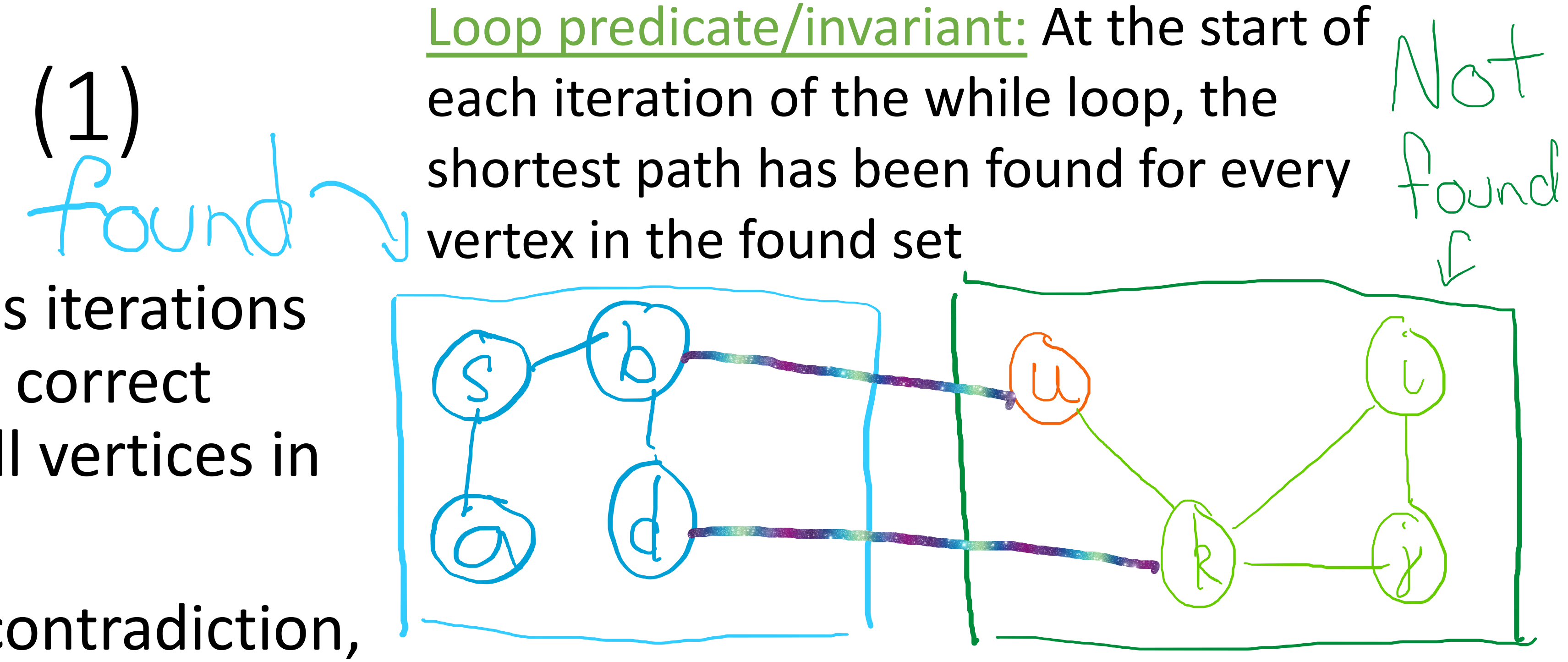


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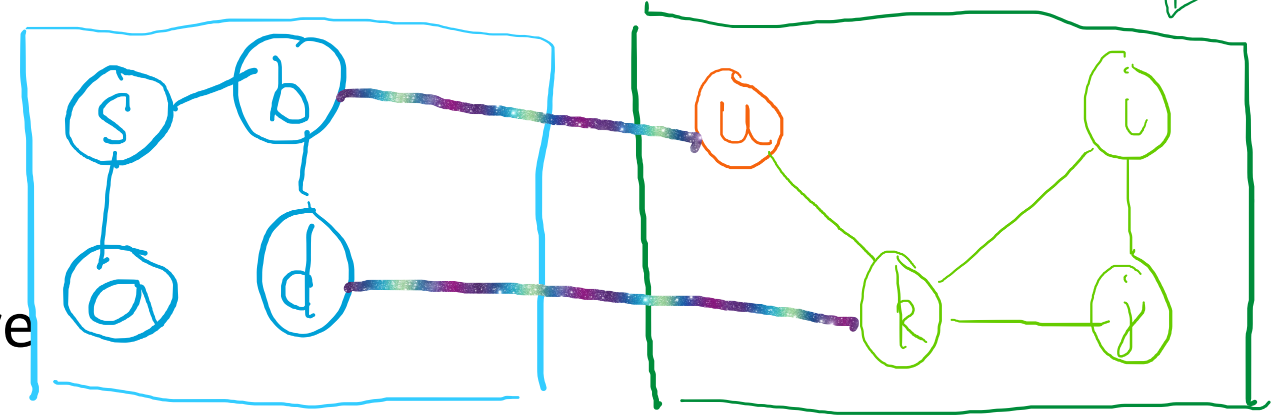
Loop predicate/invariant: At the start of each iteration of the while loop, the shortest path has been found for every vertex in the found set



- For purposes of a contradiction, assume that when a vertex u is added to the found set its path length is not optimal.
- At the time u is found we must have some path to u
- To have a shorter path to u, it must go through some vertex k not in found.
- But since we only have positive edges, a shorter path going through k, means that k must have been chosen before u. Contradiction.

```
while len(found) != n:
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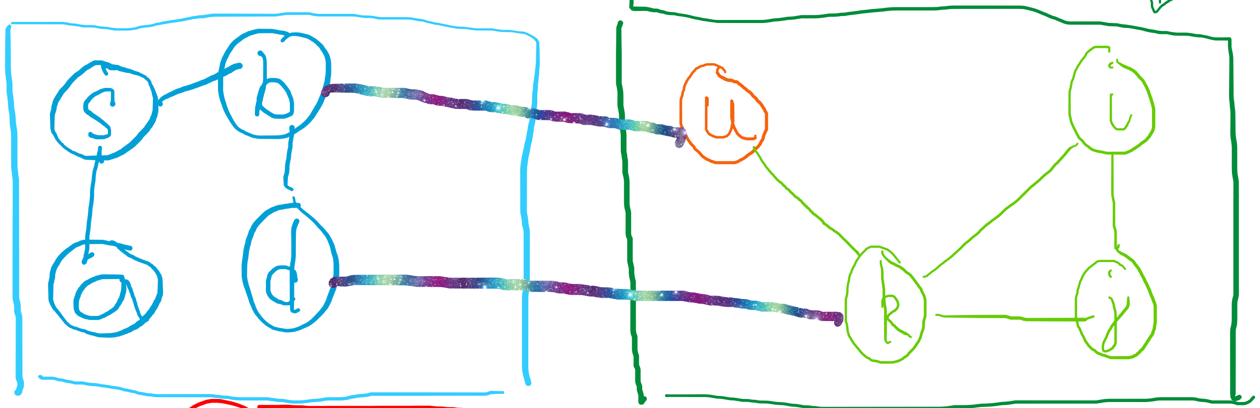
            predecessors[vto] = vfrom

            heappush(vertex_min_heap,
                     (path_lengths[vto], vto))
```

Termination

- The loop terminates when all vertices have been added to the found set.
- Given the loop invariant the shortest path to all vertices have been calculated.

Loop predicate/invariant: At the start of each iteration of the while loop, the shortest path has been found for every vertex in the found set

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