Depth First Search and Topological Orderings

https://cs.pomona.edu/classes/cs140/

Outline

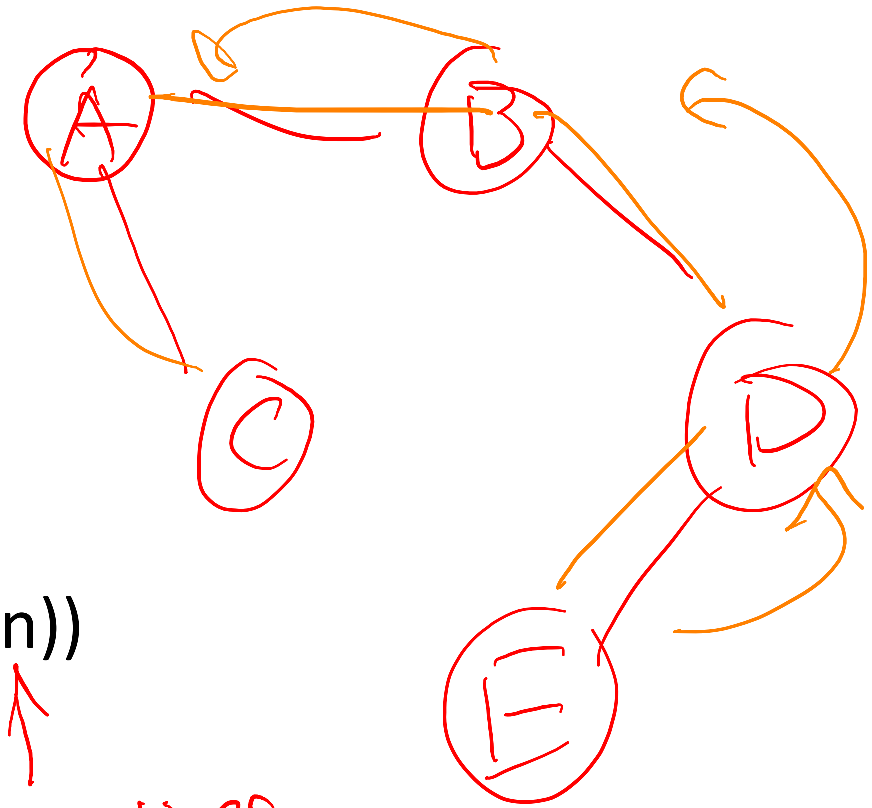
Topics and Learning Objectives

- Discuss depth first search for graphs
- Discuss topological orderings

Exercise

- DFS run through

Depth-First Search



- Explore more aggressively, and
- Backtrack when needed
- Linear time algorithm (again O(m + n))

edge vertices

- Computes topological ordering (we'll discuss this today)

```
FUNCTION DFS(G, start_vertex)
    found = {v: FALSE FOR v IN G.vertices}
    DFSRecursion(G, start_vertex, found)
    RETURN found

FUNCTION DFSRecursion(G, v, found)
    found[v] = TRUE
    FOR vOther IN G.edges[v]
        IF found[vOther] == FALSE
            DFSRecursion(G, vOther, found)
```

Why is this non-recursive function necessary?

```
FUNCTION DFS(G, start_vertex)
    found = {v: FALSE FOR v IN G.vertices}
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            DFSRecursion(G, vOther, 
```

Why is this non-recursive function necessary?

```
FUNCTION BFS(G, start_vertex)
    found = {v: FALSE FOR v IN G.vertices}
    found[start_vertex] = TRUE
    visit_queue = [start_vertex]

    WHILE visit_queue.length != 0
        vFound = visit_queue.pop()
        FOR vOther IN G.edges[vFound]
            IF found[vOther] == FALSE
                found[vOther] = TRUE
                visit_queue.add(vOther)

    RETURN found
```

```
FUNCTION DFS(G, start_vertex)
    found = {v: FALSE FOR v IN G.vertices}
    DFSRecursion(G, start_vertex, found)
    RETURN found
```

Why is this non-recursive function necessary?

```
FUNCTION DFSRecursion(G, v, found)
    found[v] = TRUE
    FOR vOther IN G.edges[v]
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```

What kind of data structure would we need for an iterative version?

```
FUNCTION DFSRecursion(G, v, found)
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    FOR vOther IN G.edges[v]
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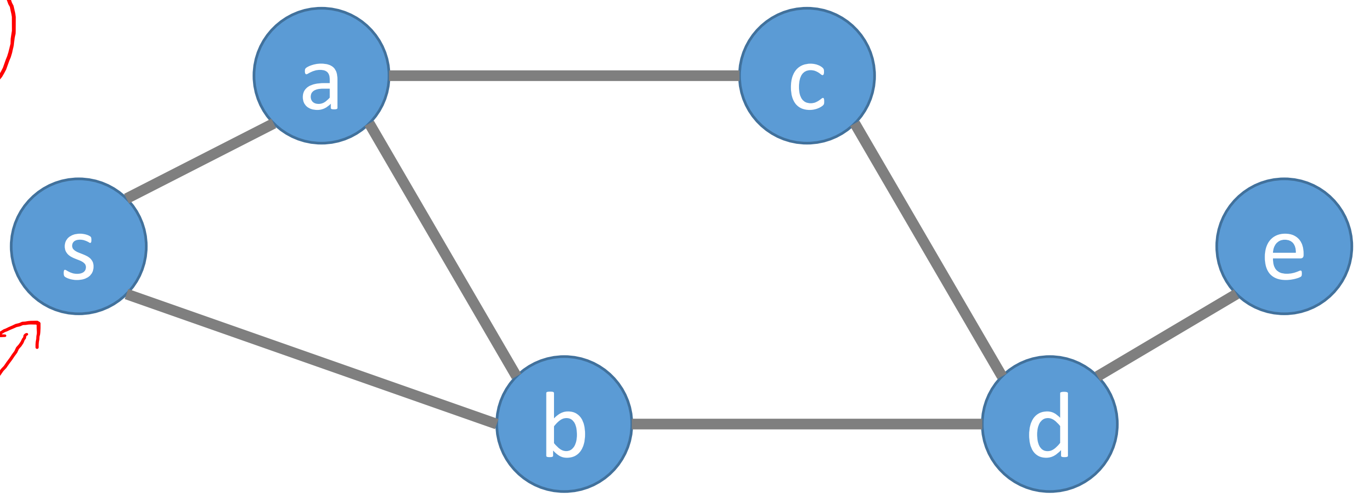

Given a tie, visit edges are in alphabetical order

Exercise

f = [~~s~~ ~~a~~ ~~b~~ ~~c~~ ~~d~~ ~~e~~]

DFSR(s)
DFSR(a)
DFSR(b)
DFSR(d)
DFSR(c)
DFSR(e)

s a b d c e

```
FUNCTION DFSRecursion(G, v, found)
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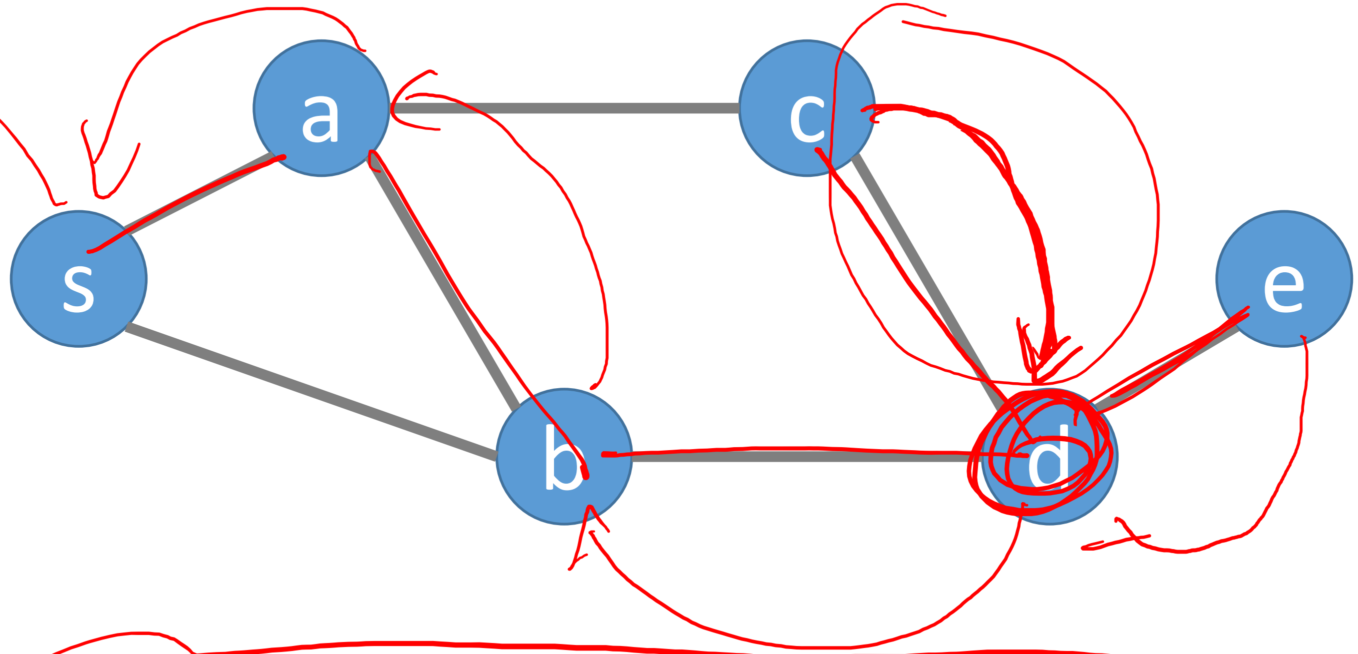
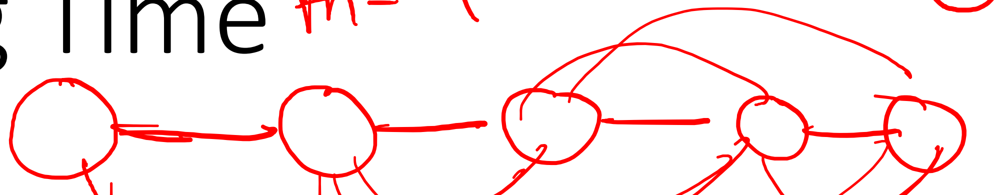


Given a tie, visit edges are in alphabetical order

Running Time

$n=5$

$\cancel{m=4}$ $m=9$



$0 \le m \le \binom{n}{2} = O(n^2)$

$\frac{n(n-1)}{2}$?

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    FOR vOther IN G.edges[v]
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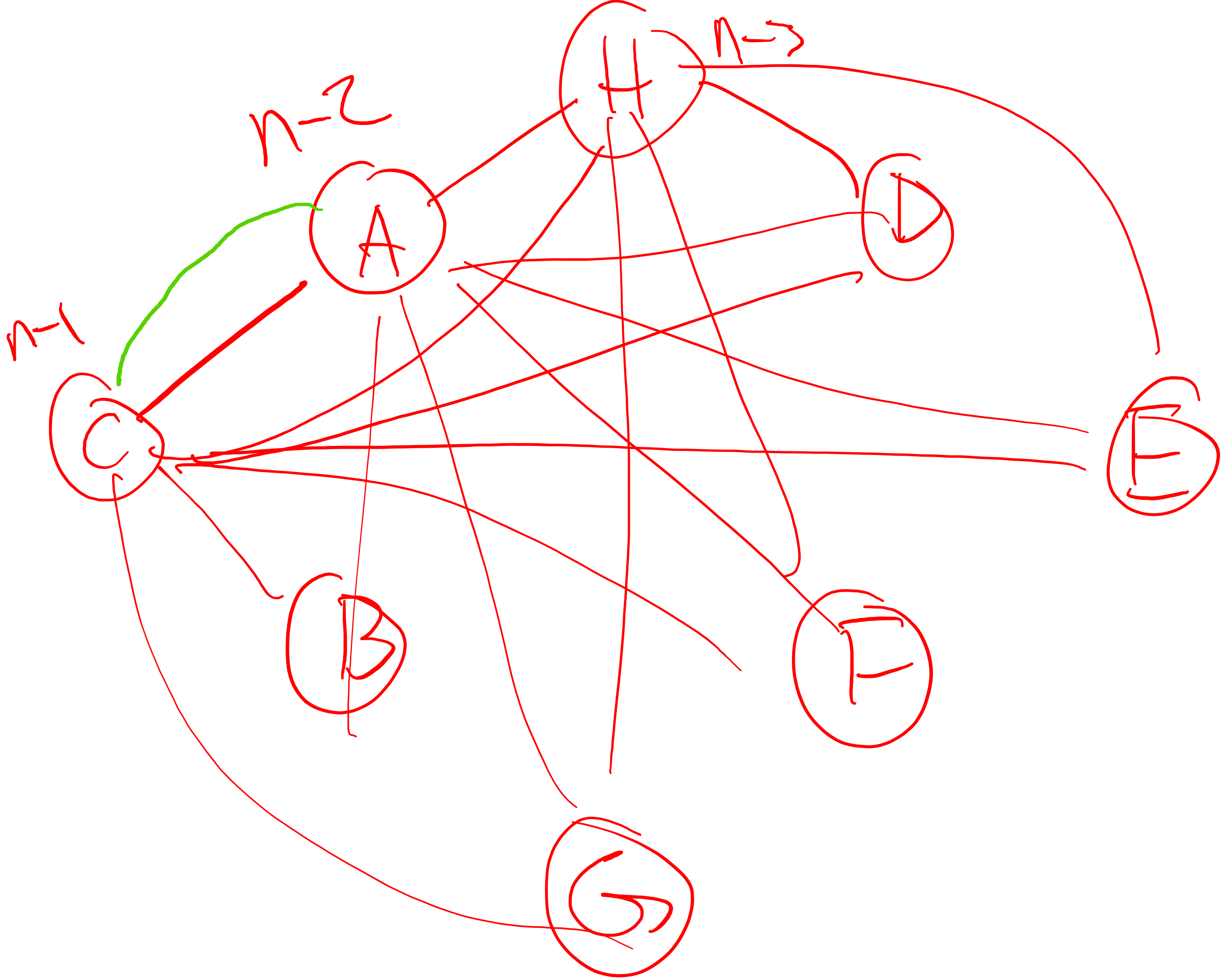
```

$O(n + m)$

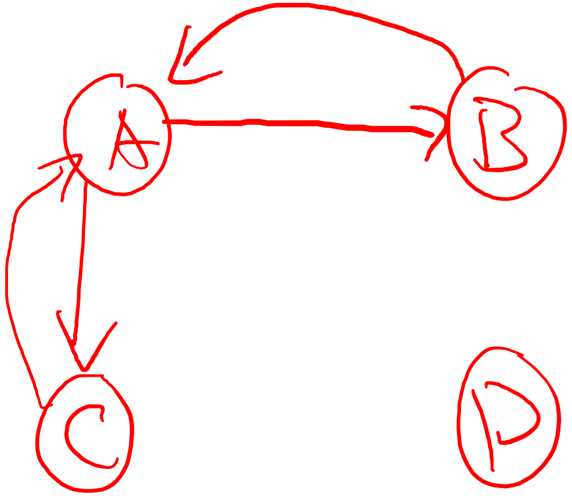
"linear"

What is the depth of the recursion tree?



$$\binom{n}{2} = \frac{n(n-1)}{2} \; ?$$

$$= O(n^2)$$



$\underline{n^2}$

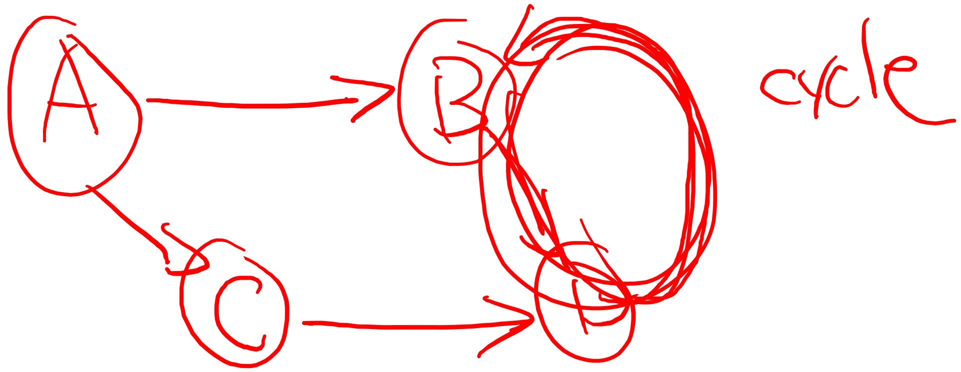
An example use case for DFS

Topological Orderings

DAG

Definition: a topological ordering of a directed acyclic graph is a labelling of the graph's vertices with "f-values" such that:

1. The f-values are of the set {1, 2, ..., n}
2. For an edge (u, v) of G, f(u) < f(v)



Topological Orderings

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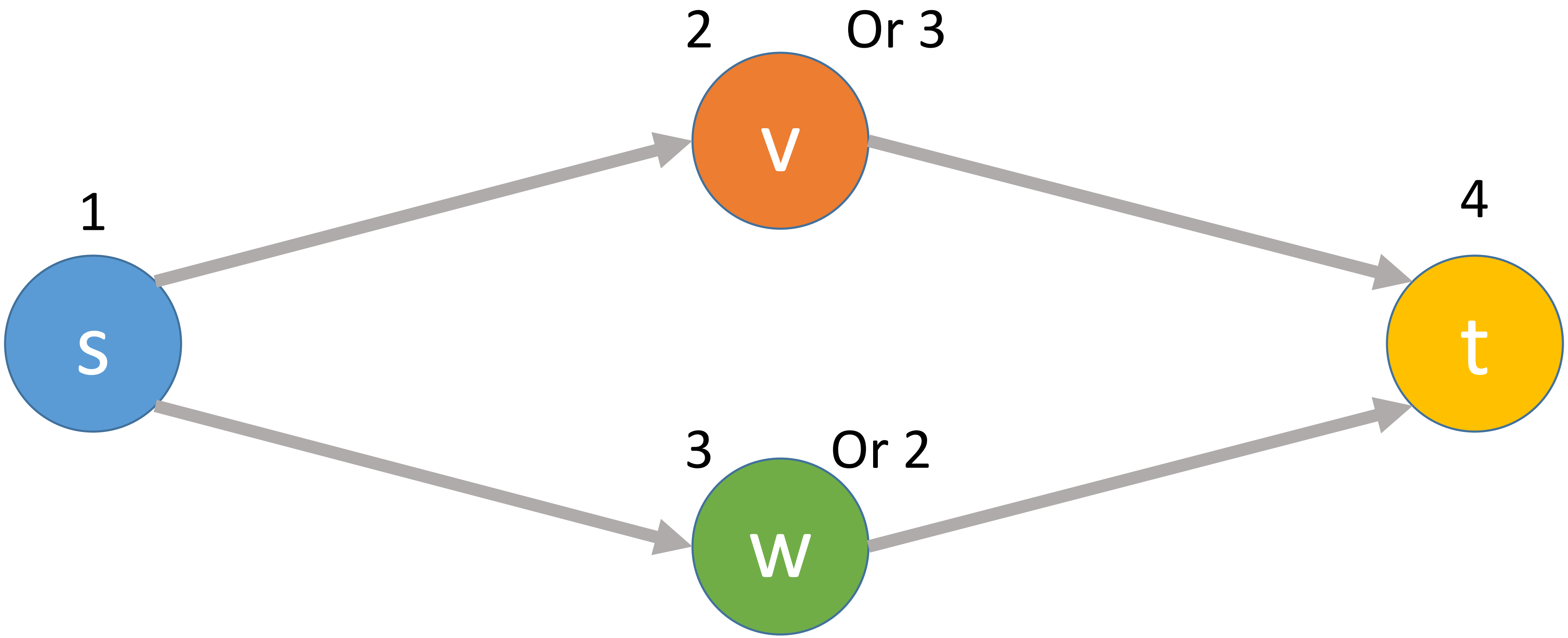
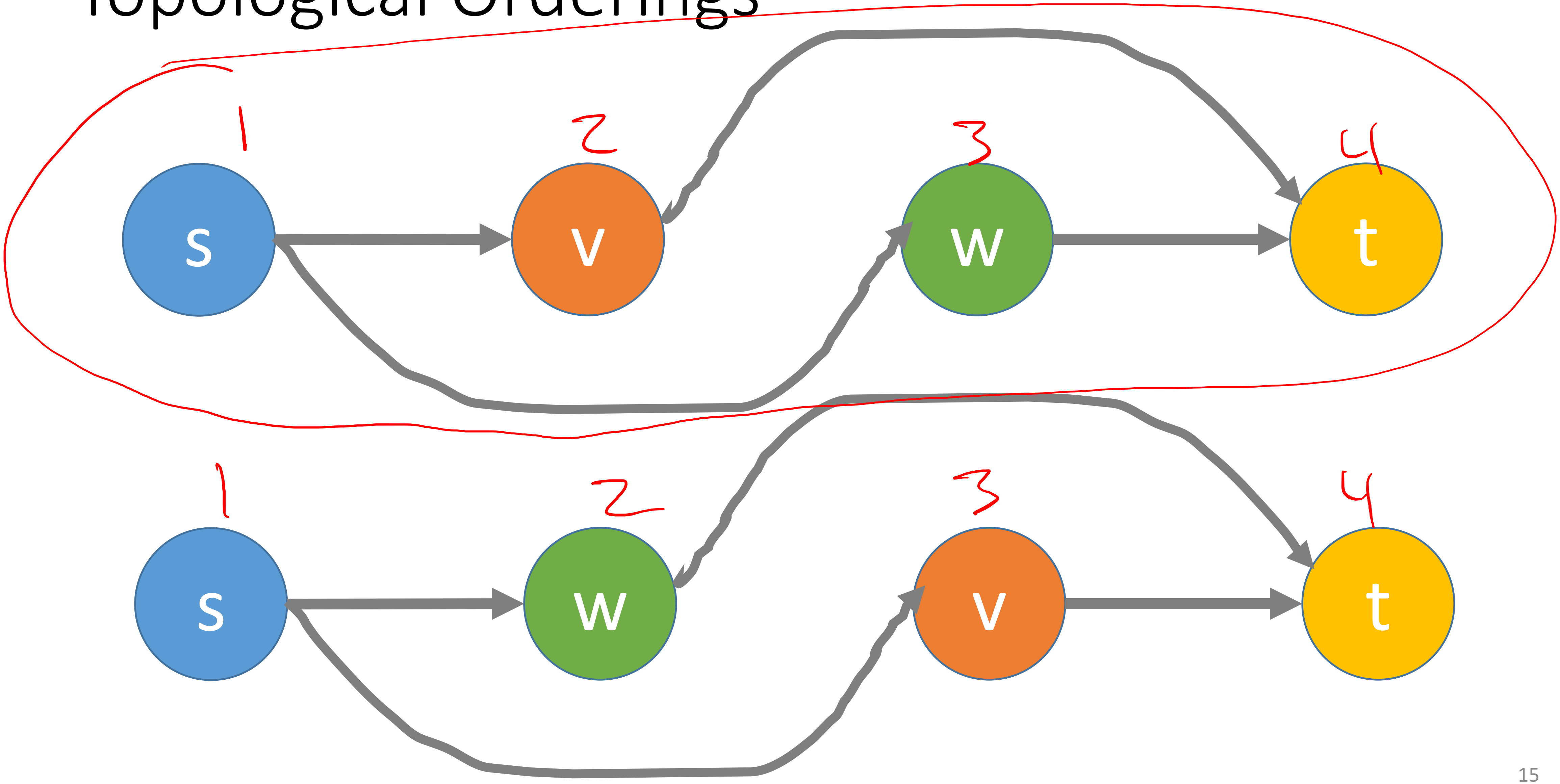
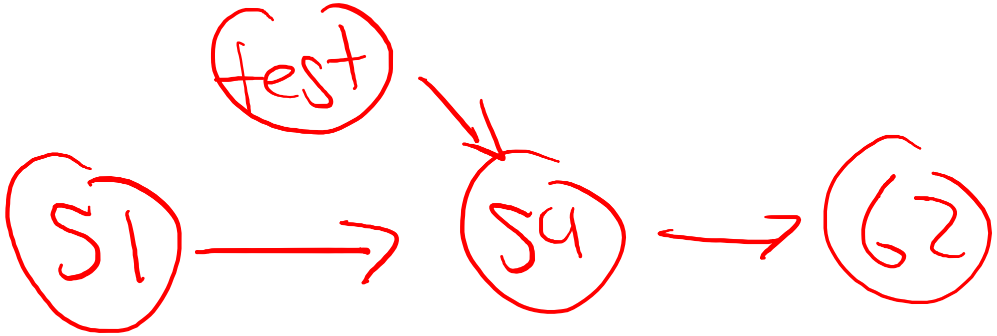


Topological Orderings



Topological Orderings



Can be used to graph a sequence of tasks while respecting all precedence constraints

- For example, a flow chart for your CS degrees
- I read a funding proposal where they were using topological orderings to schedule robot tasks for building a space station.

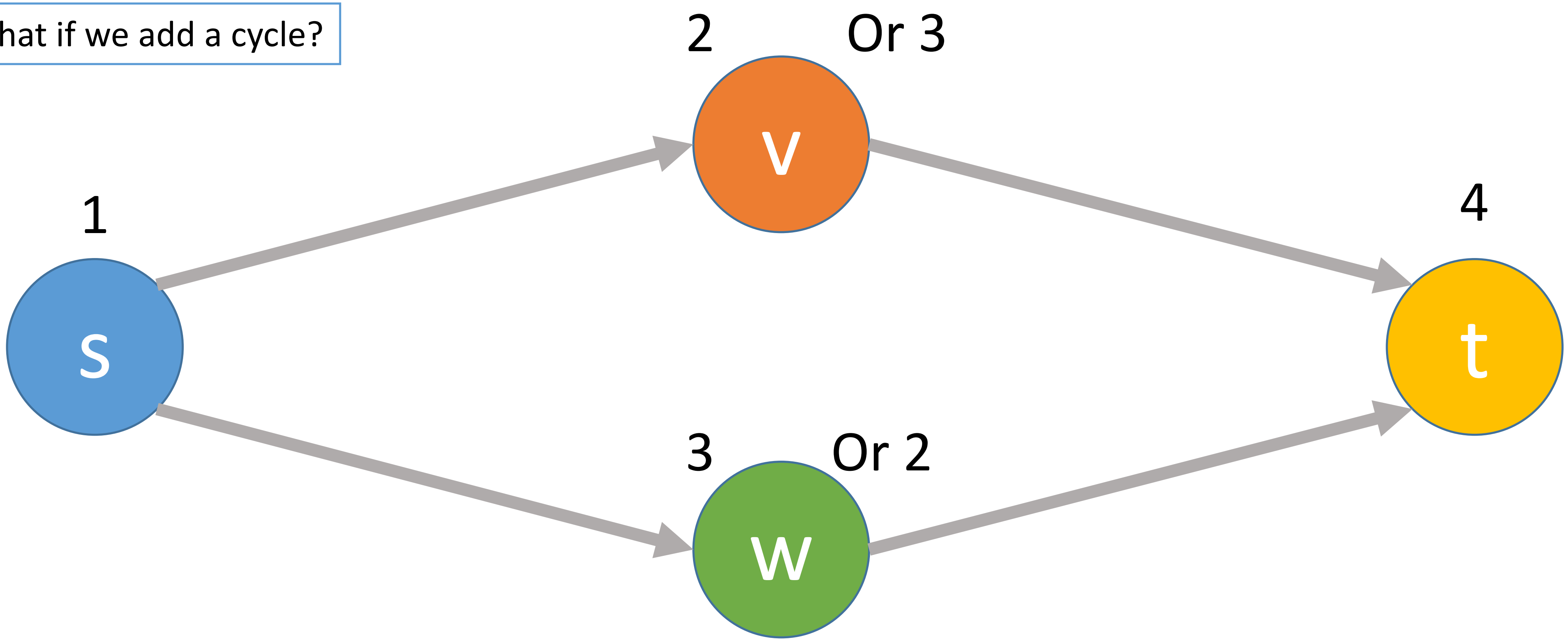
Requires the graph to be acyclic.

- Why?

Topological Orderings

1. The f-values are of the set {1, 2, ..., n}
2. For an edge (u, v) of G, f(u) < f(v)

What if we add a cycle?



How to Compute Topological Orderings?

Straightforward solution:

1. Let v be any sink of G
2. Set $f(v) = |V| = n$
3. Recursively conduct the same procedure on $G - \{v\}$

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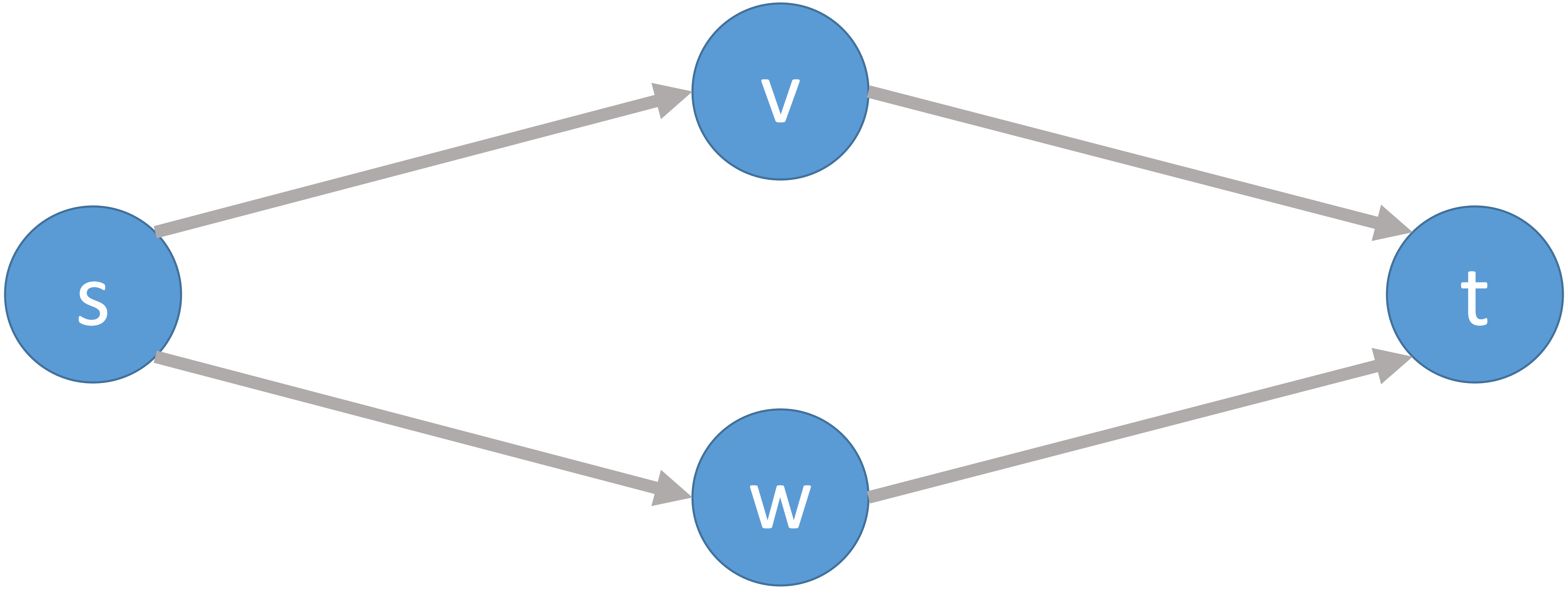
3. Recursively conduct the same procedure on G – {v}



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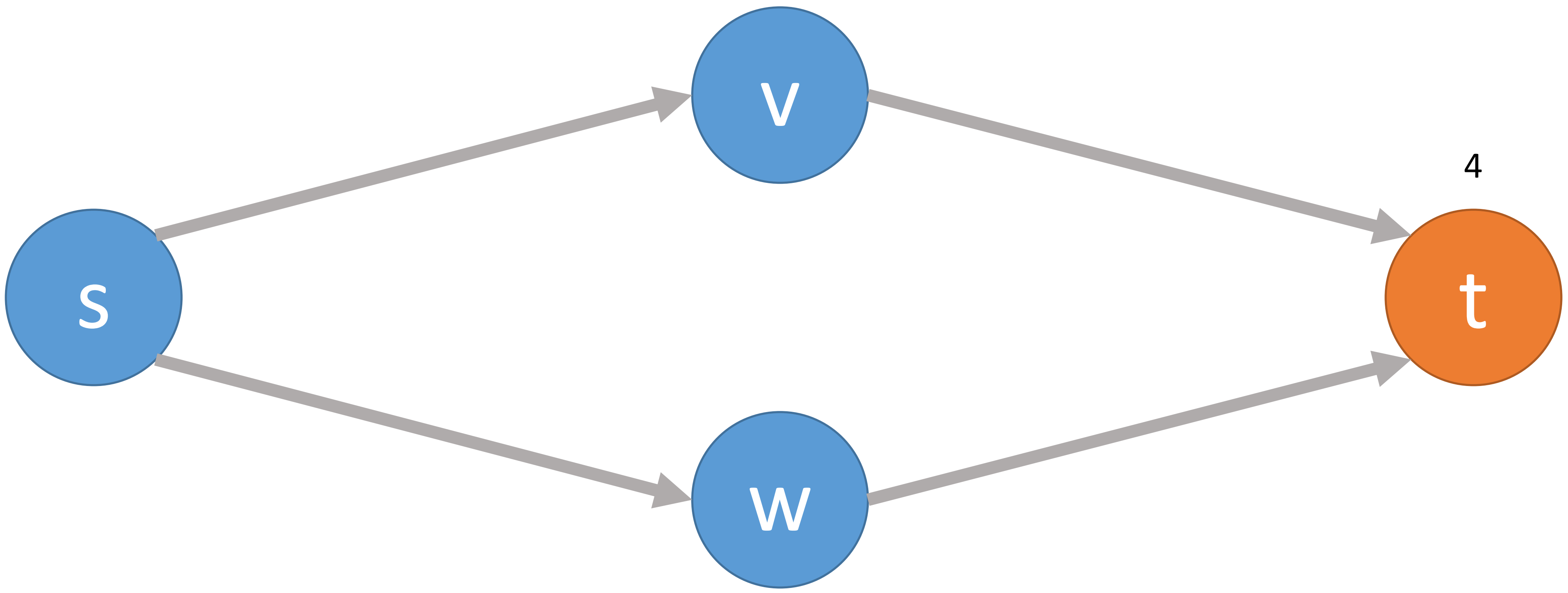

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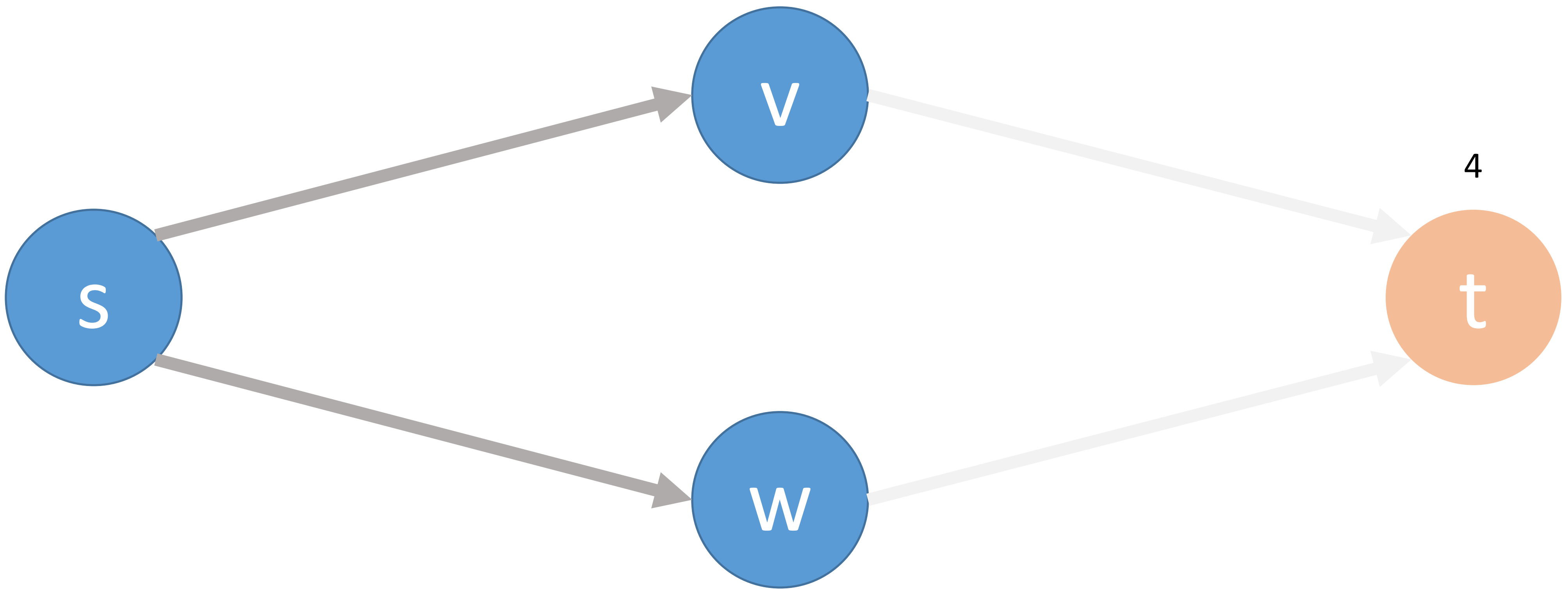

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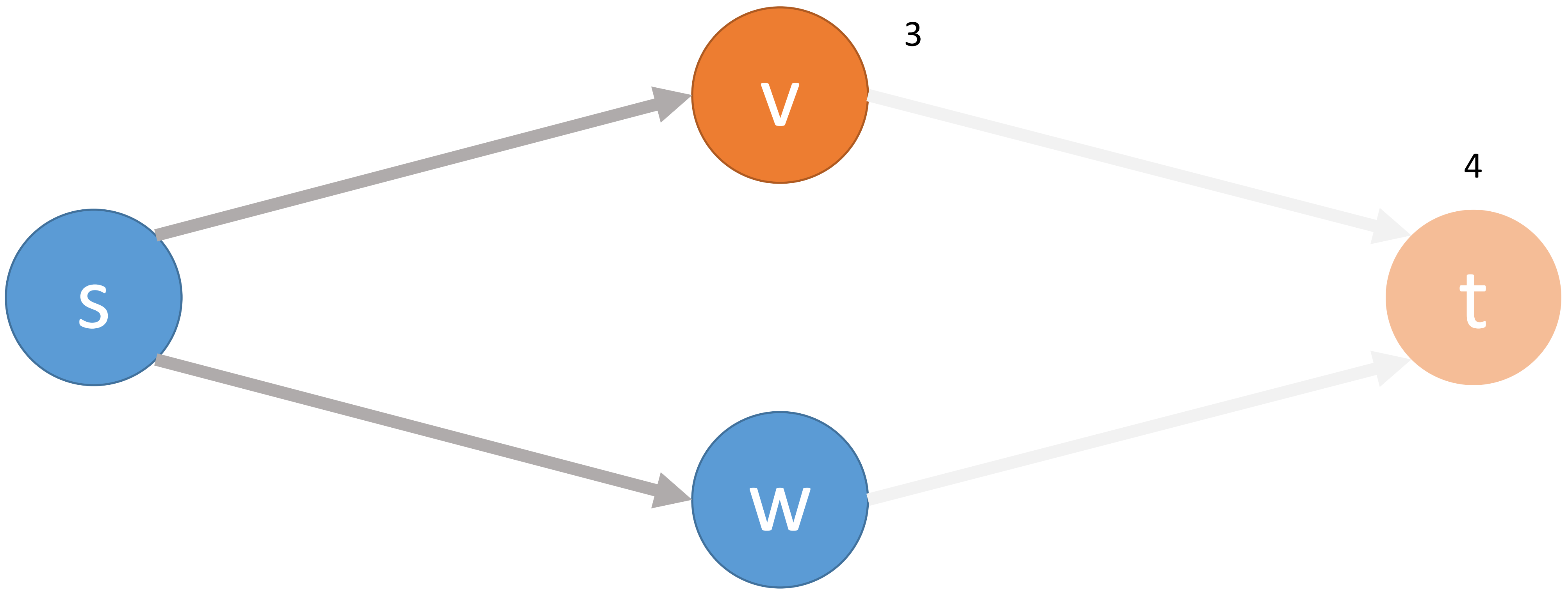

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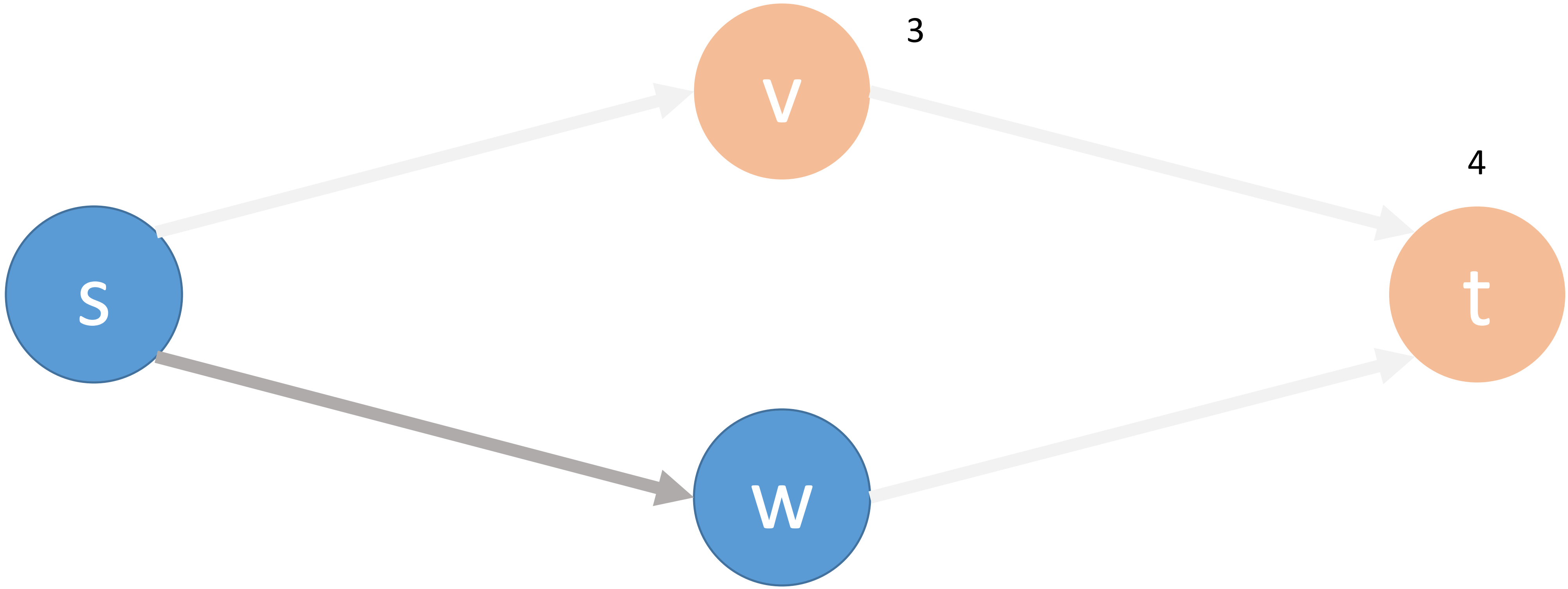

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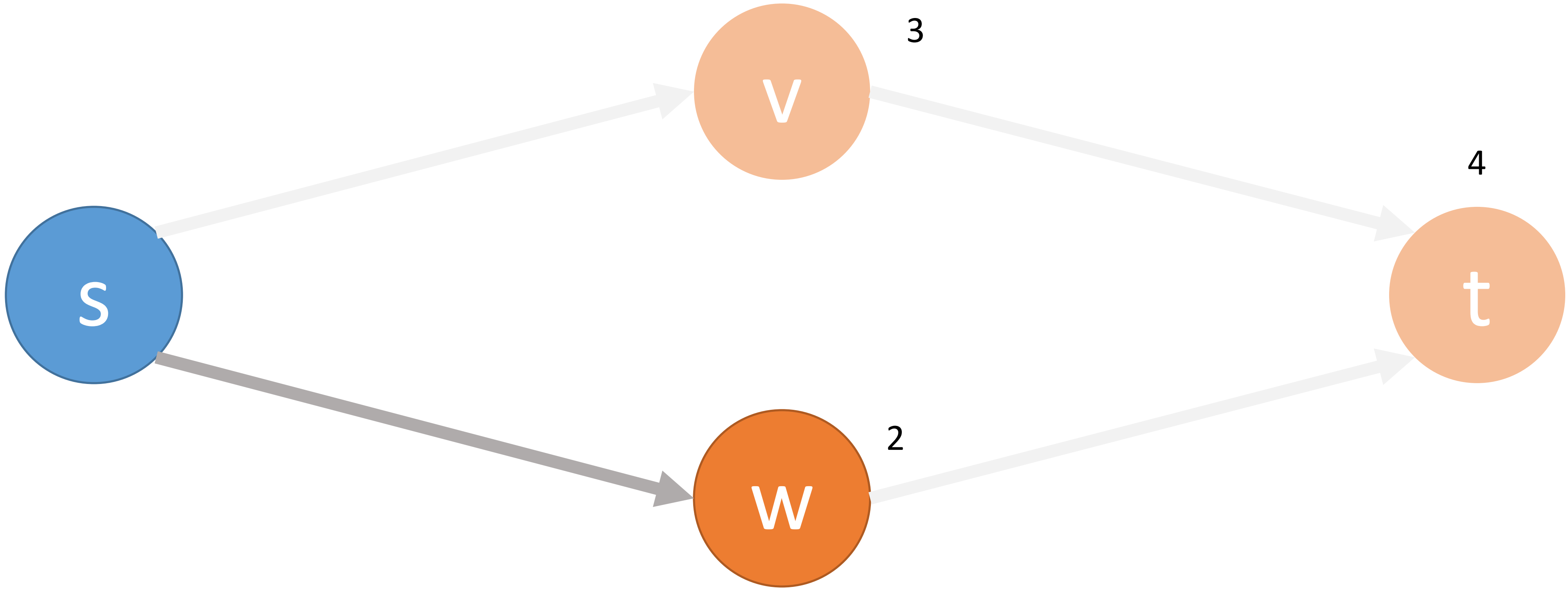

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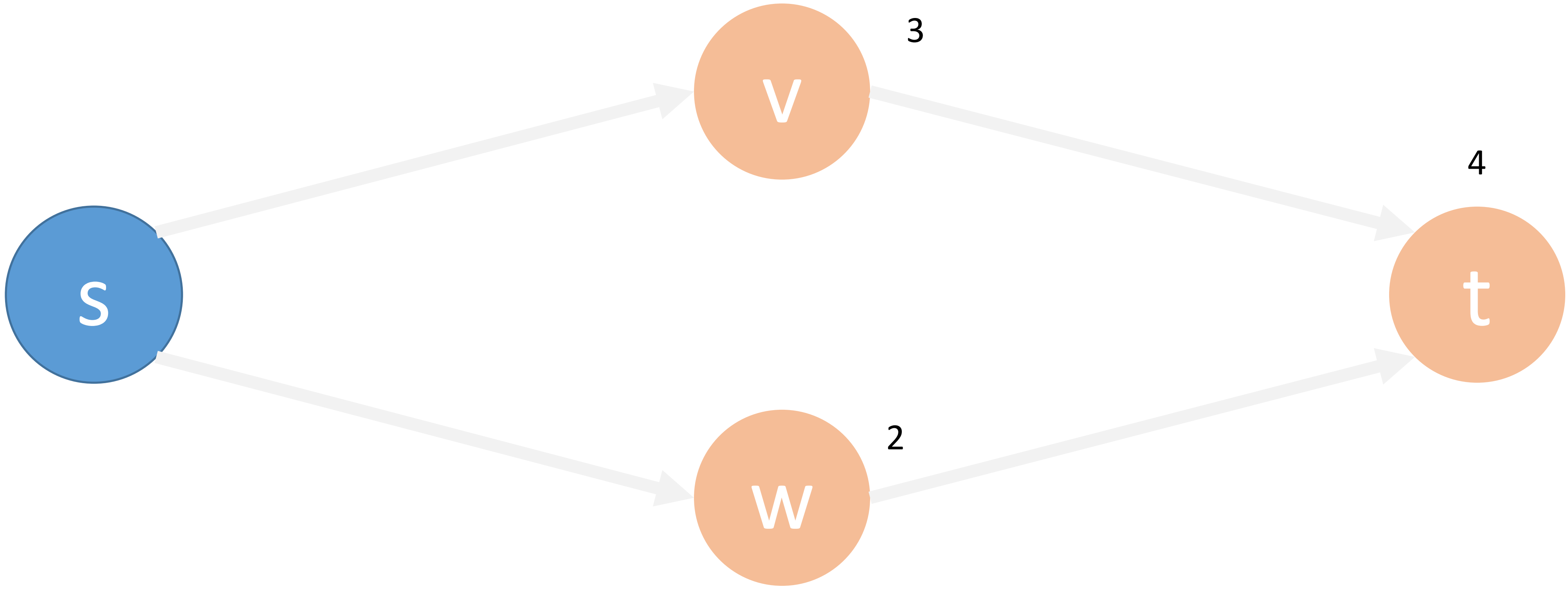

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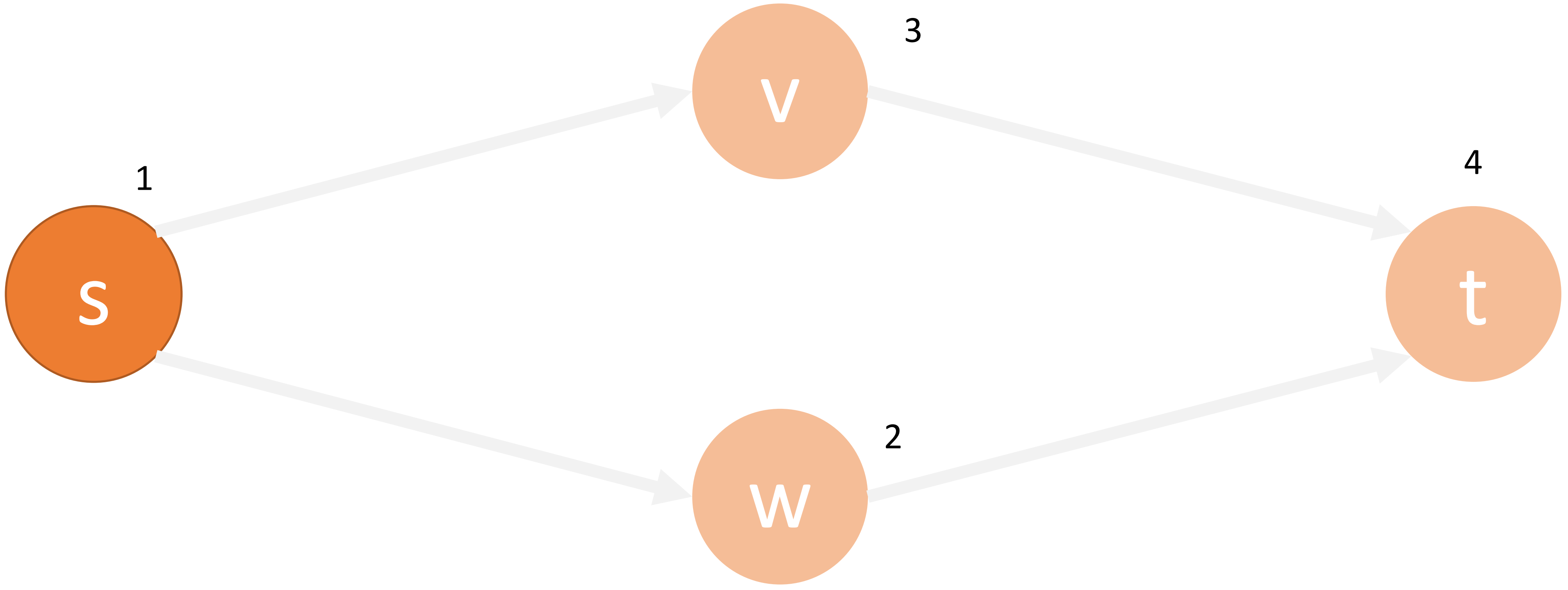

How to Compute Topological Orderings?

Straightforward solution:

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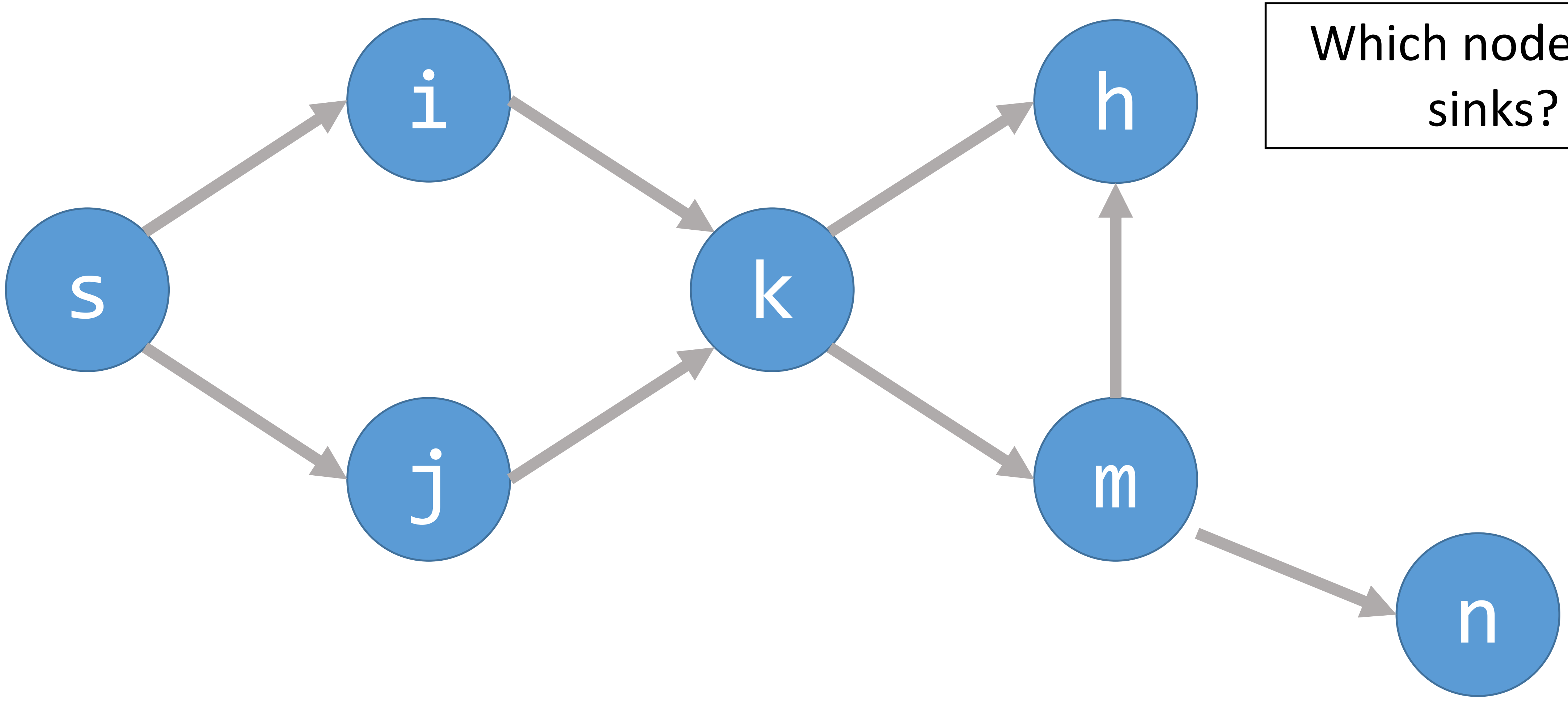
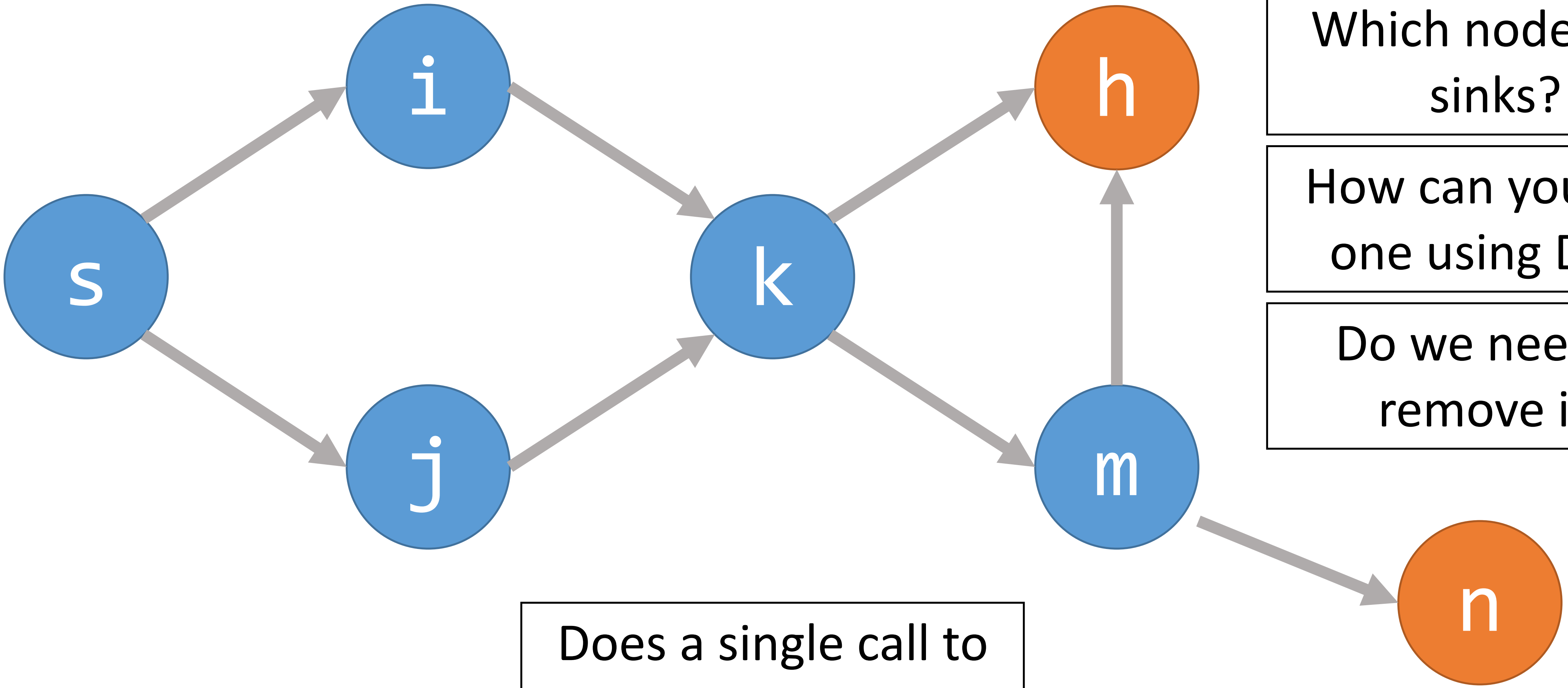
How can we do this with our DFS algorithm if we don't know which nodes are sinks?



Which nodes are sinks?



Which nodes are sinks?

How can you find one using DFS?

Do we need to remove it?

Does a single call to DFS label all nodes?

Solve with DFS

```
FUNCTION TopologicalOrdering(G)
    found = {v: FALSE FOR v IN G.vertices}
    fValues = {v: INFINITY FOR v IN G.vertices}
    f = G.vertices.length
    FOR v IN G.vertices
        IF found[v] == FALSE
            DFSTopological(G, v, found, f, fValues)
    RETURN fValues
```

```
FUNCTION DFSTopological(G, v, found, f, fValues)
    found[v] = TRUE
    FOR vOther IN G.edges[v]
        IF found[vOther] == FALSE
            DFSTopological(G, vOther, found, f, fValues)
    fValues[v] = f
    f = f - 1
```

```
FUNCTION TopologicalOrdering(G)
    found = {v: FALSE FOR v IN G.vertices}
    fValues = {v: INFINITY FOR v IN G.vertices}
    f = G.vertices.length = 7 6 5 4 3 2
    FOR v IN G.vertices    i, s
        IF found[v] == FALSE
            DFSTopological(G, v, found, f, fValues)
    RETURN fValues

FUNCTION DFSTopological(G, v, found, f, fValues)
    found[v] = TRUE
    FOR vOther IN G.edges[v]
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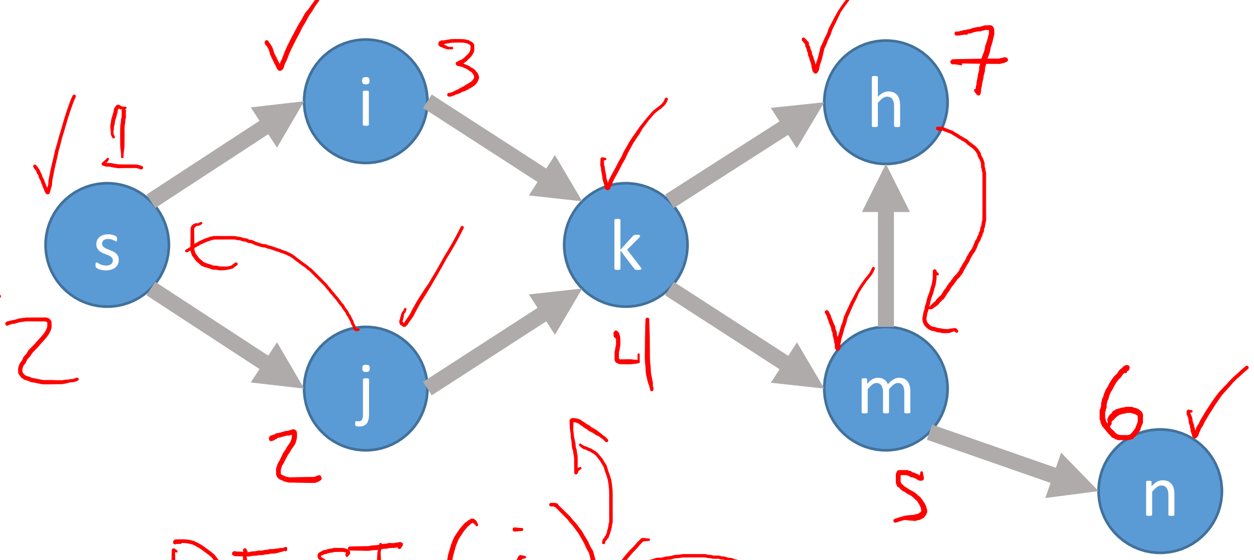


DFST(i)

DFST(k)

DFST(m)

DFST(h)

DFST(n)

DFST(s)

DFST(j)

Running Time

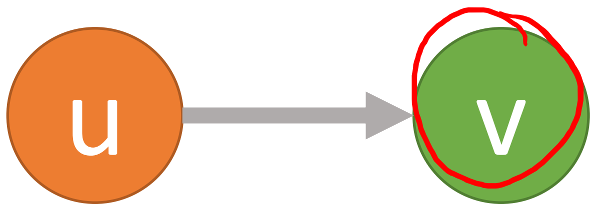
Again, this algorithm is O(n + m)

We only consider each vertex once, and

We only consider each edge once (twice if you consider backtracking)

Correctness of DFS Topological Ordering

We need to show that for any (u, v) that f(u) < f(v)



1. Consider the case when u is visited first
    1. We recursively look at all paths from u and **label those vertices first**
    2. So, f(u) must be less than f(v)

2. Now consider the case when v is visited first
    1. There is **no path back** to u, so v gets labeled before we explore u
    2. Thus, f(u) must be less than f(v)

```
FUNCTION DFSTopological(G, v, found,
                        f, fValues)
  found[v] = TRUE
  FOR vOther IN G.edges[v]
    IF found[vOther] == FALSE
      DFSTopological(G, vOther,
        found, f, fValues)
  fValues[v] = f
  f = f - 1
```

How do we know that there is no path from v to u?

Topological Ordering

- We can use DFS to find a topological ordering since a DFS will search as far as it can until it needs to backtrack

- It only needs to backtrack when it finds a sink

- Sinks are the first values that must be labeled

$$E[X_{2D}] = \underbrace{\frac{1}{36}}\cdot 2 + \frac{2}{36}\cdot 3 + \frac{3}{36}\cdot 4 + \cdots$$

$$E[X_{2D}] = E[X_1] + E[X_2]$$

$$3.5 + 3.5 = 7$$