Quicksort Running Time

https://cs.pomona.edu/classes/cs140/

Outline

Topics and Learning Objectives

- Learn how quicksort works
- Learn how to partition an array

Exercise

- Running time

Extra Resources

- https://me.dt.in.th/page/Quicksort/
- https://www.youtube.com/watch?v=ywWBy6J5gz8
- CLRS Chapter 7
- Algorithms Illuminated Chapter 5

Choosing a Pivot

What is Quicksort's running time? (can we use master theorem?)

- <u>It depends on the pivot</u>

What is a recurrence for Quicksort?

Choosing a Pivot

What is Quicksort's running time? (can we use master theorem?)

- It depends on the pivot

What is the worst case for Quicksort, and what is its running time?

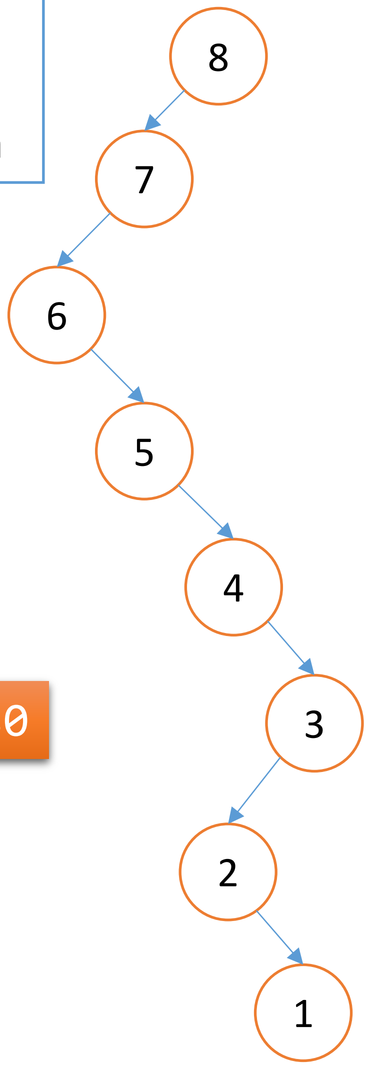
- Always select the smallest (or largest) possible pivot and it takes $O(n^2)$
- Think of a one-sided tree

What is the best case for Quicksort, and what is its running time?

- Always select the median element as a pivot leading to $O(n \lg n)$
- Think of a balanced tree

Recursion tree for the worst and best cases of Quicksort

Let's assume the cost of Partition is 5m



5n = 40

+ 5 (n-1) = 35

+ 5 (n-2) = 30

+ 5 (n-3) = 25

+ 5 (n-4) = 20

+ 5 (n-5) = 15

+ 5 (n-6) = 10

+ 5 (n-7) = 5

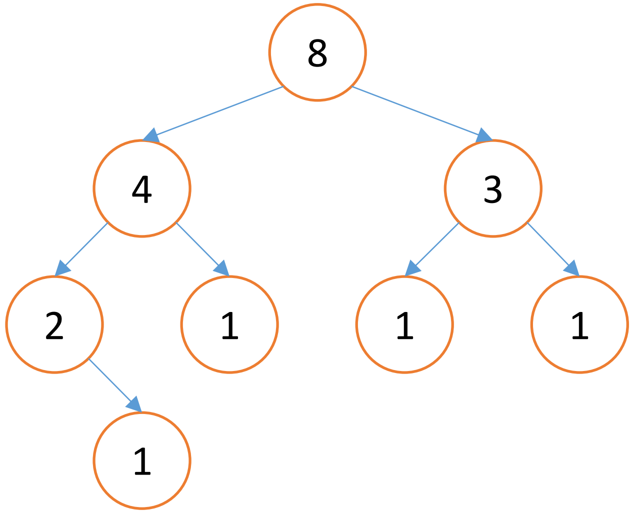
T(n) = 180

5n = 40

+ 5 (n-1) = 35

+ 5 (n-3) = 25

+ 5 (n-7) = 5



T(n) = 105

How would you select a pivot?

- If pivot selection is so important, how should we do it?
- Shouldn’t we take great care in selecting the pivot?
- **Key idea for Quicksort: select the pivot uniformly at random!**
  - Easy
  - Fast
  - Gets good results as long as the pivots are “decent” fairly “often”

Random Pivots

- Some foreshadowing:

  If the randomly chosen pivot is close to the median (in the middle 25-75 percentile range) we will get an average running time of O(n lg n)

- We cannot use the master theorem

- We are going to show the runtime of quicksort another way

What is our Quicksort Theorem?

Theorem: For every input of the array of length n, the **average** running time of quicksort with random pivots is O(n lg n).

This is a big deal; it means that the average running time is closer to the best-case than it is to the worst-case.

Note: here, average refers to the algorithm itself–it does not depend on the input.

- If we re-run quicksort on the same input we will get different pivots each time, and we are talking about the average running time of quicksort for these different sequences of pivots <u>on the same input array</u>.

Quicksort

Most of the work is done inside Partition

We are going to count the number of comparisons performed inside the for-loop.

```
FUNCTION QuickSort(array, left_index, right_index)

  IF left_index ≥ right_index
    RETURN

  MovePivotToLeft(left_index, right_index)

  pivot_index = Partition(array, left_index, right_index)

  QuickSort(array, left_index, pivot_index)
  QuickSort(array, pivot_index + 1, right_index)
```

```
FUNCTION Partition(array, left_index, right_index)

  pivot_value = array[left_index]

  i = left_index + 1
  FOR j IN [left_index + 1 ..< right_index]
    IF array[j] < pivot_value
      swap(array, i, j)
      i = i + 1

  swap(array, left_index, i − 1)
  RETURN i − 1
```

Some notation

Let Z_i = i^{th} smallest element of A <u>(not the i^{th} element)</u>

Z_i

What is Z_1

Z								
Index	0	1	2	3	4	5	6	7
Value	51	43	17	83	79	23	61	37

Z_i

What is Z_2

Z			Z_1					
Index	0	1	2	3	4	5	6	7
Value	51	43	17	83	79	23	61	37

Z_i

Z			Z_1			Z_2		
Index	0	1	2	3	4	5	6	7
Value	51	43	17	83	79	23	61	37

Z_i

z	Z_5	Z_4	Z_1	Z_8	Z_7	Z_2	Z_6	Z_3
Index	0	1	2	3	4	5	6	7
Value	51	43	17	83	79	23	61	37

Some notation

Let Z_i = i^{th} smallest element of A <u>(not the i^{th} element)</u>

Let $X_{i,j}$ be a random variable for the number of times Z_i and Z_j get compared during a call to Quicksort

i and j can be any indices, but I'll normally use i for the lower index

How many times can Z_i and Z_j possibly be compared?

$X_{2,4}$

z	Z_5	Z_4	Z_1	Z_8	Z_7	Z_2	Z_6	Z_3
Index	0	1	2	3	4	5	6	7
Value	51	43	17	83	79	23	61	37

Exercise Question 1

How many times can two elements be compared by a single run of the Quicksort algorithm?

Some notation

Let Z_i = i^{th} smallest element of A <u>(not the i^{th} element)</u>

Let $X_{i,j}$ be a random variable for the number of times Z_i and Z_j get compared during a call to Quicksort

How many times can Z_i and Z_j possibly be compared?

- Can only be compared 0 or 1 times!
- Every comparison involves the pivot, but the pivot is excluded from recursive calls.

$X_{2,4}$ =

$X_{1,7}$ =

`left_index` `right_index`

z	Z_5	Z_4	Z_1	Z_8	Z_7	Z_2	Z_6	Z_3
Index	0	1	2	3	4	5	6	7
Value	51	43	17	83	79	23	61	37

Every comparison involves the pivot, but the pivot is excluded from recursive calls.

```
FUNCTION QuickSort(array, left_index, right_index)

    IF left_index ≥ right_index
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    MovePivotToLeft(left_index, right_index)

    pivot_index = Partition(array, left_index, right_index)

    QuickSort(array, left_index, pivot_index)
    QuickSort(array, pivot_index + 1, right_index)
```

```
FUNCTION Partition(array, left_index, right_index)

    pivot_value = array[left_index]

    i = left_index + 1
    FOR j IN [left_index + 1 ..< right_index]
        IF array[j] < pivot_value
            swap(array, i, j)
            i = i + 1

    swap(array, left_index, i − 1)
    RETURN i − 1
```

The upper index is exclusive

`right_index` is not included in comparisons

Exercise Question 2

How many comparisons will be performed by Quicksort if we always pick the median element as the pivot? You only need to consider the case when **n = 8**. You should draw a recursion tree and note how many comparisons are performed at each subproblem.

Considering $X_{i,j}$

- Space of all possible outcomes is Ω
  - A comparison happens (1)
  - Or it doesn't (0)
  - This is an indicator variable

- What is the expected value of $X_{i,j}$ ($E[X_{i,j}]$)?
  - We need to know the probability of a comparison

$$p(X_{i,j} = 1)$$

Z_i, Z_j

Z_5	Z_4	Z_1	Z_8	Z_7	Z_2	Z_6	Z_3
51	43	17	83	79	23	61	37

What is the probability that Z_3 (37) and Z_7 (79) are compared?

Probability that Z_i, Z_j get compared

$p(X_{i,j} = 1)$

Consider any Z_i, Z_{i+1} , …, Z_{j-1}, Z_j from the array

- Remember that these are not contiguous in the array, they are numbers in increasing order

What can you tell me about this group of numbers? (Hint: consider different values for the pivot element)

If none of these are chosen as a pivot, all are passed to the same recursive call.

Probability that Z_i, Z_j get compared

$$p(X_{i,j} = 1)$$

Consider any Z_i, Z_{i+1} , …, Z_{j-1}, Z_j from the array

Among these values, consider the first one that gets chosen

1. If Z_i or Z_j are chosen first, then Z_i and Z_j are compared.
2. If one of Z_{i+1}, …, Z_{j-1} is chosen, then Z_i and Z_j are **NEVER** compared.

Why?

1. If is chosen, then they become a pivot and the two values get compared
2. If a value in the middle gets chosen, then they go to separate calls

Z_i, Z_j

What is the probability that Z_3 (37) and Z_7 (79) are compared?

Z_5	Z_4	Z_1	Z_8	Z_7	Z_2	Z_6	Z_3
51	43	17	83	79	23	61	37

Each recursive call we have three options for Z_3 and Z_7:

1. One is selected as pivot, and they are compared. ($X_{3,7} = 1$)
2. An item between them is selected and they are split apart. ($X_{3,7} = 0$)
3. An item outside their range is selected and they are partitioned together.

Probability that Z_i, Z_j get compared

$$p(X_{i,j} = 1) = \frac{2}{total\ \#\ of\ relevant\ choices} = \frac{2}{j - i + 1}$$

- What does this mean for two values that are close to each other?

- What does this mean for two values that are far from each other?

Counting the total number of comparisons

What is the equation for the total number of comparisons?

$$\mathrm{T} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{i,j}$$

Every possible comparison

$$E[T] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[X_{i,j}]$$

Linearity of expectations

Counting the total number of comparisons

$$E[T] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[X_{i,j}]$$

$$E[T] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \mathrm{p}(X_{i,j} = 1) \cdot 1 + \mathrm{p}(X_{i,j} = 0) \cdot 0$$

Counting the total number of comparisons

$$E[T] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[X_{i,j}]$$

$$E[T] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \mathrm{p}(X_{i,j} = 1) \cdot 1 + \mathrm{p}(X_{i,j} = 0) \cdot 0$$

Counting the total number of comparisons

$$p(X_{i,j} = 1) = \frac{2}{total\ \#\ of\ relevant\ choices} = \frac{2}{j - i + 1}$$

$$E[T] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} p(X_{i,j} = 1) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j - i + 1}$$

Simplifying the Inner Summation

$$\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{2}{j-i+1}$$

Consider a fixed value for i (i=1)

$$\sum_{j=i+1}^{n}\frac{2}{j-i+1}=\sum_{j=2}^{n}\frac{2}{j}=2\cdot\left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n-1+1}\right)$$

Consider another fixed value for i (i=5)

The inner summation is maximized with i=1

$$\sum_{j=i+1}^{n}\frac{2}{j-i+1}=\sum_{j=6}^{n}\frac{2}{j-4}=2\cdot\left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n-5+1}\right)$$

Counting the total number of comparisons

$$p(X_{i,j} = 1) = \frac{2}{total\ \#\ of\ relevant\ choices} = \frac{2}{j\ - i + 1}$$

$$E[T] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} p(X_{i,j} = 1) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j - i + 1}$$

Simplify by turning this into an inequality and taking the value for i that results in the biggest number

$$E[T] \leq 2 \sum_{i=1}^{n-1} \sum_{j=2}^{n} \frac{1}{j - 1 + 1}$$

Summations no longer depends on i

Counting the total number of comparisons

$$E[T] \leq 2 \sum_{i=1}^{n-1} \sum_{j=2}^{n} \frac{1}{j - 1 + 1}$$

Summations no longer depends on i

$$E[T] \leq 2n \sum_{j=2}^{n} \frac{1}{j}$$

Change of base for logarithms

$$E[T] \leq 2n \int_{2}^{n} \frac{1}{x} dx = 2n \ln(x) \Big|_{2}^{n} = 2n(\ln(n) - \ln(2)) \leq 2n \ln(n) = O(n \lg(n))$$


1
0.8
0.6
0.4
0.2
0
1
2
3
4
5
6
7
8
9
10

Counting the total number of comparisons

$$E[T] \leq 2 \sum_{i=1}^{n-1} \sum_{j=2}^{n} \frac{1}{j - 1 + 1}$$

Summations no longer depends on i

$$E[T] \leq 2n \sum_{j=2}^{n} \frac{1}{j}$$

Change of base for logarithms

$$E[T] \leq 2n \int_{2}^{n} \frac{1}{x} dx = 2n \ln(x) \Big|_{2}^{n} = 2n(\ln(n) - \ln(2)) \leq 2n \ln(n) = O(n \lg(n))$$

Summary

$$E[T] \leq O(n \lg(n))$$

- The expected number of comparisons is O(n lg n)
- The expected number of comparisons is directly proportional to the total running time of Quicksort
- The **average** asymptotic running time of Quicksort of O(n lg n)
- **Theorem:** For every input of the array of length n, the **average** running time of Quicksort with random pivots is O(n lg n).