Deterministic Selection

https://cs.pomona.edu/classes/cs140/

Outline

Topics and Learning Objectives

- Analyze an irregular, deterministic, and divide and conquer algorithm for the selection problem.

Exercise

- Recursion Tree

Resources

- Algorithms Illuminated Chapter 6

Selection Problem

Input: A set of n numbers and an integer i, with $1 \leq i \leq n$

Output: The element that is larger than exactly $i - 1$ other elements

- Known as the i^{th} order statistic or the i^{th} smallest number
- The minimum element is the 1^{st} order statistic ($i = 1$)
- The maximum element is the n^{th} order statistic ($i = n$)

- What is "i" for the median? (an expression base on n)
  - If n is even, then the medians are the $n/2$ and $n/2 + 1$ order statistics
  - If n is odd, then the median is the $(n + 1)/2$ order statistic

Selection Problem

Find the i^{th} smallest number in an array

We can reduce this to sorting:

- O(n lg n)

We can use Quickselect (randomized selection):

- Best Case: O(n)
- Average Case: O(n)
- Worst Case: $O(n^2)$

Key Component of Quickselect: Partitioning

Pivot

3	8	2	5	1	4	7	6

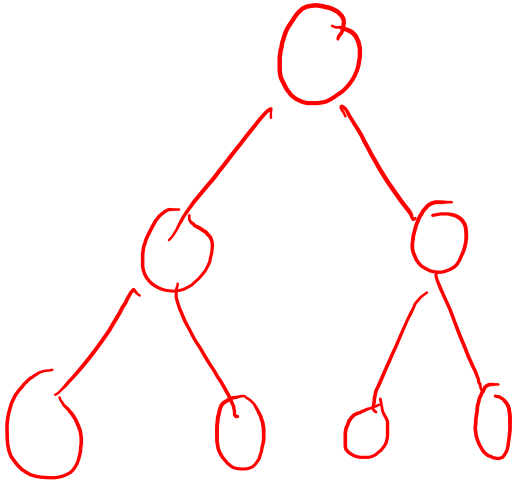
Pivot

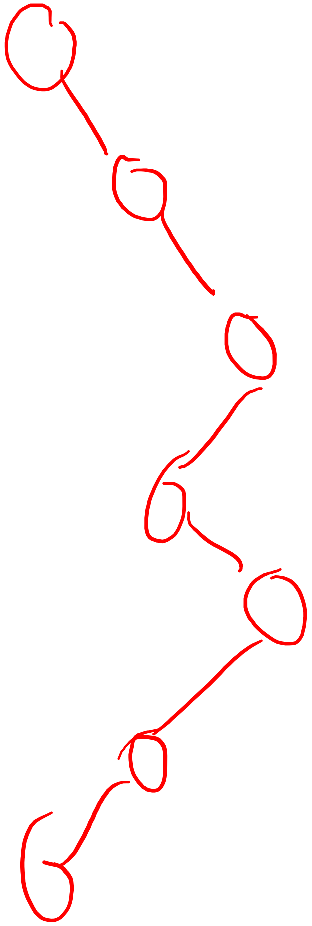
2	1	3	6	7	4	5	8

<u>What if we are looking for the 5th order statistic?</u>

- What is the fifth order statistic?
- Do we need to recursively look on both sides of the pivot?

Deterministic Selection





Works like Quicksort

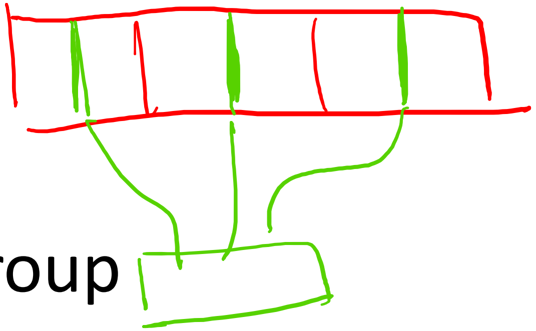
Deterministically choose "good" pivot (*close* to a 50-50 split)

- The pivot is some value near to the median

Goal: select a pivot that is guaranteed to be pretty good

Key idea: find the median of medians

Deterministic Selection, Pivot Selection



- Break input into groups of size 5 (n/5 total groups)
- Sort each group
- Copy the n/5 medians (middle elements) from each group
- Recursively compute the median of medians
- Use the median of medians as the pivot
- Partition using this pivot
- Return
  - the pivot element, or
  - recursively search the left and right

You can call this higher-level pseudocode

```
FUNCTION DSelect(array, i)
    # Base 1 indexing (makes it easier to interpret indices)
    n = array.length
    IF n == 1, RETURN A[1]

    groups = CreateGroupsOfFive(array)
    groups_sorted = SortGroupsOfFive(groups)
    medians = GetMediansGroupsOfFive(groups_sorted)

    # Get median of medians and call it the pivot
    pivot = DSelect(medians, n/5/2)

    left, right, pivot_index = Partition(array, pivot)

    IF pivot_index == i, RETURN pivot
    IF pivot_index  < i, RETURN DSelect(left, i)
    IF pivot_index  > i, RETURN DSelect(right, i - pivot_index)
```

Base Case

Recursively find median of medians

Partition

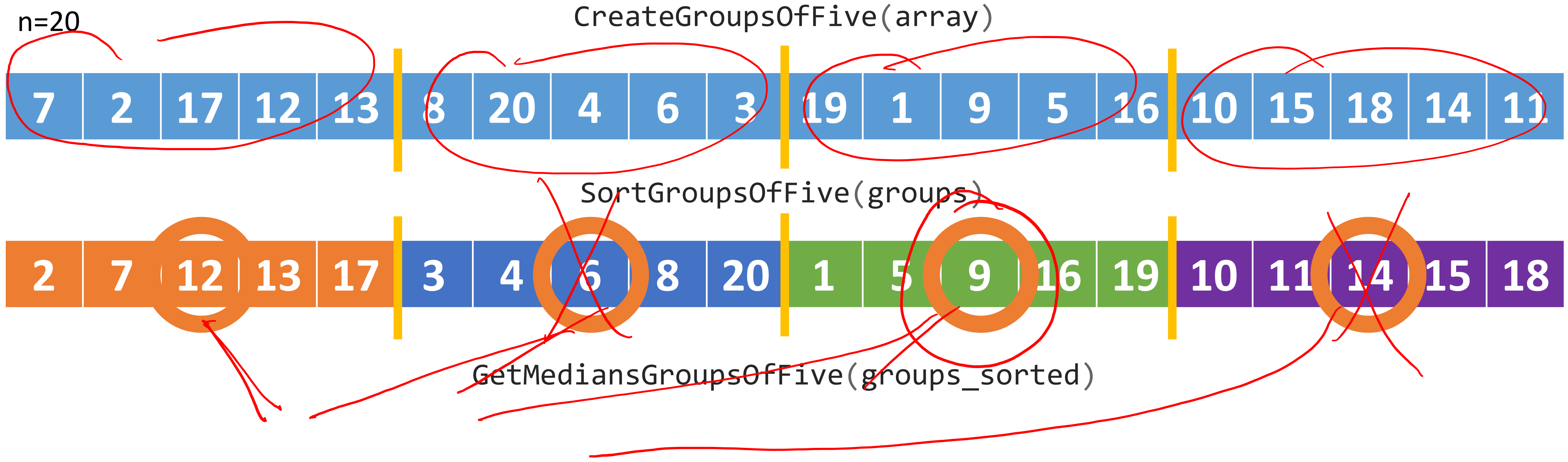
Recursion

n=20

`CreateGroupsOfFive(array)`

7	2	17	12	13	8	20	4	6	3	19	1	9	5	16	10	15	18	14	11

7	2	17	12	13	8	20	4	6	3	19	1	9	5	16	10	15	18	14	11



What is the median of medians?

$\frac{n}{5} = \frac{20}{5} = 4$

$\frac{100}{5} = 20$

$\frac{n}{5}$

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FUNCTION DSelect(array, i)
    # Base 1 indexing
    n = array.length
    IF n == 1, RETURN A[1]

    groups = CreateGroupsOfFive(array)
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    # Get median of medians and call it the pivot
    pivot = DSelect(medians, n/5/2)

    left, right, pivot_index = Partition(array, pivot)

    IF pivot_index == i, RETURN pivot
    IF pivot_index  < i, RETURN DSelect(left, i)
    IF pivot_index  > i, RETURN DSelect(right, i - pivot_index)
```

What is the running time T(n) of DSelect?

Base Case

Recursively find median of medians

Partition

Recursion

21m

constant

faster

15m

100+

DSelect Running Time

Theorem:

- for every input array of length n, DSelect returns the i^{th} order statistic in O(n)

Seems impossible since (compared with quicksort) we've added

- another recursive call (to find the pivot) and
- a bunch of work to find the median of medians

T(n)

```
FUNCTION DSelect(array, i)
    # Base 1 indexing
```

O(1)

```
    n = array.length
    IF n == 1, RETURN A[1]
```

Base Case

```
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    pivot = DSelect(medians, n/5/2)
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Recursively find median of medians

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    left, right, pivot_index = Partition(array, pivot)
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Partition

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    IF pivot_index == i, RETURN pivot
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```

Recursion

T(n)

```
FUNCTION DSelect(array, i)
    # Base 1 indexing
    n = array.length
    IF n == 1, RETURN A[1]
```

O(1)

Base Case

```
    groups = CreateGroupsOfFive(array)
    groups_sorted = SortGroupsOfFive(groups)
    medians = GetMediansGroupsOfFive(groups_sorted)
```

< O(n)

```
    # Get median of medians and call it the pivot
    pivot = DSelect(medians, n/5/2)
```

Recursively find median of medians

What must be the running time of this work?

```
    left, right, pivot_index = Partition(array, pivot)
```

Partition

```
    IF pivot_index == i, RETURN pivot
    IF pivot_index  < i, RETURN DSelect(left, i)
    IF pivot_index  > i, RETURN DSelect(right, i - pivot_index)
```

Recursion

T(n)

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FUNCTION DSelect(array, i)
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O(1)

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Base Case

```
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```

Recursively find median of medians

What must be the running time of this work?

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    left, right, pivot_index = Partition(array, pivot)
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Partition

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    IF pivot_index == i, RETURN pivot
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    IF pivot_index  > i, RETURN DSelect(right, i - pivot_index)
```

Recursion

Sorting 5 elements

A	B	C	D	E
19	11	17	7	13

1

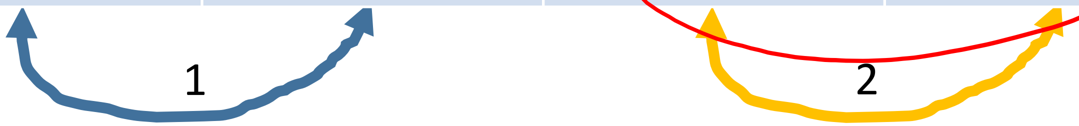
compare

Steps:

1. A > B
   - Swap(A, B)

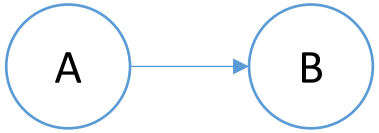
Sorting 5 elements

A	B	C	D	E
11	19	17	7	13



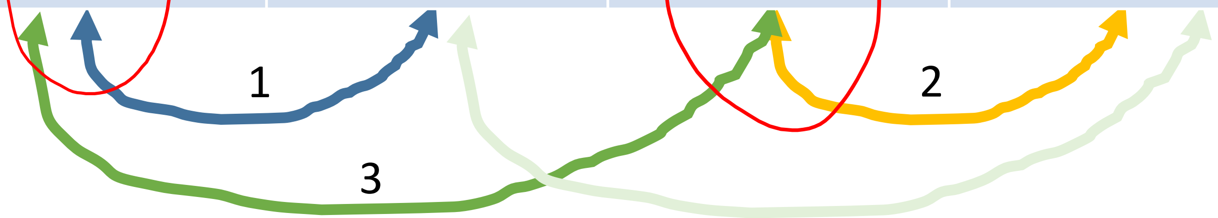
Steps:

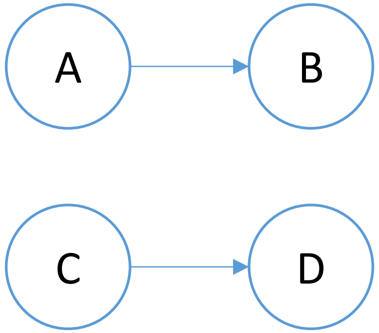
1. A > B
   - Swap(A, B)
2. C > D
   - Swap(C, D)



Sorting 5 elements

A	B	C	D	E
11	19	7	17	13



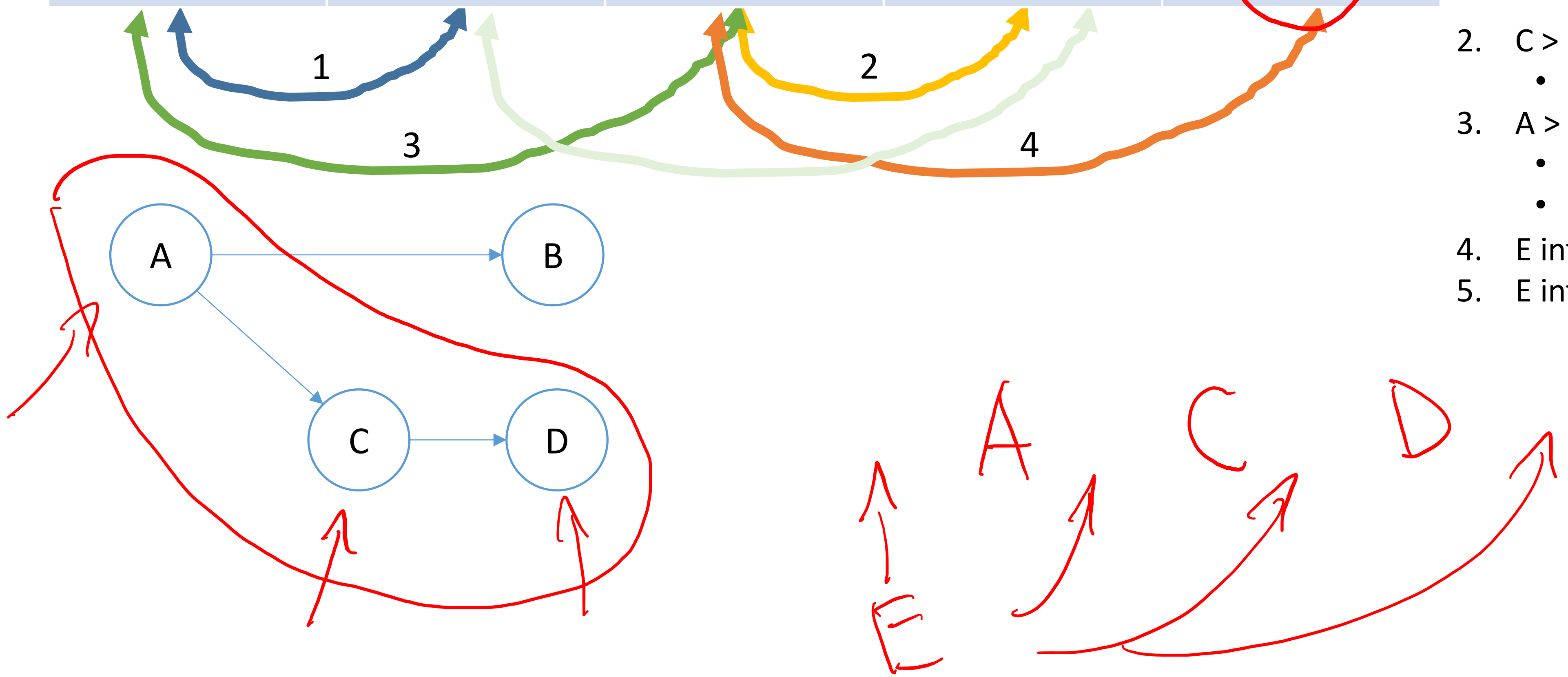


Steps:

1. A > B
   - Swap(A, B)
2. C > D
   - Swap(C, D)
3. A > C
   - Swap(A, C)
   - Swap(B, D)

Sorting 5 elements

A	B	C	D	E
7	17	11	19	13

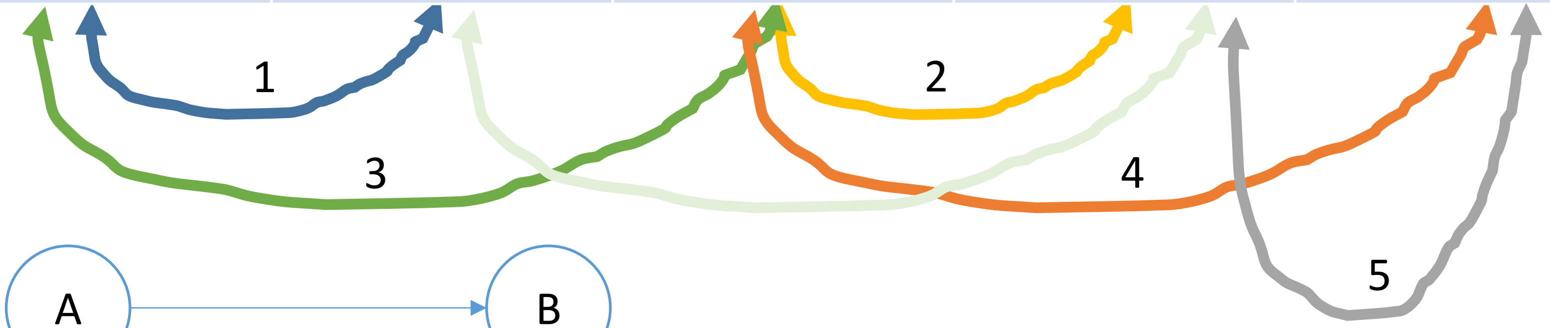


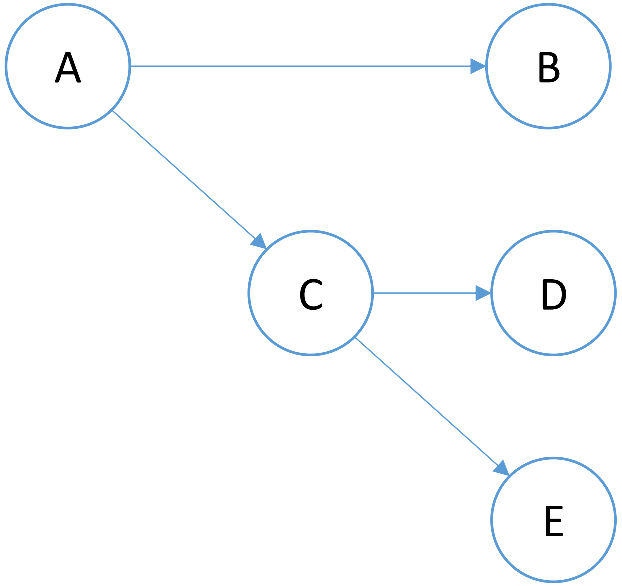
Steps:

1. A > B
   - Swap(A, B)
2. C > D
   - Swap(C, D)
3. A > C
   - Swap(A, C)
   - Swap(B, D)
4. E into A-C-D
5. E into A-C-D

Sorting 5 elements

A	B	C	D	E
7	17	11	19	13



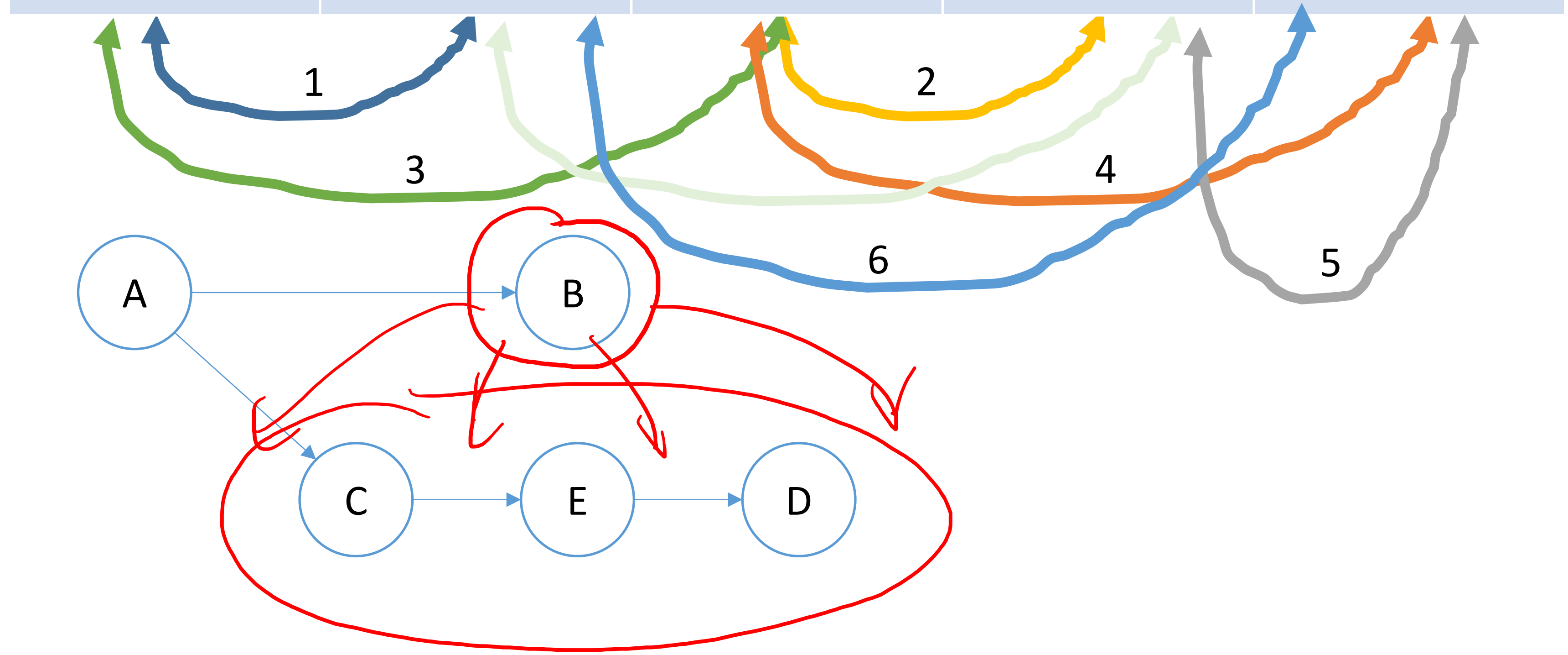


Steps:

1. A > B
   - Swap(A, B)
2. C > D
   - Swap(C, D)
3. A > C
   - Swap(A, C)
   - Swap(B, D)
4. E into A-C-D
5. E into A-C-D

Sorting 5 elements

A	B	C	D	E
7	17	11	19	13

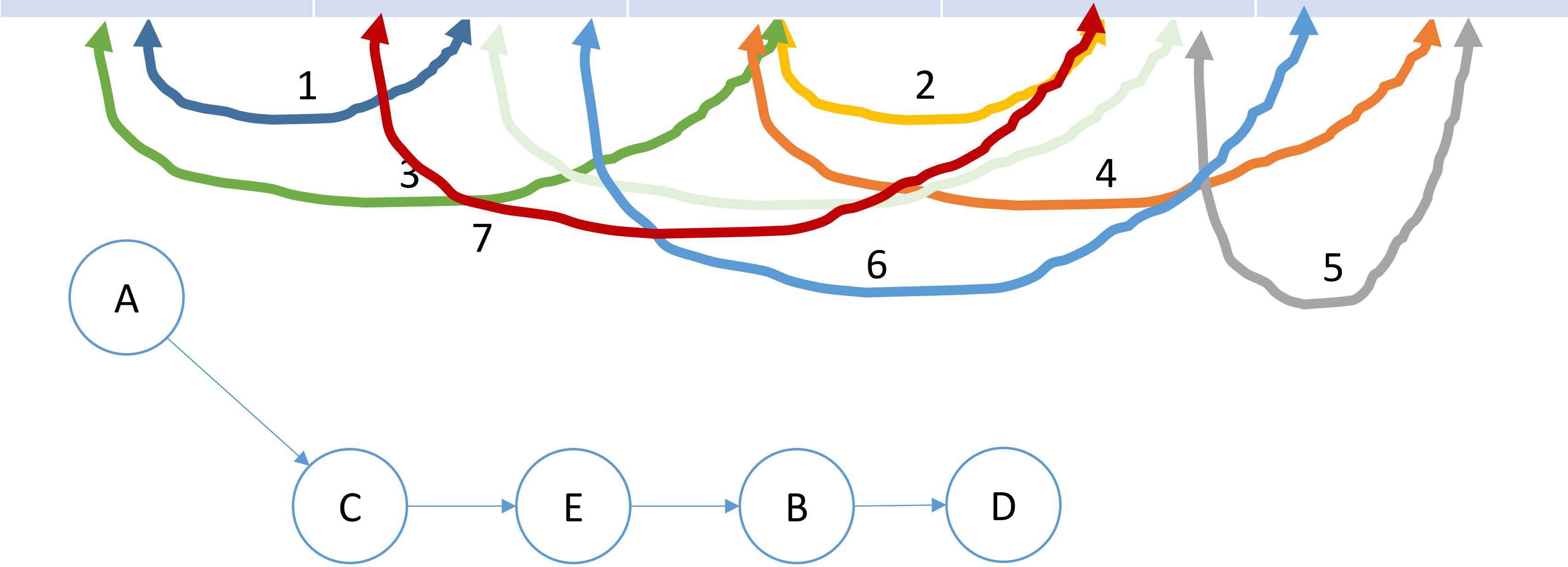


Steps:

1. A > B
   - Swap(A, B)
2. C > D
   - Swap(C, D)
3. A > C
   - Swap(A, C)
   - Swap(B, D)
4. E into A-C-D
5. E into A-C-D
6. B into C-E-D
7. B into C-E-D

Sorting 5 elements

A	B	C	D	E
7	17	11	19	13



Steps:

1. A > B
   - Swap(A, B)
2. C > D
   - Swap(C, D)
3. A > C
   - Swap(A, C)
   - Swap(B, D)
4. E into A-C-D
5. E into A-C-D
6. B into C-E-D
7. B into C-E-D

Return: [A, C, E, B ,D]

At most 7 comparisons!

T(n)

```
FUNCTION DSelect(array, i)
    # Base 1 indexing
```

O(1)

```
    n = array.length
    IF n == 1, RETURN A[1]
```

Base Case

O(n)

```
    groups = CreateGroupsOfFive(array)
    groups_sorted = SortGroupsOfFive(groups)
    medians = GetMediansGroupsOfFive(groups_sorted)

    # Get median of medians and call it the pivot
    pivot = DSelect(medians, n/5/2)
```

Recursively find median of medians

```
    left, right, pivot_index = Partition(array, pivot)
```

Partition

```
    IF pivot_index == i, RETURN pivot
    IF pivot_index  < i, RETURN DSelect(left, i)
    IF pivot_index  > i, RETURN DSelect(right, i - pivot_index)
```

Recursion

T(n)

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Base Case

O(n)

```
    groups = CreateGroupsOfFive(array)
    groups_sorted = SortGroupsOfFive(groups)
    medians = GetMediansGroupsOfFive(groups_sorted)
```

T(n/5)

```
    # Get median of medians and call it the pivot
    pivot = DSelect(medians, n/5/2)
```

Recursively find median of medians

How can we denote this recursive running time?

```
    left, right, pivot_index = Partition(array, pivot)
```

Partition

```
    IF pivot_index == i, RETURN pivot
    IF pivot_index  < i, RETURN DSelect(left, i)
    IF pivot_index  > i, RETURN DSelect(right, i - pivot_index)
```

Recursion

T(n)

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    IF n == 1, RETURN A[1]
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Base Case

O(n)

```
    groups = CreateGroupsOfFive(array)
    groups_sorted = SortGroupsOfFive(groups)
    medians = GetMediansGroupsOfFive(groups_sorted)
```

T(n/5)

```
    # Get median of medians and call it the pivot
    pivot = DSelect(medians, n/5/2)
```

Recursively find median of medians

O(n)

```
    left, right, pivot_index = Partition(array, pivot)
```

Partition

```
    IF pivot_index == i, RETURN pivot
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    IF pivot_index  > i, RETURN DSelect(right, i - pivot_index)
```

Recursion

T(n)

```
FUNCTION DSelect(array, i)
    # Base 1 indexing
```

O(1)

Base Case

```
    n = array.length
    IF n == 1, RETURN A[1]
```

O(n)

T(n/5)

Recursively find median of medians

```
    groups = CreateGroupsOfFive(array)
    groups_sorted = SortGroupsOfFive(groups)
    medians = GetMediansGroupsOfFive(groups_sorted)

    # Get median of medians and call it the pivot
    pivot = DSelect(medians, n/5/2)
```

O(n)

Partition

```
    left, right, pivot_index = Partition(array, pivot)
```

T(?)

Recursion

```
    IF pivot_index == i, RETURN pivot
    IF pivot_index  < i, RETURN DSelect(left, i)
    IF pivot_index  > i, RETURN DSelect(right, i - pivot_index)
```

O(1) T(?)

DSelect Running Time

T(n) = maximum # of operations required for input of length n

$$T(1) = O(1)$$

Finding a good pivot

$$T(n) \leq cn + T\left(\frac{n}{5}\right) + T(?)$$

Sorting, partitioning, copying, etc.

Recursively searching one side

On what does the "?" depend?

DSelect Running Time

T(n) = maximum # of operations required for input of length n

$T(1) = O(1)$

Finding a good pivot

$$T(n) \leq cn + T\left(\frac{n}{5}\right) + T(?)$$

Sorting, partitioning, copying, etc.

Lemma:
the recursive search is guaranteed to be on an array of size ≤ 7n/10

T(n)

```
FUNCTION DSelect(array, i)
    # Base 1 indexing
```

O(1) — Base Case

```
    n = array.length
    IF n == 1, RETURN A[1]
```

O(n), T(n/5) — Recursively find median of medians

```
    groups = CreateGroupsOfFive(array)
    groups_sorted = SortGroupsOfFive(groups)
    medians = GetMediansGroupsOfFive(groups_sorted)

    # Get median of medians and call it the pivot
    pivot = DSelect(medians, n/5/2)
```

O(n) — Partition

```
    left, right, pivot_index = Partition(array, pivot)
```

T(7n/10) — Recursion

```
    IF pivot_index == i, RETURN pivot
    IF pivot_index  < i, RETURN DSelect(left, i)
    IF pivot_index  > i, RETURN DSelect(right, i - pivot_index)
```

Selecting the pivot

$$T(n) \leq cn + T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right)$$

- We can now replace the "?" with 7n/10
- Let k = n/5 be the number of groups of size 5
- Let x_i = i^{th} smallest element of the k medians
- So, the pivot is $x_{k/2}$ (the median of medians)

- Our goal is to show that:

  $\leq 30\ \%$ of the input array is smaller than $x_{k/2}$

  $\leq 30\ \%$ of the input array is larger than $x_{k/2}$

This means that we must search at most 70% (7/10^{ths}) of the remaining input

n=20

`CreateGroupsOfFive(array)`

7	2	17	12	13	8	20	4	6	3	19	1	9	5	16	10	15	18	14	11

`SortGroupsOfFive(groups)`

2	7	12	13	17	3	4	6	8	20	1	5	9	16	19	10	11	14	15	18

`GetMediansGroupsOfFive(groups_sorted)`

What is the median of medians?

n=20

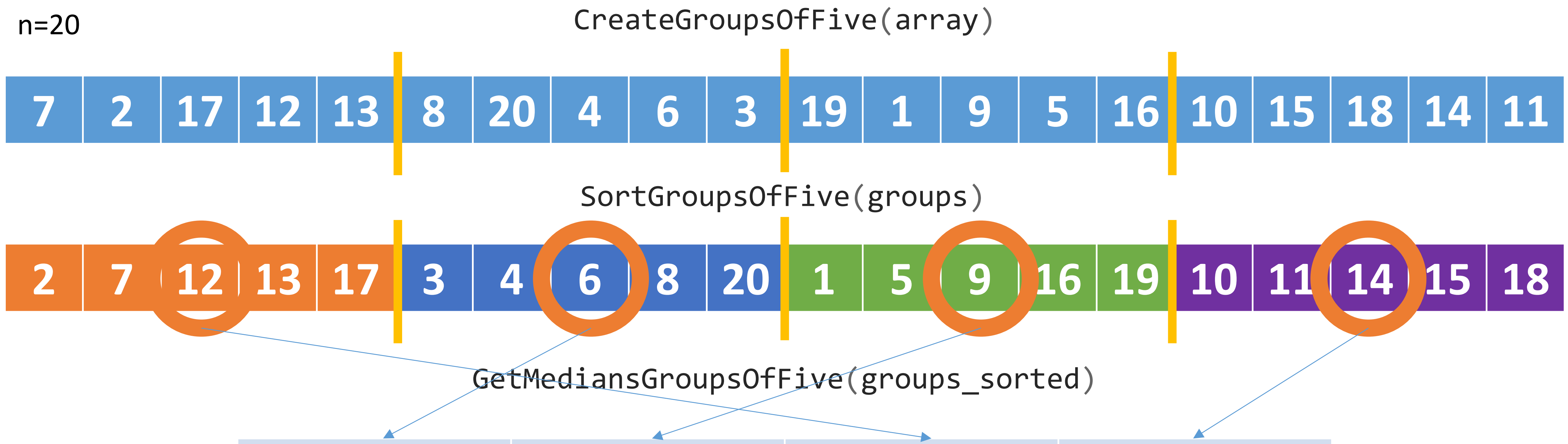
`CreateGroupsOfFive(array)`

`SortGroupsOfFive(groups)`

`GetMediansGroupsOfFive(groups_sorted)`

What is the median of medians?

From where do we get 30%?



20	19	17	18
8	16	13	15
6	9	12	14
4	5	7	11
3	1	2	10

This is a just a diagram to show what we’re looking at

Guaranteed bigger than (or equal to) (at least) 3/5 of 1/2 of the groups = **30%**

	20	19	17	18
	8	16	13	15
$x_{k/2}$	6	9	12	14
	4	5	7	11
	3	1	2	10

Guaranteed smaller than (or equal to) (at least) 3/5 of 1/2 of the groups = **30%**

	20	19	17	18
	8	16	13	15
$x_{k/2}$	6	9	12	14
	4	5	7	11
	3	1	2	10

Guaranteed **bigger** than (or equal to) (at least)

3/5 of 1/2 of the groups = **<u>30%</u>**

Guaranteed **smaller** than (or equal to) (at least)

3/5 of 1/2 of the groups = **<u>30%</u>**

	20	19	17	18
	8	16	13	15
$x_{k/2}$	6	9	12	14
	4	5	7	11
	3	1	2	10

So, we need to search either the (at most) upper 70% of the array or the (at most) lower 70% of the array.

	20	19	17	18
	8	16	13	15
$x_{k/2}$	6	9	12	14
	4	5	7	11
	3	1	2	10

Total Running Time

$$T(n) \leq cn + T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right)$$

- Can we use the master method?
- Not all subproblems are the same size
- We are going to use the substitution method

Fix animations

$$T(n) \le c_{DS}n + T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right)$$

Guess and Check

$T(n) = O(\lg n)$

Claim: $T(n) = O(n)$

$$c_{DS}n + T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right) \le an \qquad \forall n \ge n_0$$

Let $a = 10c_{DS}$, and $n_0 = 1$

Proof by induction

1. Base Case: $T(1) \le a \cdot n = a \cdot 1 = 10c_{DS}$
2. Inductive Hypothesis: Assume $T(k) \le ak$ for $k < n$
3. Induction Step

$$T(n) \le c_{DS}n + T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right) \le c_{DS}n + a\frac{n}{5} + a\frac{7n}{10} \le an$$

Applying IH

$$T(n) \le c_{DS}n + T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right) \le c_{DS}n + a\frac{n}{5} + a\frac{7n}{10} \le an$$

Let $a = 10c_{DS}$, and $n_0 = 1$

$$T(n) = O(n)$$

$$n c_{DS} + a\frac{n}{5} + a\frac{7n}{10} \le an$$

$$n\left(c_{DS} + a\frac{1}{5} + a\frac{7}{10}\right) \le an$$

$$c_{DS} + a\frac{9}{10} \le a$$

$$c_{DS} + 10c_{DS}\frac{9}{10} \le 10c_{DS}$$

$$10c_{DS} \le 10c_{DS}$$

Selection

Randomized selection (**average** O(n) runtime)

- Fast and practical
- All operations done in-place
- Small constant factors

$O(n) = T(n) \le 80n$

Deterministic selection (**guaranteed** O(n) runtime)

- Slower in practice
- Extra memory required
- Large constant factors (extra non-recursive work)

$O(n) = T(n) \le 200n$