Closest Pair Algorithm

https://cs.pomona.edu/classes/cs140/

Outline

Topics and Learning Objectives

- Learn more about Divide and Conquer paradigm
- Learn about the closest-pair problem and its O(n lg n) algorithm
  - Gain experience analyzing the run time of algorithms
  - Gain experience proving the correctness of algorithms

Exercise

- Closest Pair

Extra Resources

- Algorithms Illuminated: Part 1: Chapter 3

Closest Pair Problem

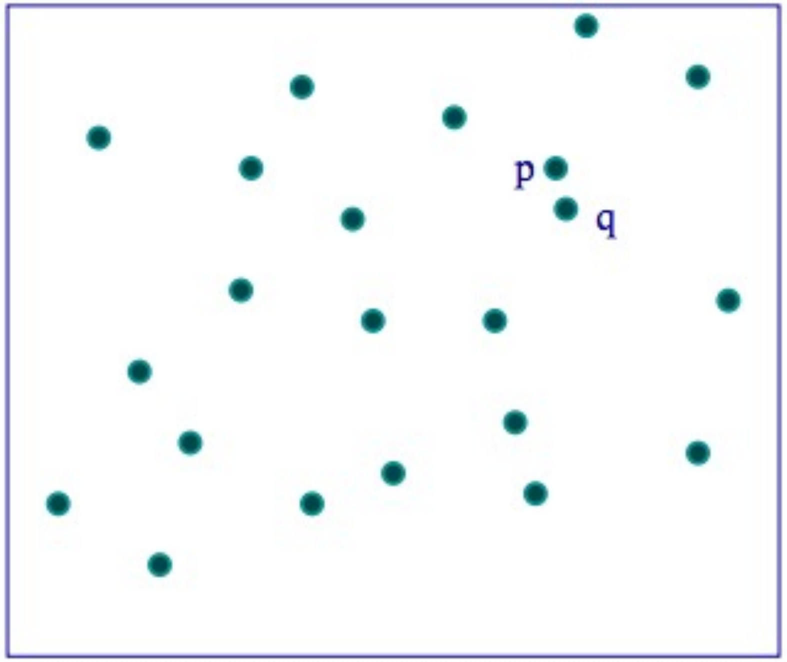
- **Input**: P, a set of n points that lie in a (two-dimensional) plane

- **Output**: a pair of points `(p, q)` that are the "closest"
  - Distance is measured using Euclidean distance:

$$d(p, q) = sqrt((p_x - q_x)^2 + (p_y - q_y)^2)$$

- **Assumptions**: None

Closest Pair Problem

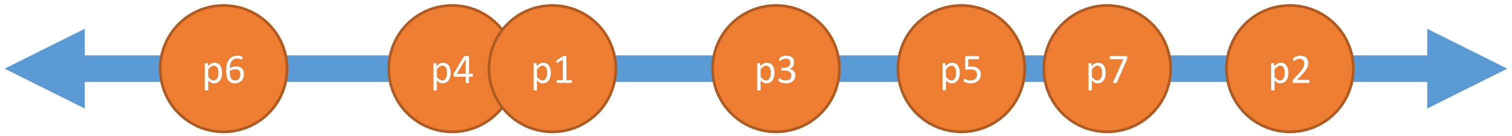


Can we do better than $O(n^2)$?

- What is the brute force method for this search?
- What is the asymptotic running time of the brute force method?

Input	p1	p2	p3	p4	p5	p6	p7

One-dimensional closest pair



How would you find the closest two points?

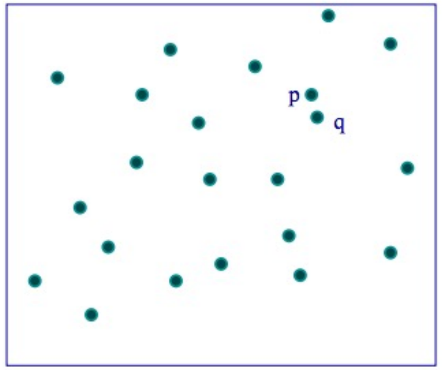
- Sort by position : O(n lg n)

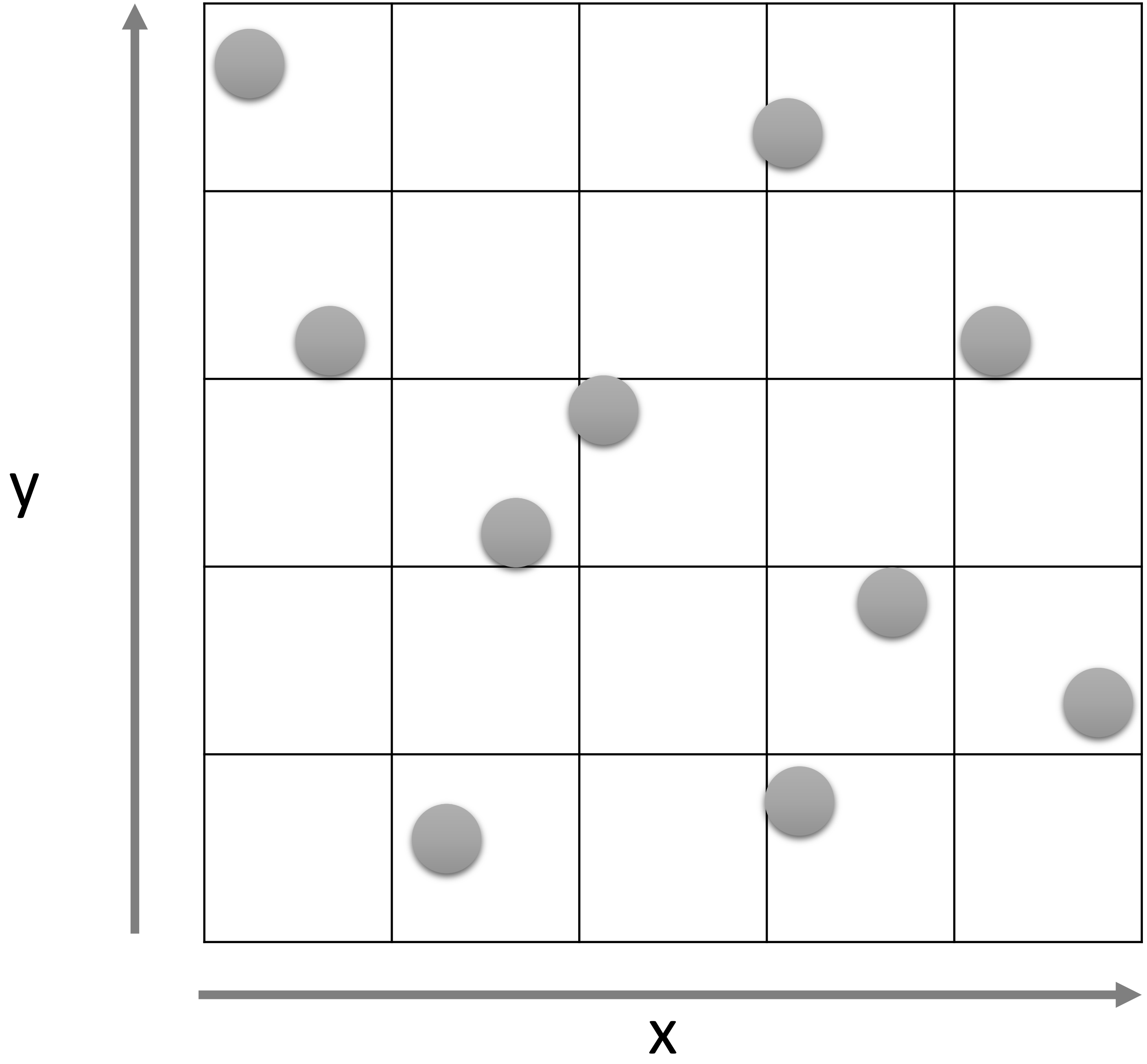
p6	p4	p1	p3	p5	p7	p2

- Return the closest two using a linear scan : O(n)
- Total time : O(n lg n) + O(n) = O(n lg n)

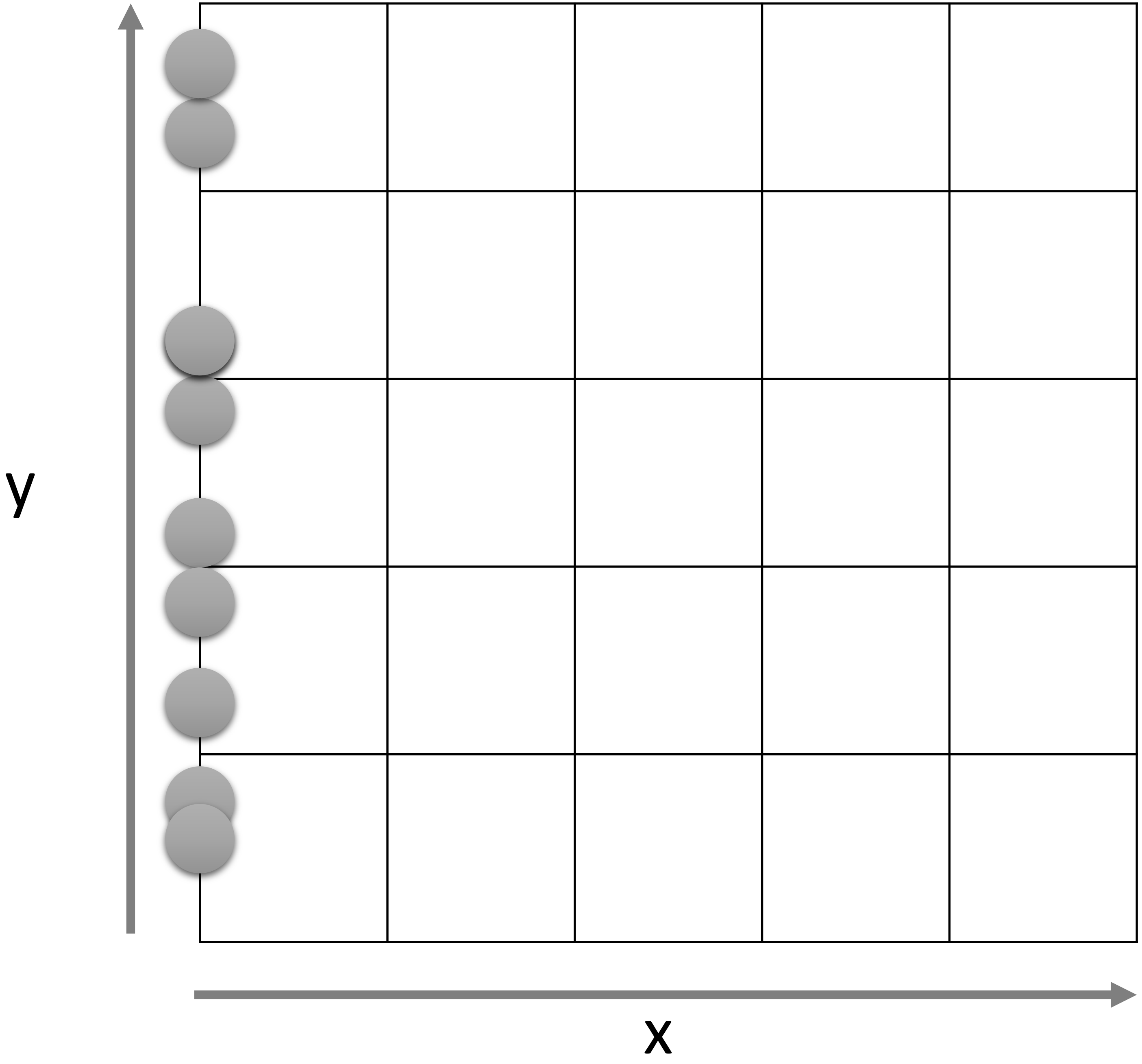
Any problems using this approach for the two-dimensional case?

- <u>Sorting does not generalize to higher dimensions!</u>
- How do you sort the points?





1. Which two are closest on the y-axis?



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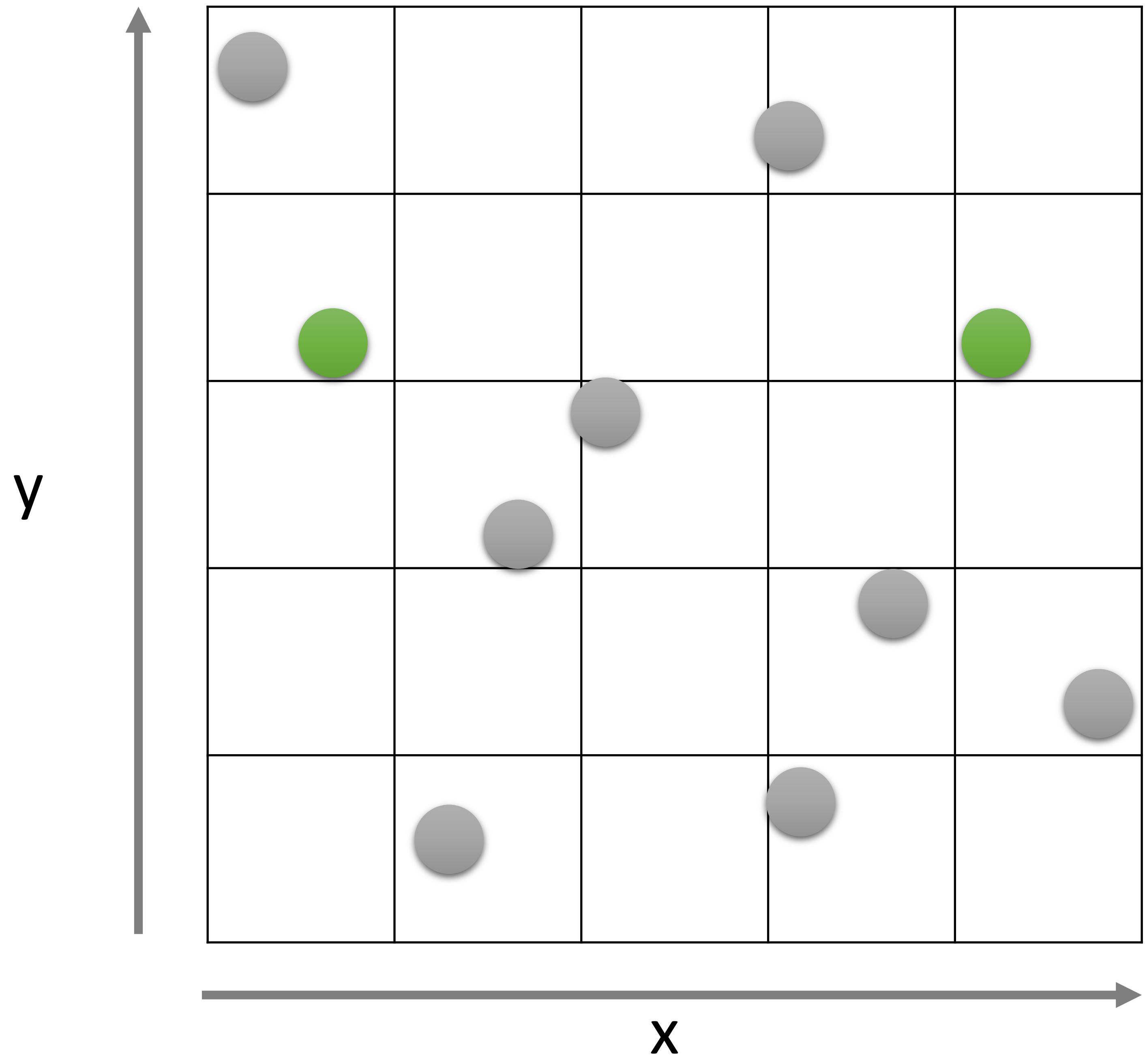

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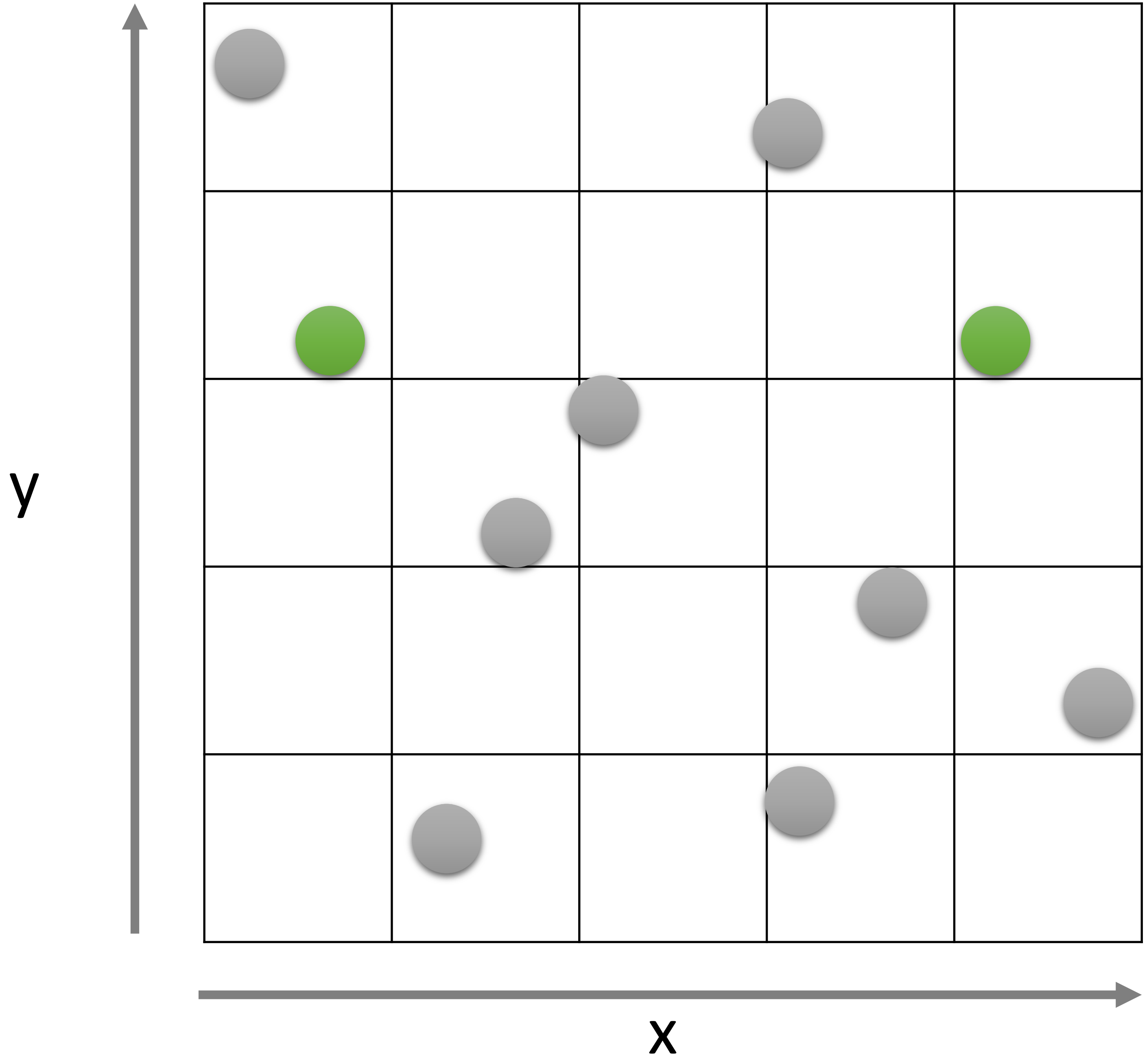

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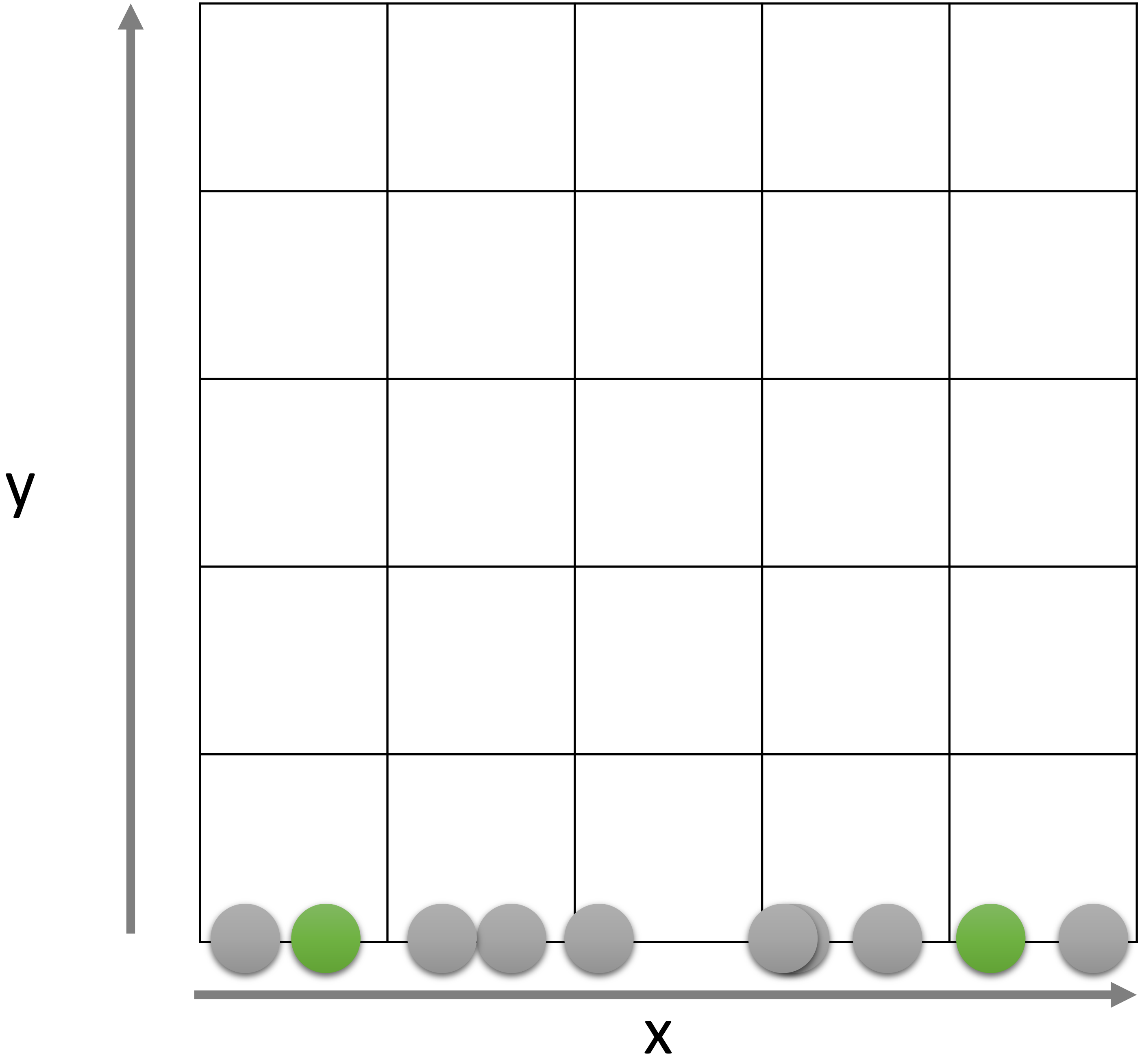
2. Which two are closest on the x-axis?



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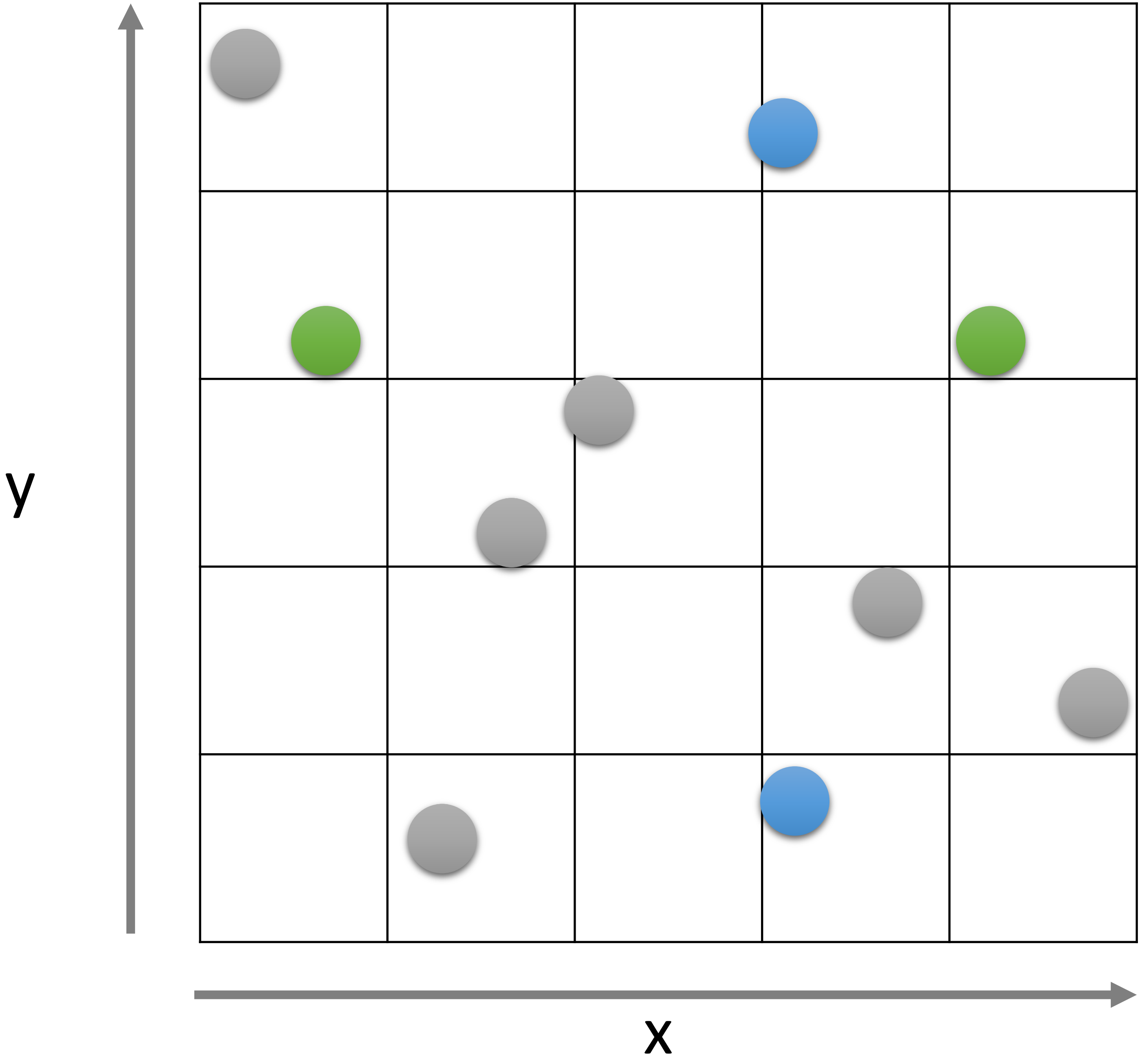

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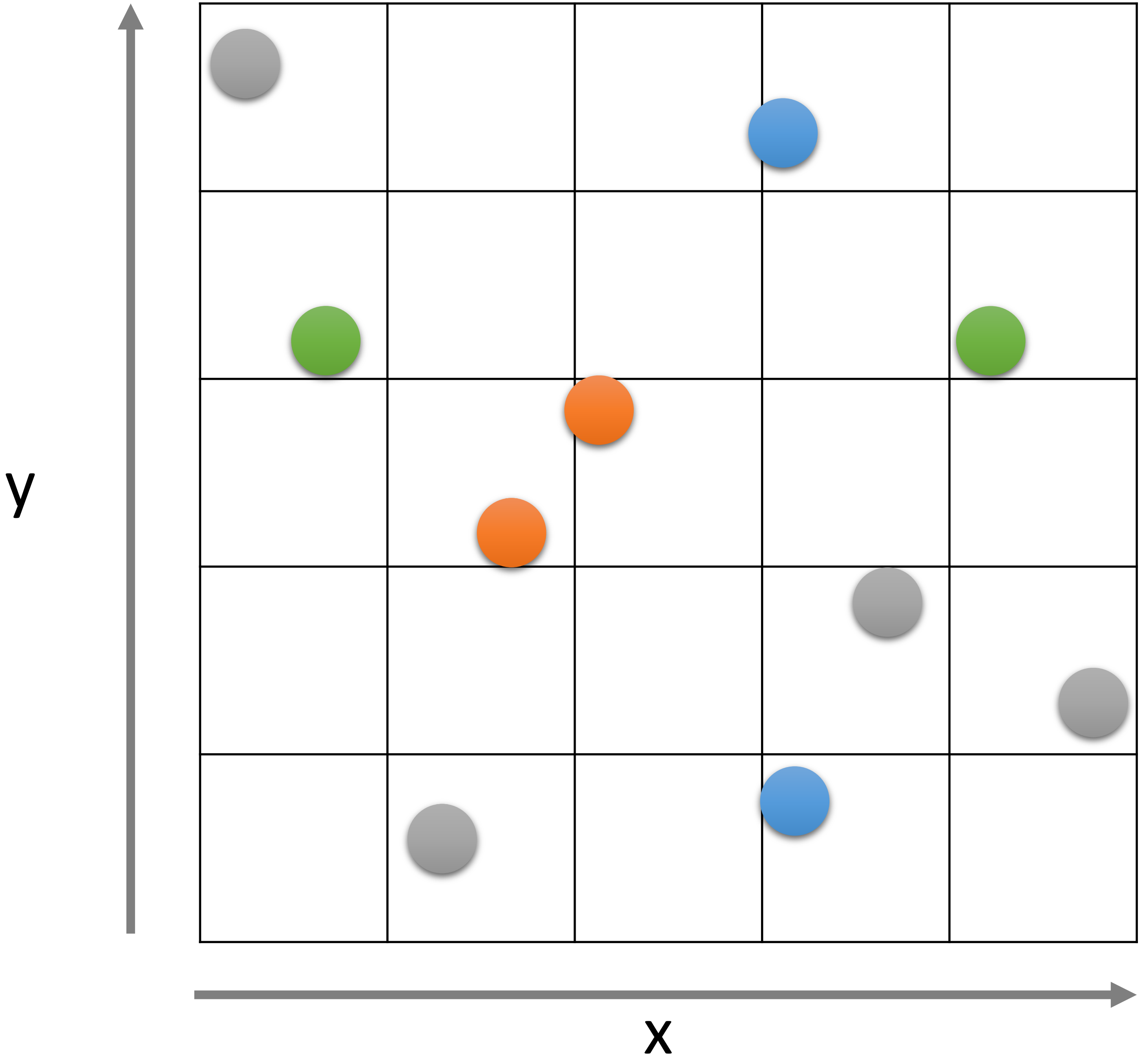

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Closet Pair—Two-Dimensions

1. Create a copy of the points (we now have two separate copies of P)
    1. Sort by x-coordinate
    2. Sort other by y-coordinate

O(n lg n)

Now we know we can't do better than O(n lg n)

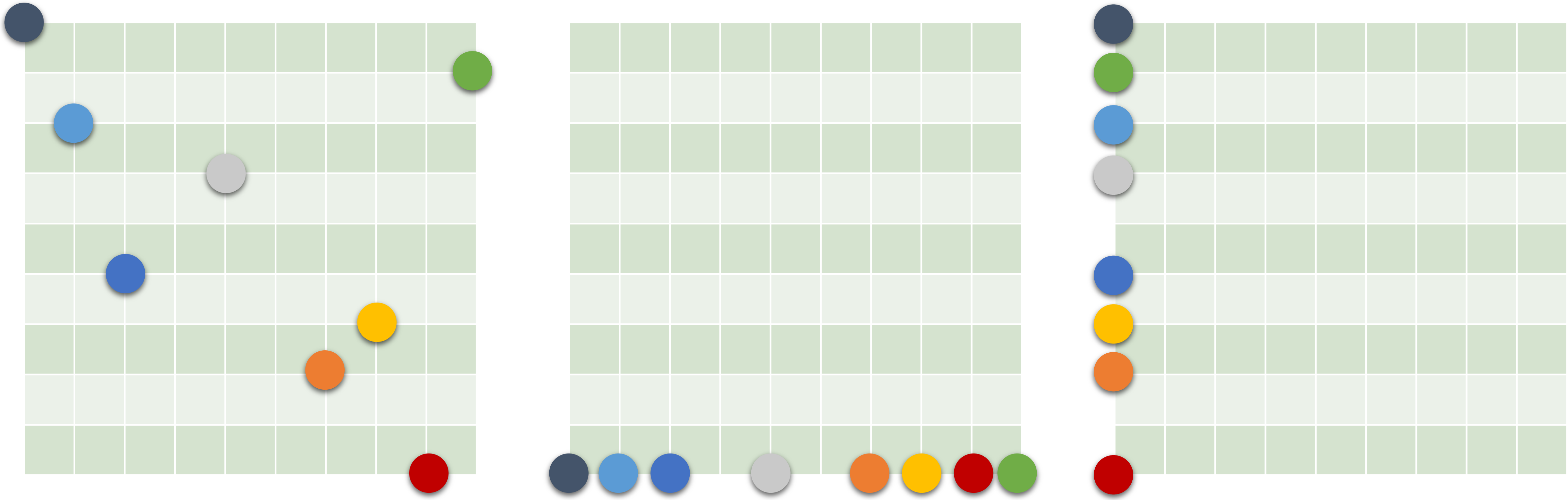
```
P  : [p0(1,10), p1(2,8), p2(7,3), p3(5,7), p4(8,4), p5(3,5), p6(10,9), p7(9,1)]
```

Sorted by x coordinate

```
Px : [p0(1,10), p1(2,8), p5(3,5), p3(5,7), p2(7,3), p4(8,4), p7(9,1), p6(10,9)]
```

Sorted by y coordinate

```
Py : [p7(9,1), p2(7,3), p4(8,4), p5(3,5), p3(5,7), p1(2,8), p6(10,9), p0(1,10)]
```



Closet Pair—Two-Dimensions

1. Create a copy of the points (we now have two separate copies of P)
    1. Sort by x-coordinate
    2. Sort other by y-coordinate

O(n lg n)

- Can we still end up with a O(n lg n) algorithm for finding the closest pair?
- Does the closeness of two points on one axis matter?

```
FUNCTION FindClosestPair(points)
    points_x = copy_and_sort_by_x(points)
    points_y = copy_and_sort_by_y(points)
    RETURN ClosestPair(points_x, points_y)
```

Closet Pair—Two-Dimensions

1. Create a copy of the points (we now have two separate copies of P)
   1. Sort by x-coordinate
   2. Sort other by y-coordinate

O(n lg n)

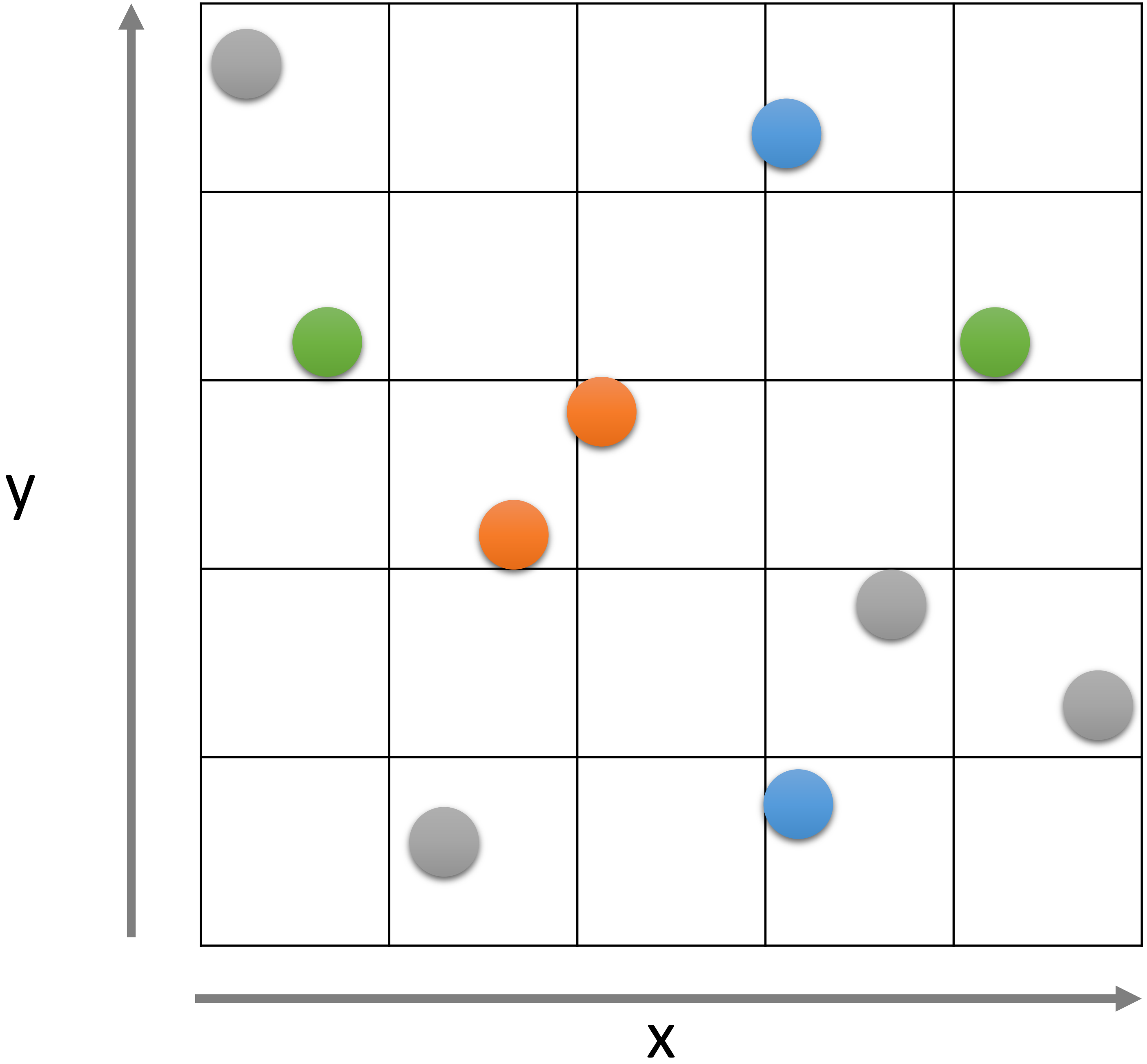
- Can we still end up with a O(n lg n) algorithm for finding the closest pair?
- Does the closeness of two points on one axis matter?

2. Apply the Divide-and-Conquer method

Divide-and-Conquer

1. DIVIDE into smaller subproblems
2. CONQUER the subproblems via recursive calls
3. COMBINE solutions from the subproblems

- How would you divide the problems?

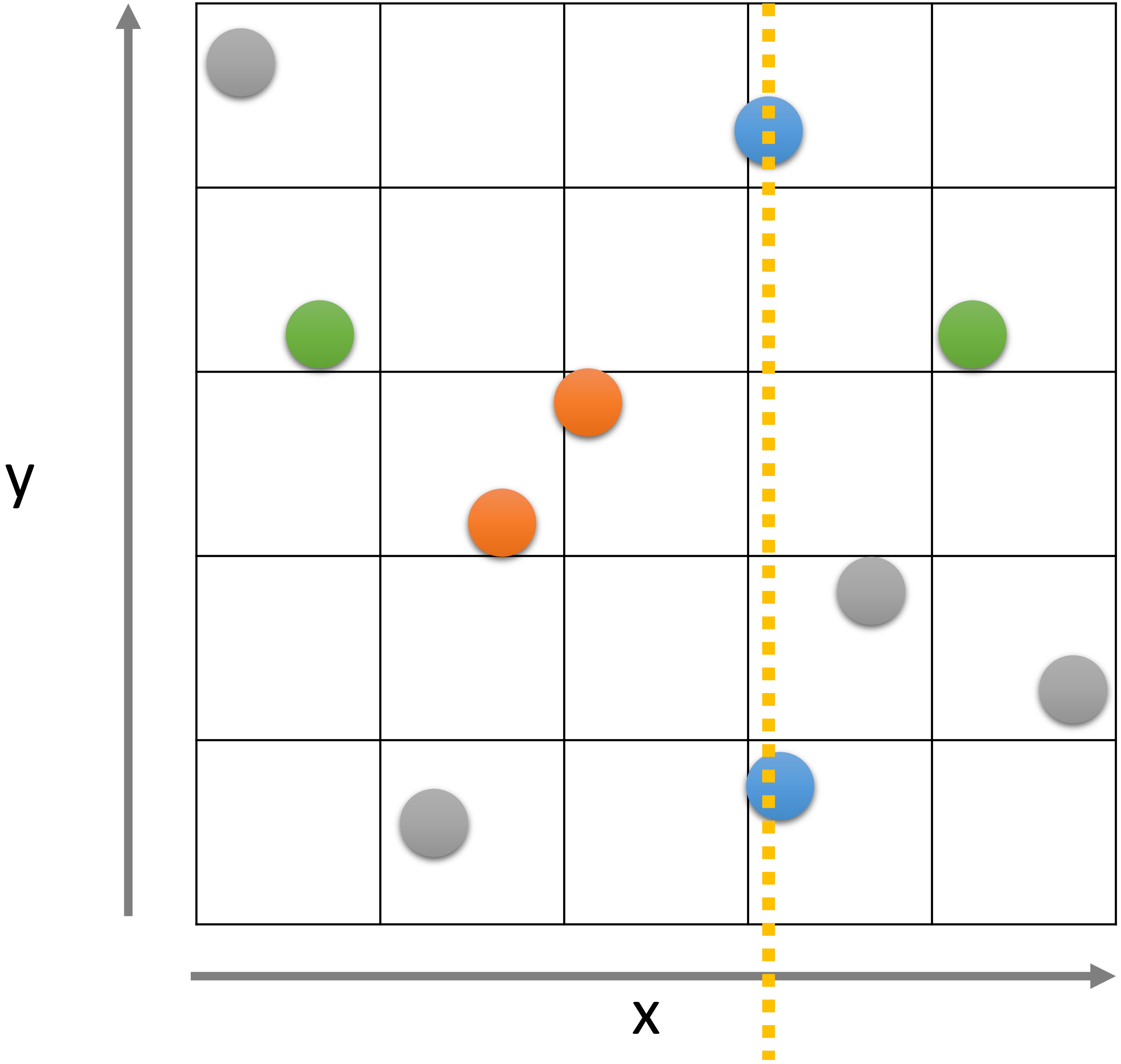


1. Which two are closest on the y-axis?

2. Which two are closest on the x-axis?

3. Which two are closest?

4. How would you divide the search space?



1. Which two are closest on the y-axis?
2. Which two are closest on the x-axis?
3. Which two are closest?
4. How would you divide the search space?

This is the median x-value
This is not the average x-value

```
FUNCTION FindClosestPair(points)
    points_x = copy_and_sort_by_x(points)
    points_y = copy_and_sort_by_y(points)
    RETURN ClosestPair(points_x, points_y)
```

```
FUNCTION ClosestPair(px, py)
    n = px.length
    # What is the base case?
    IF n == 2
        RETURN px[0], px[1], dist(px[0], px[1])
    # What are the recursive cases?
    pl, ql, dl = ClosestPair(left_px, left_py)
    pr, qr, dr = ClosestPair(right_px, right_py)
```

How do we create these arrays?

P : [p0(1,10), p1(2,8), p2(7,3), p3(5,7), p4(8,4), p5(3,5), p6(10,9), p7(9,1)]

left_px

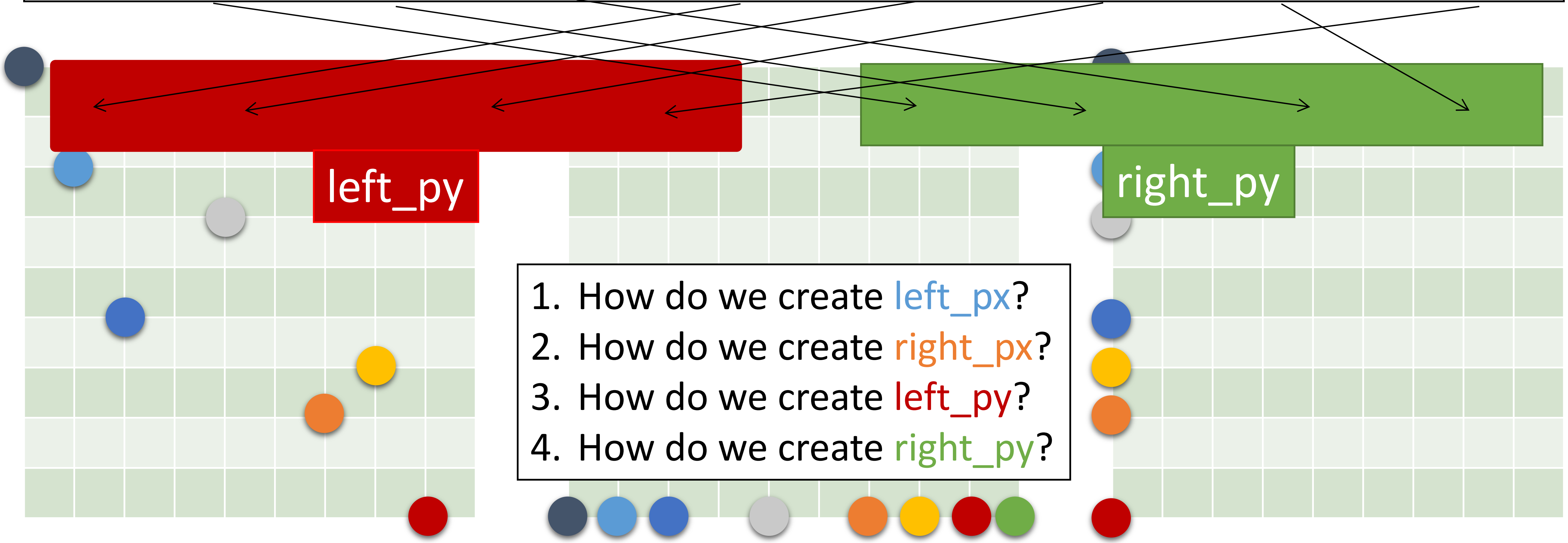
right_px

Sorted by x coordinate

Px : [p0(1,10), p1(2,8), p5(3,5), p3(5,7), p2(7,3), p4(8,4), p7(9,1), p6(10,9)]

Sorted by y coordinate

Py : [p7(9,1), p2(7,3), p4(8,4), p5(3,5), p3(5,7), p1(2,8), p6(10,9), p0(1,10)]



1. How do we create left_px?
2. How do we create right_px?
3. How do we create left_py?
4. How do we create right_py?

```
FUNCTION ClosestPair(px, py)
    n = px.length
    IF n == 2
        RETURN px[0], px[1], dist(px[0], px[1])

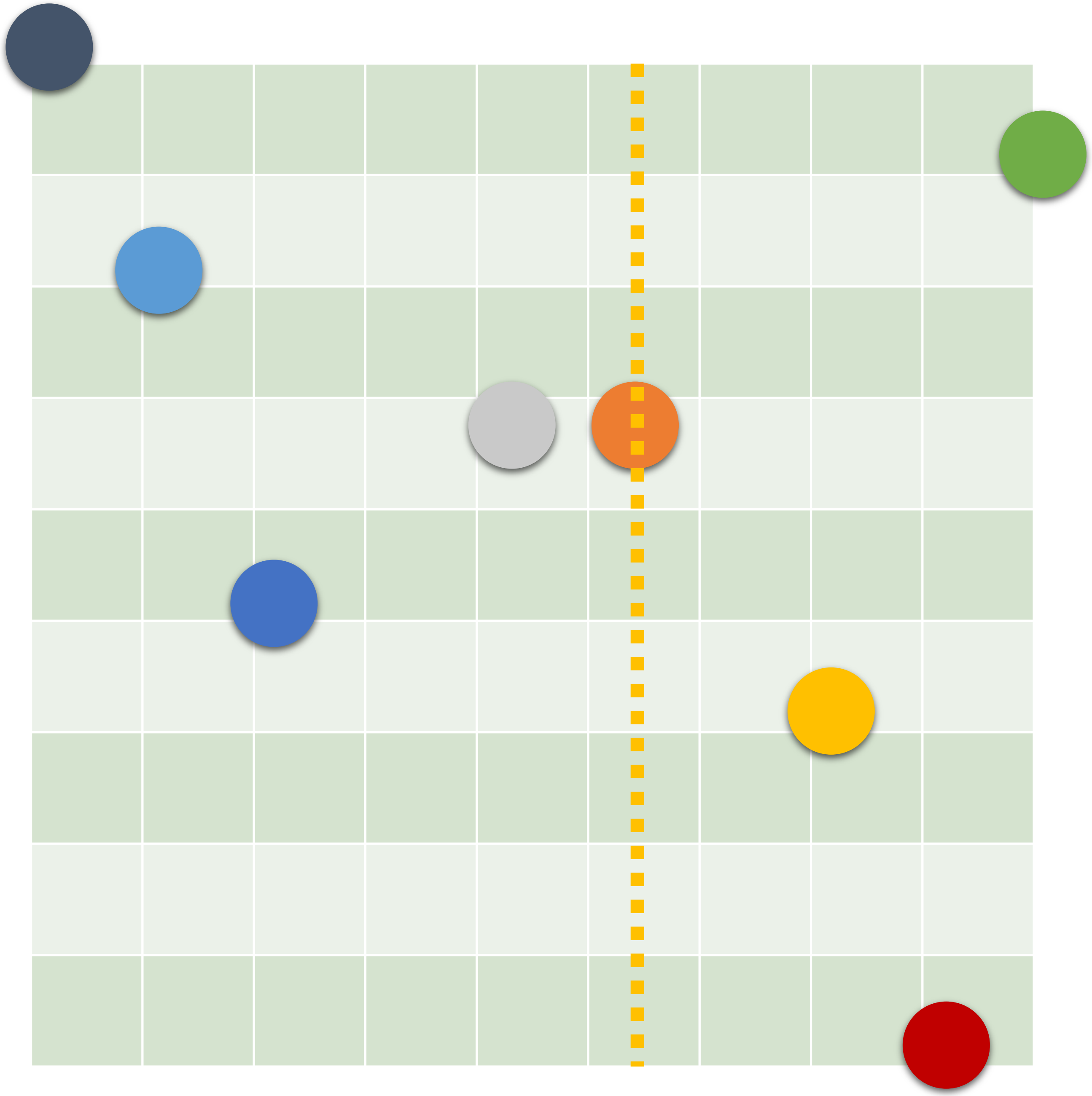
    left_px = px[0 ..< n//2]
    left_py = [p FOR p IN py IF p.x < px[n//2].x]
    pl, ql, dl = ClosestPair(left_px, left_py)

    right_px = px[n//2 ..< n]
    right_py = [p FOR p IN py IF p.x ≥ px[n//2].x]
    pr, qr, dr = ClosestPair(right_px, right_py)
```

Median x value

What is the running time of these operations?

Any problems with our current approach?



```
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    left_px = px[0 ..< n//2]
    left_py = [p FOR p IN py IF p.x < px[n//2].x]
    pl, ql, dl = ClosestPair(left_px, left_py)

    right_px = px[n//2 ..< n]
    right_py = [p FOR p IN py IF p.x
    pr, qr, dr = ClosestPair(right_p

    d = min(dl, dr)
    ps, qs, ds = ClosestSplitPair(px, py, d)

    RETURN Closest(pl, ql, dl, pr, qr, dr, ps, qs, ds)
```

What time complexity does this process need such that the overall algorithm runs in O(n lg n)?
Hint: think about Merge Sort.

Exercise Question 1

1. What must be the running time of `ClosestSplitPair` if the `ClosestPair` algorithm is to have a running time of O(n lg n)?

```
FUNCTION ClosestPair(px, py)
    n = px.length
    IF n == 2
        RETURN px[0], px[1], dist(px[0], px[1])

    left_px = px[0 ..< n//2]
    left_py = [p FOR p IN py IF p.x < px[n//2].x]
    pl, ql, dl = ClosestPair(left_px, left_py)

    right_px = px[n//2 ..< n]
    right_py = [p FOR p IN py IF p.x ≥ px[n//2].x]
    pr, qr, dr = ClosestPair(right_px, right_py)

    d = min(dl, dr)
    ps, qs, ds = ClosestSplitPair(px, py, d)

    RETURN Closest(pl, ql, dl, pr, qr, dr, ps, qs, ds)
```

Merge Sort and It’s Recurrence

```
FUNCTION RecursiveFunction(some_input)
   IF base_case:
      # Usually O(1)
      RETURN base_case_work(some_input)

   # Two recursive calls, each with half the data
   one = RecursiveFunction(some_input.first_half)
   two = RecursiveFunction(some_input.second_half)

   # Combine results from recursive calls (usually O(n))
   one_and_two = Combine(one, two)

   RETURN one_and_two
```

```
FUNCTION ClosestPair(px, py)
    n = px.length
    IF n == 2
        RETURN px[0], px[1], dist(px[0], px[1])

    left_px = px[0 ..< n//2]
    left_py = [p FOR p IN py IF p.x < px[n//2].x]
    pl, ql, dl = ClosestPair(left_px, left_py)

    right_px = px[n//2 ..< n]
    right_py = [p FOR p IN py IF p.x
    pr, qr, dr = ClosestPair(right_p

    d = min(dl, dr)
    ps, qs, ds = ClosestSplitPair(px, py, d)

    RETURN Closest(pl, ql, dl, pr, qr, dr, ps, qs, ds)
```

How do we find the closest pair that splits the two sides?

Key Idea

- **In** `ClosestSplitPair` **we only need to check for pairs that are closer than those found in the recursive calls to** `ClosestPair`

- This is easier (faster) than trying to find the closest split pair without any extra information!

```
d = min[d(pl, ql), d(pr, qr)]
```

```
FUNCTION ClosestSplitPair(px, py, d)
   n = px.length
   x_median = px[n//2].x
   middle_py = [p FOR p IN py IF x_median - d < p.x < x_median + d]

   closest_d = INFINITY, closest_p = closest_q = NONE
   FOR i IN [0 ..< middle_py.length - 1]
      FOR j IN [1 ..= min(7, middle_py.length - i)]
         p = middle_py[i], q = middle_py[i + j]
         IF dist(p, q) < closest_d
            closest_d = dist(p, q)
            closest_p = p, closest_q = q

   RETURN closest_p, closest_q, closest_d
```

At most 6 points vertically “between” the two closest points.

Exercise Question 2

2. What is the running time of the nested for-loop (looping over j)?

```
FUNCTION ClosestSplitPair(px, py, d)
   n = px.length
   x_median = px[n//2].x
   middle_py = [p FOR p IN py IF x_median - d < p.x < x_median + d]

   closest_d = INFINITY, closest_p = closest_q = NONE
   FOR i IN [0 ..< middle_py.length - 1]
      FOR j IN [1 ..= min(7, middle_py.length - i)]
         p = middle_py[i], q = middle_py[i + j]
         IF dist(p, q) < closest_d
            closest_d = dist(p, q)
            closest_p = p, closest_q = q

   RETURN closest_p, closest_q, closest_d
```

Loop Unrolling

```
FOR j IN [1 ..= min(7, middle_py.length - i)]
   p = middle_py[i], q = middle_py[i + j]
   IF dist(p, q) < closest_d
      closest_d = dist(p, q)
      closest_p = p, closest_q = q
```

```
IF dist(middle_py[i], middle_py[i + 1]) < closest_d
   closest_d = dist(middle_py[i], middle_py[i + 1])
   closest_p = middle_py[i]
   closest_q = middle_py[i + 1]

IF dist(middle_py[i], middle_py[i + 2]) < closest_d
   closest_d = dist(middle_py[i], middle_py[i + 2])
   closest_p = middle_py[i]
   closest_q = middle_py[i + 2]

…
```

```
FUNCTION ClosestSplitPair(px, py, d)
   n = px.length
   x_median = px[n//2].x
   middle_py = [p FOR p IN py
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   closest_d = INFINITY, closest_p = closest_q = NONE
   FOR i IN [0 ..< middle_py.length − 1]
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         p = middle_py[i], q = middle_py[i + j]
         IF dist(p, q) < closest_d
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   RETURN closest_p, closest_q, closest_d
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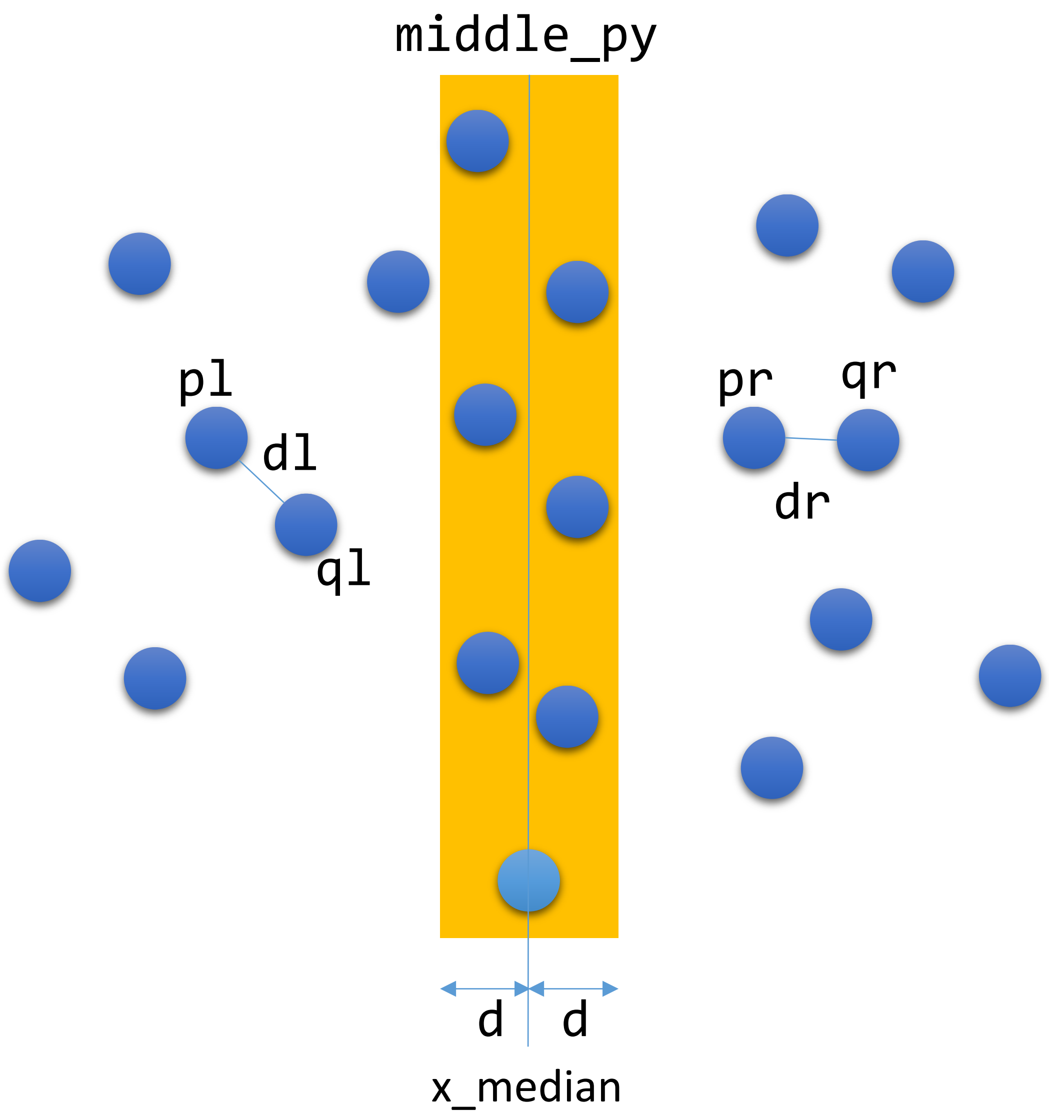

Theorem for correctness of ClosestPair

Theorem:

Provided a set of n points called P, the `ClosestPair` algorithm find the closest pair of points according to their pairwise Euclidean distances.

`ClosestPair` finds the closest pair

Let p `∈ left, q ∈ right` be a split pair with `d(p, q) < d`

Then

A. p and q ∈ `middle_py`, and
B. p and q are at most **7** positions apart in `middle_py`

If the claim is true:

Corollary 1: If the closest pair of P is in a split pair, then our `ClosestSplitPair` procedure finds it.

Corollary 2: `ClosestPair` is correct and runs in O(n lg n) since it has the same recursion tree as merge sort

Proof—Part A

Let p ∈ `left`, q ∈ `right` be a split pair with `d(p, q) < d`
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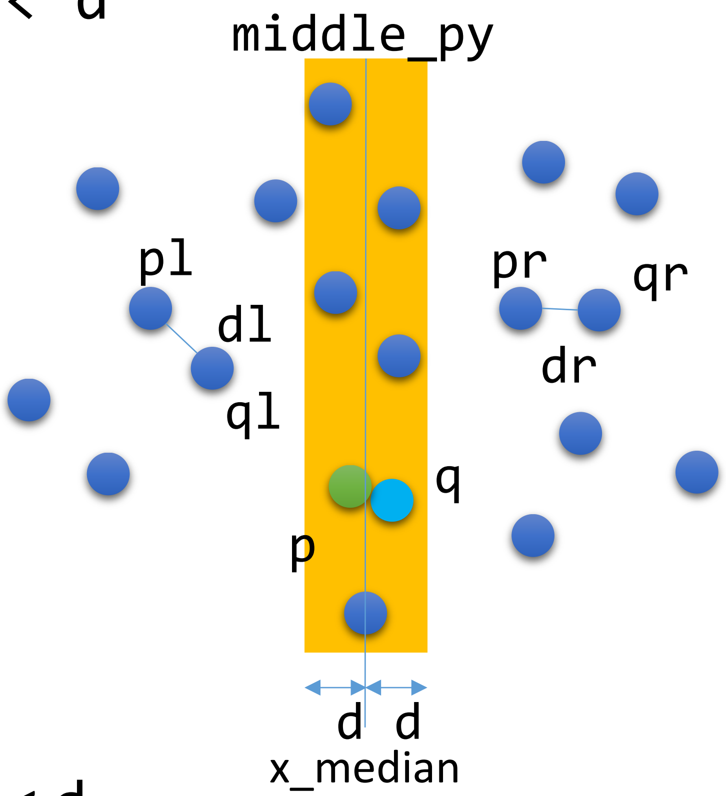
$\boxed{\text{A. p and q} \in \texttt{middle\_py}\text{, and}}$



If p = (x1,y1) ∈ left AND q = (x2,y2) ∈ right AND d(p,q) < d
Then

```
x_median - d < x1 ≤ x_median and
x_median     ≤ x2 < x_median + d
```

Otherwise, p and q would not be the closest pair with d(p, q) < d

Proof—Part A

Let p ∈ `left`, q ∈ `right` be a split pair with `d(p, q) < d`
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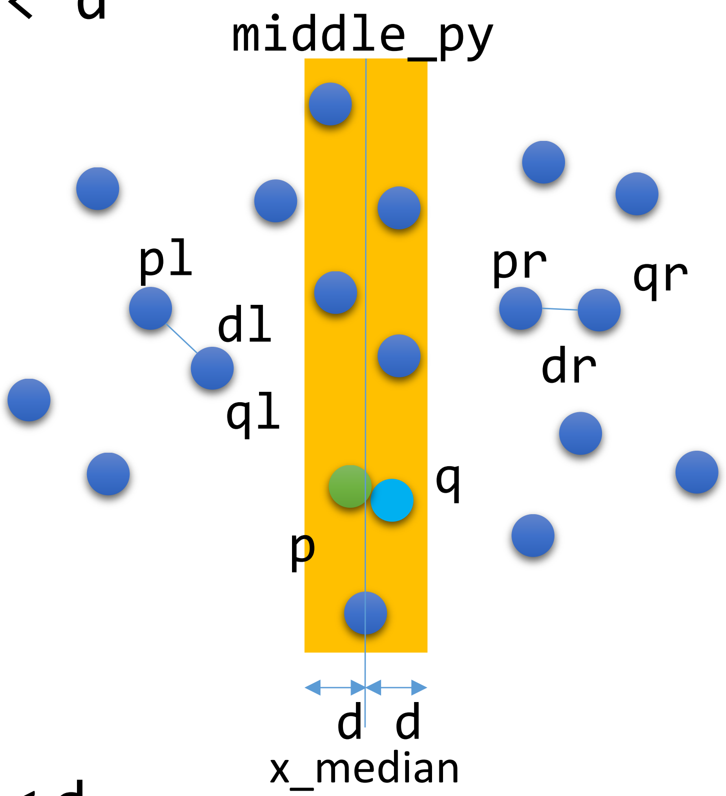

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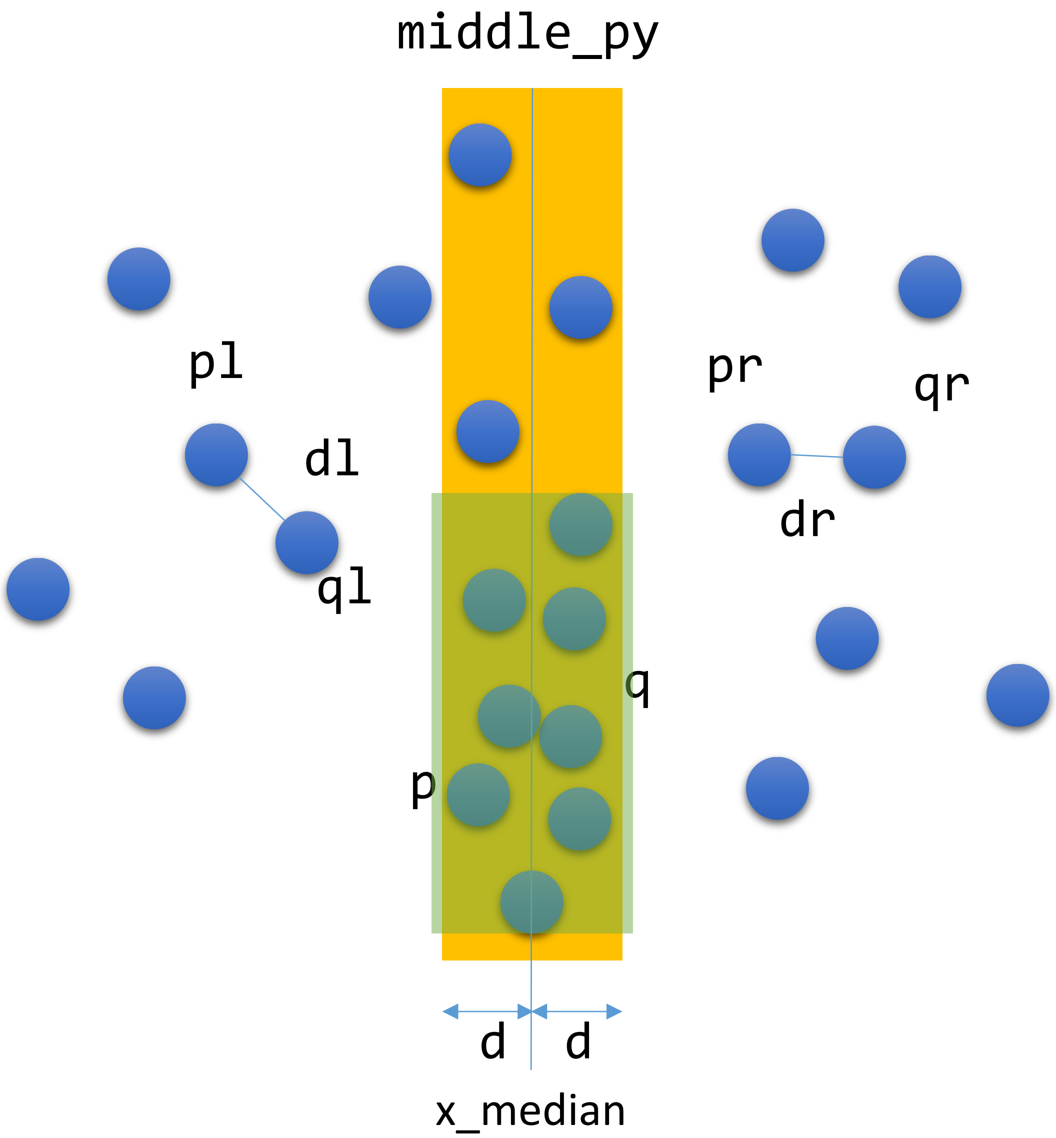
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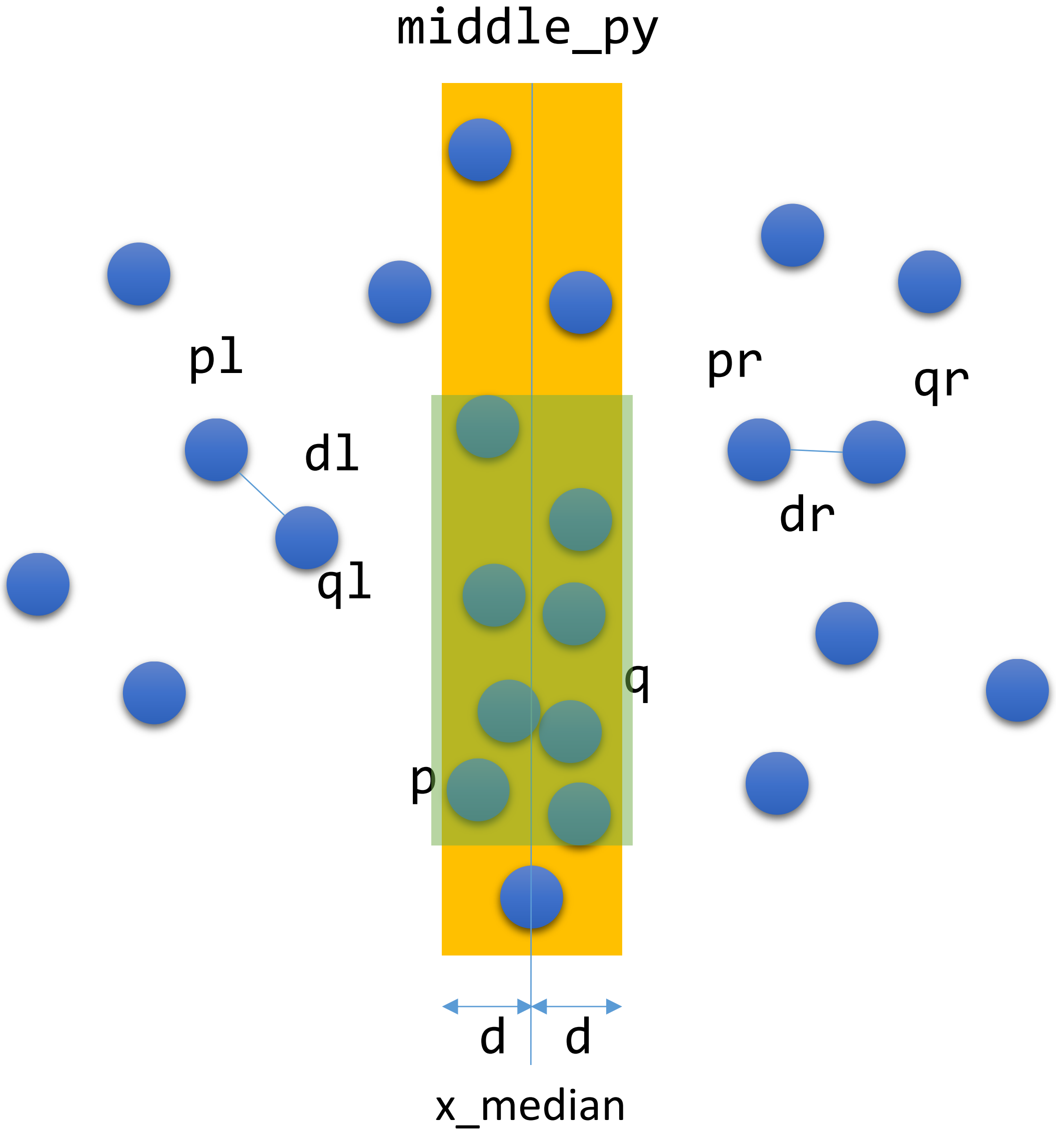
x_median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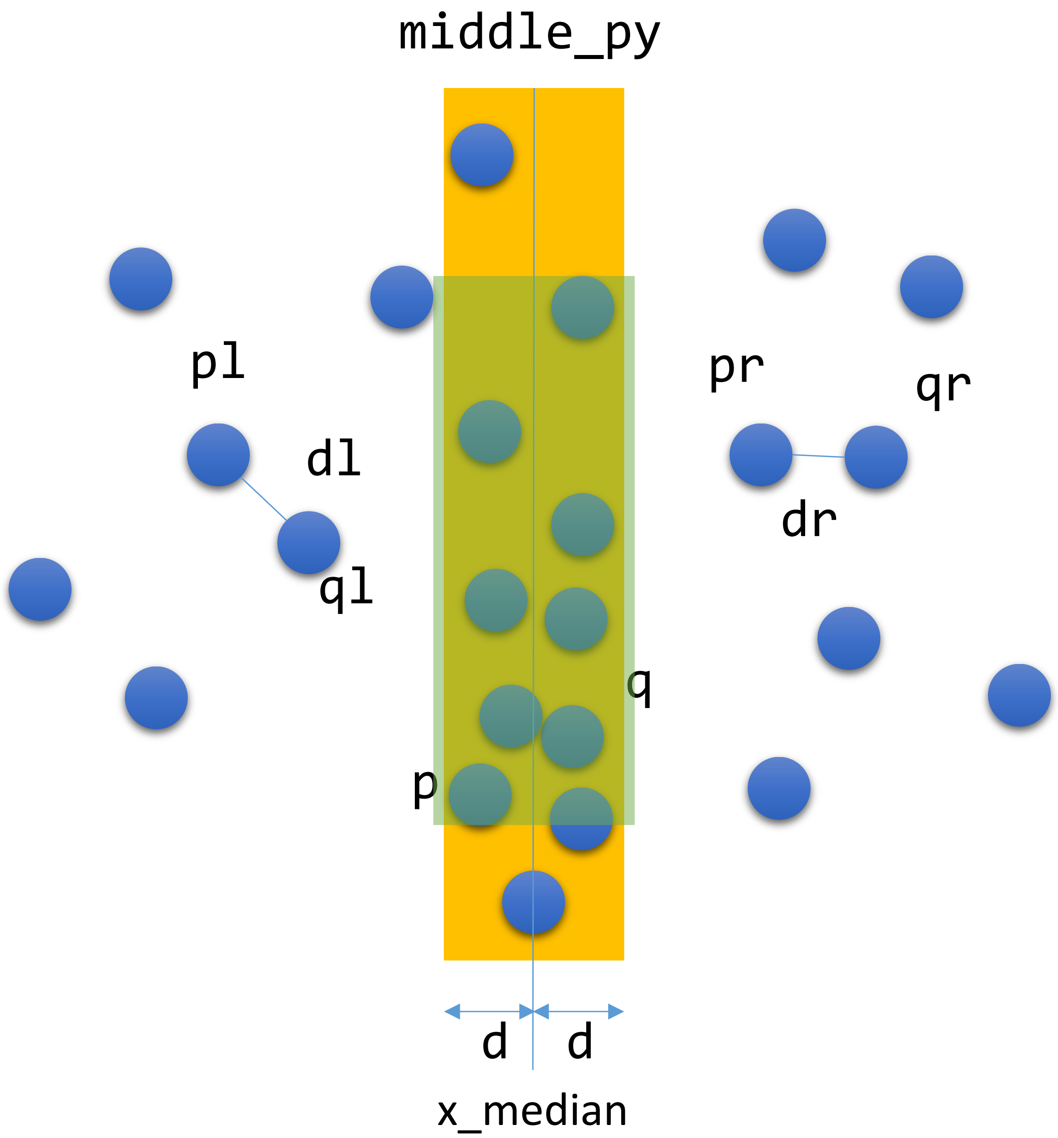
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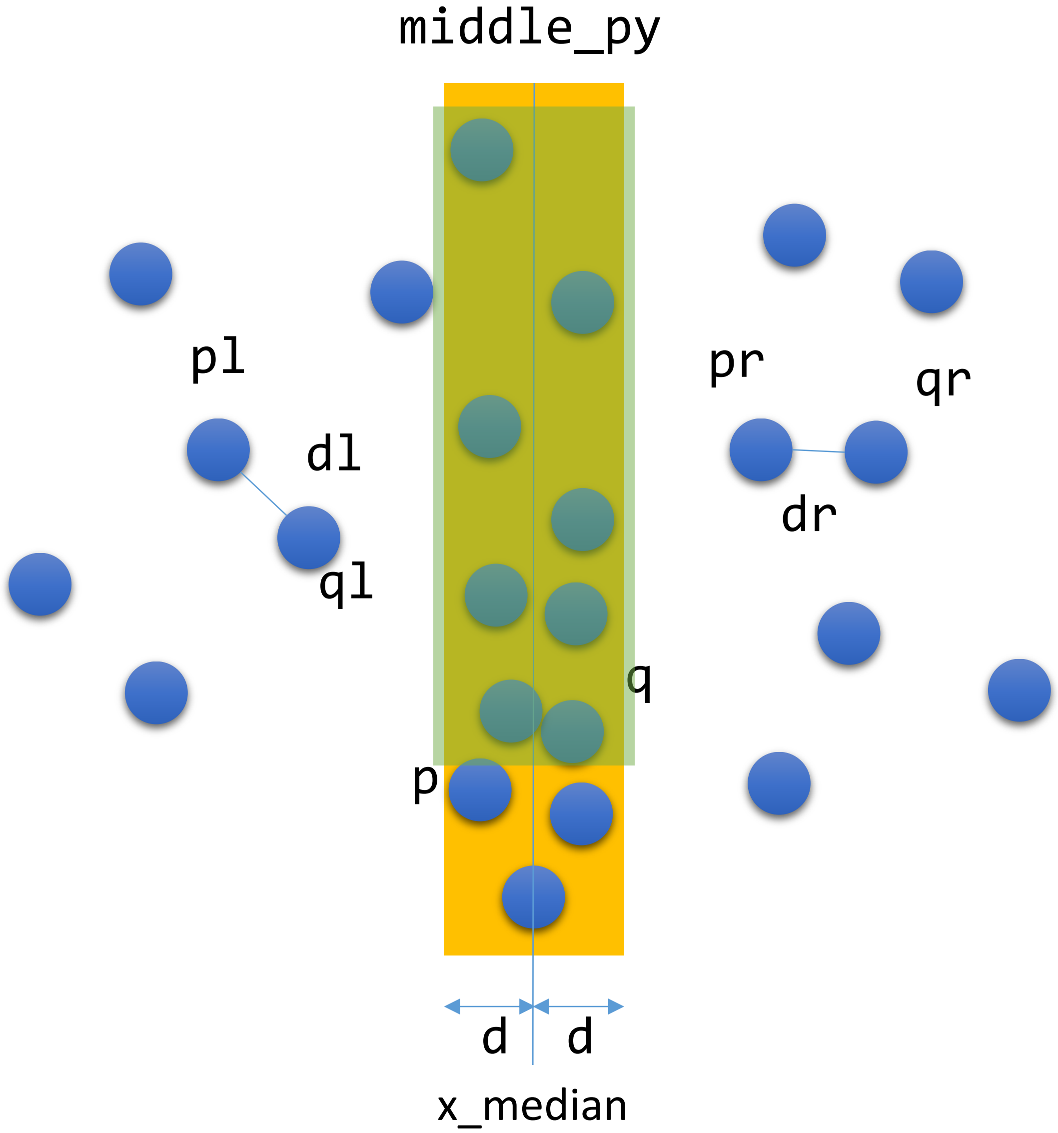
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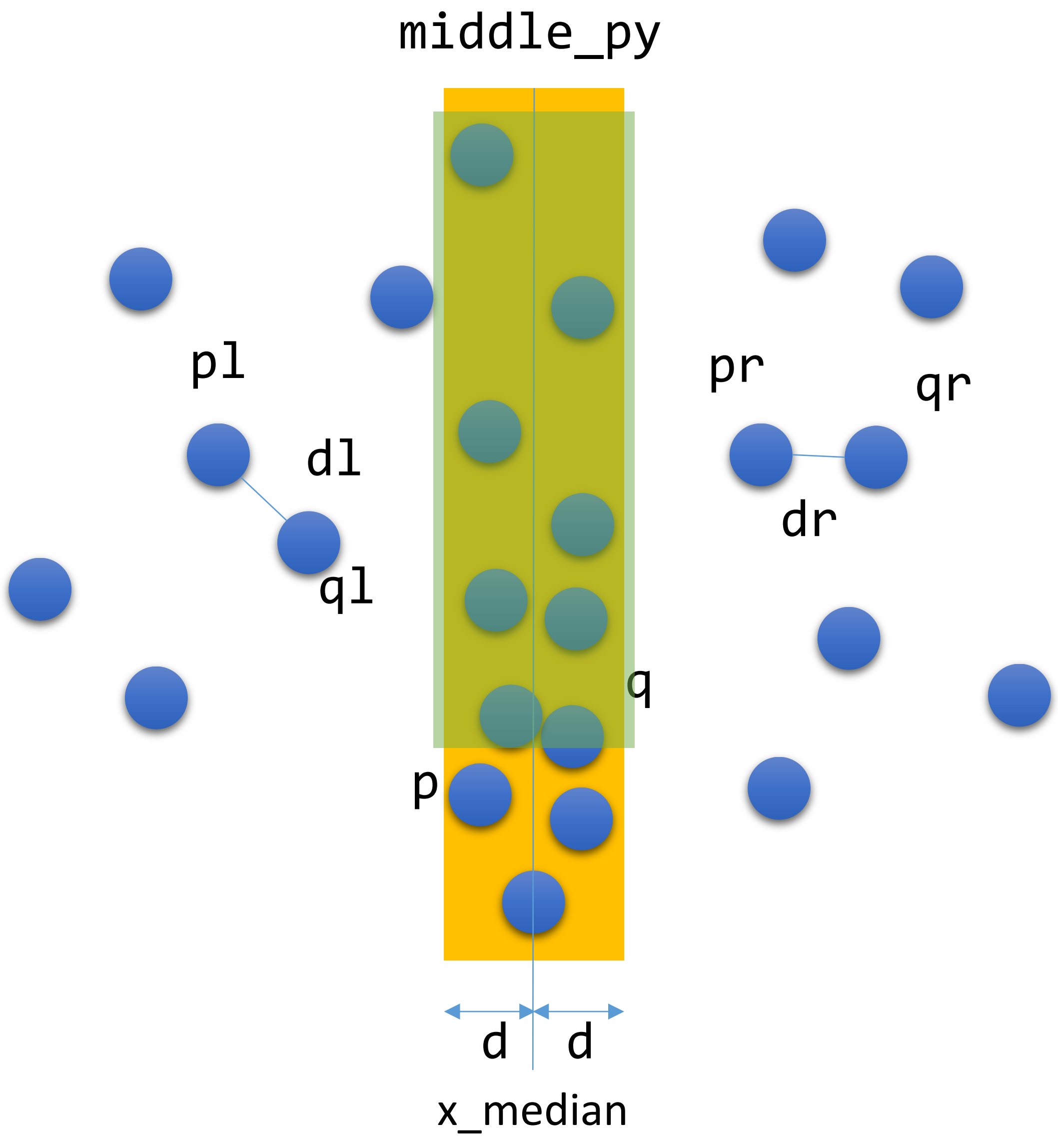
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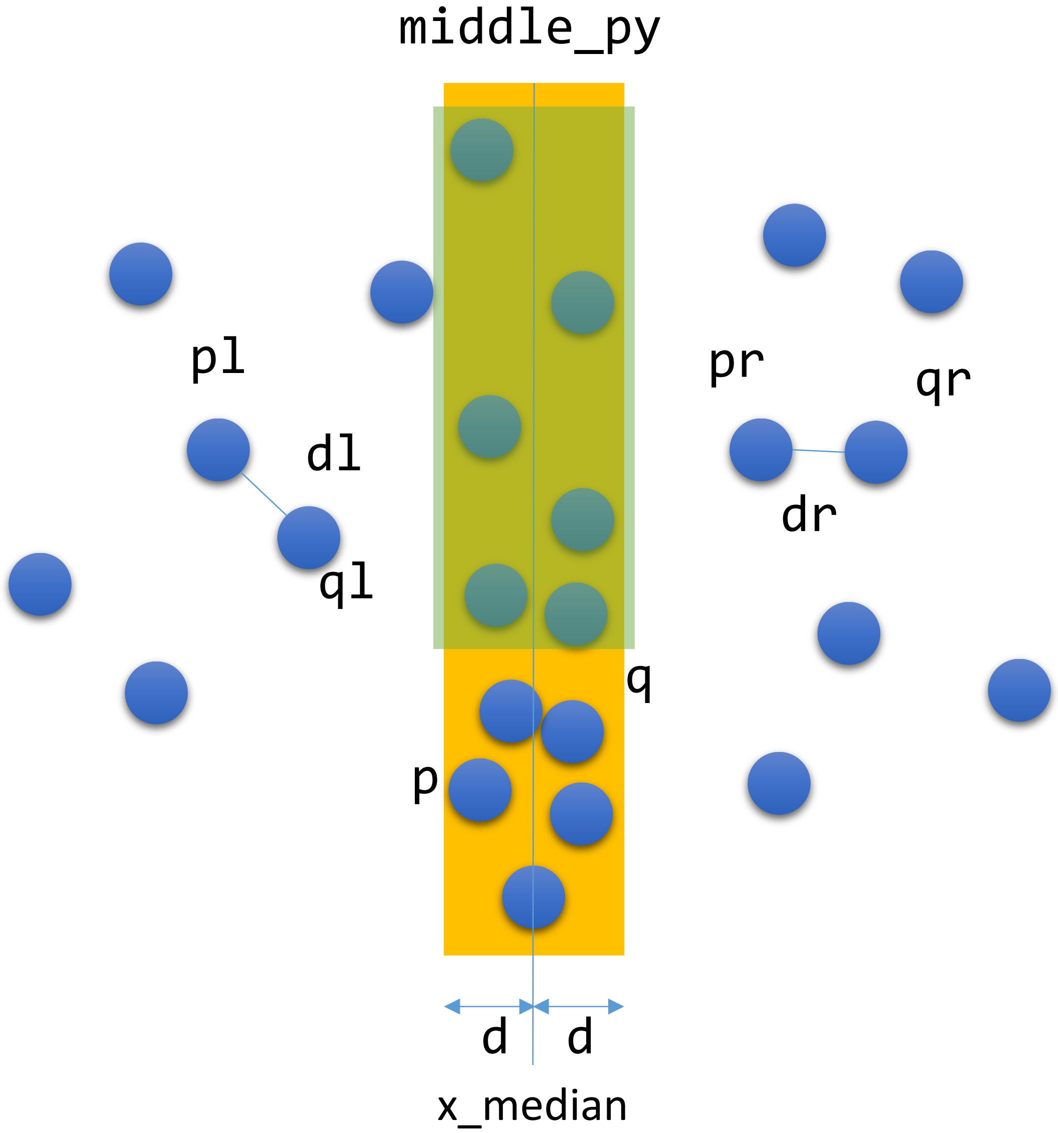
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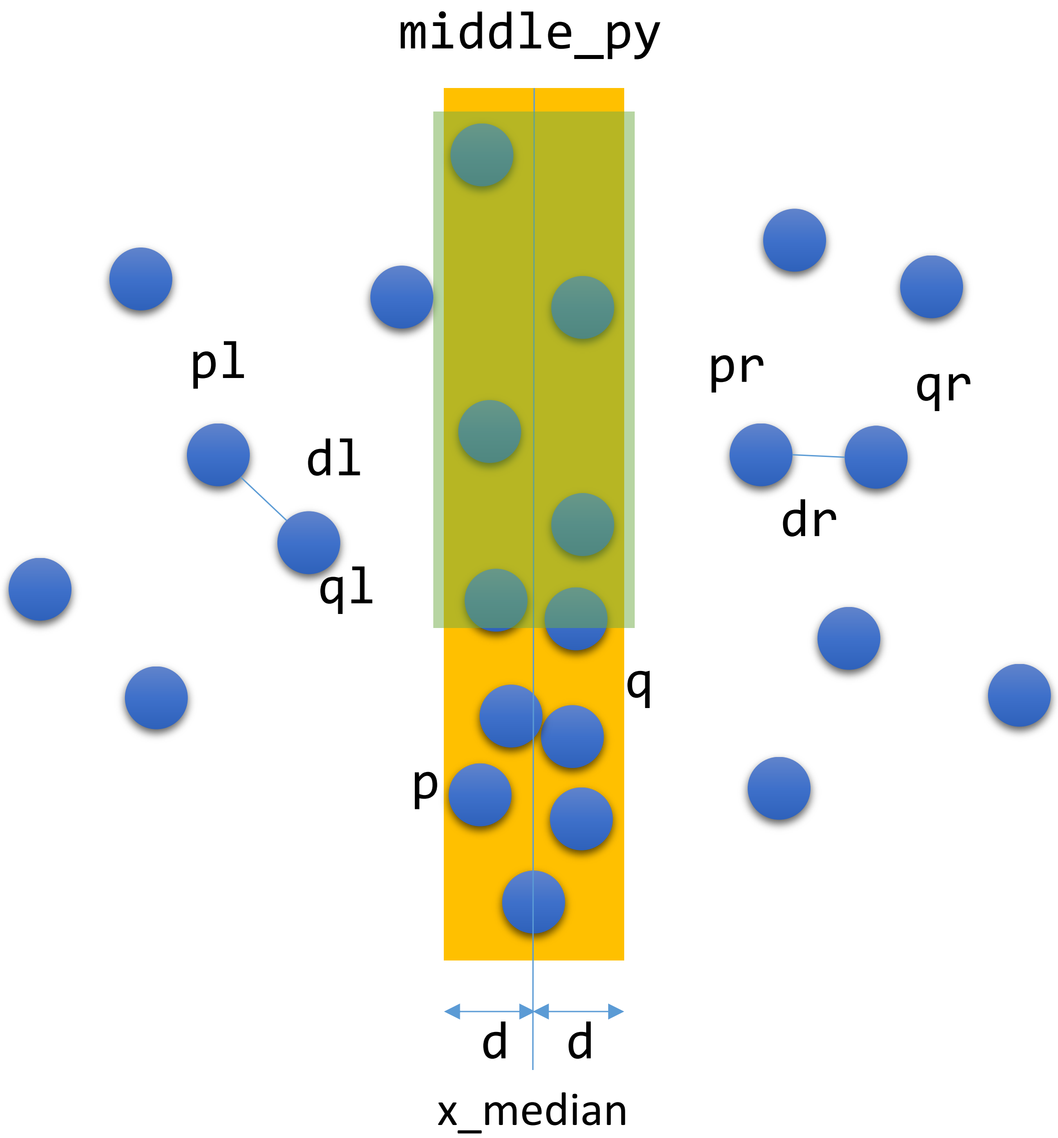
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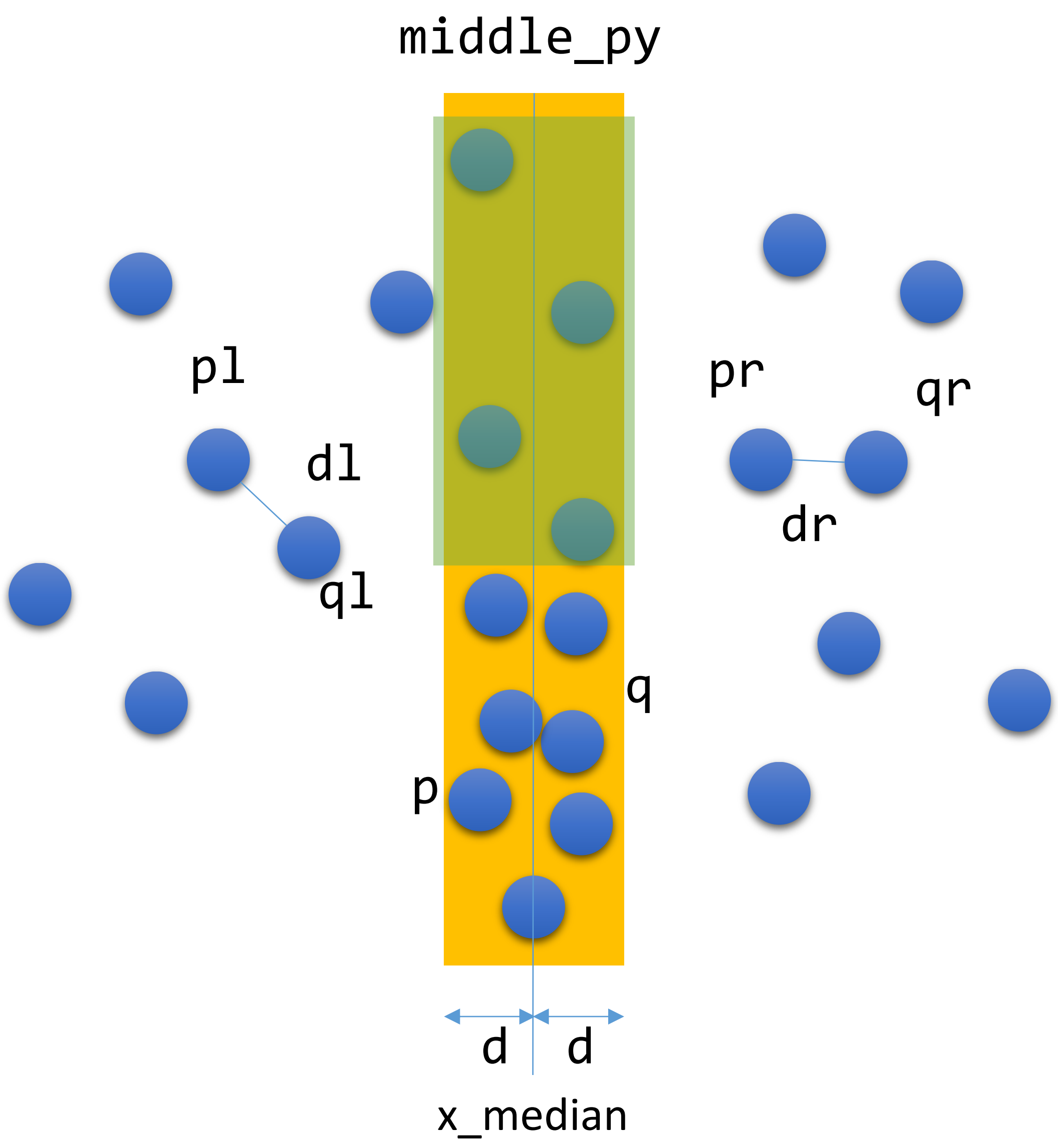
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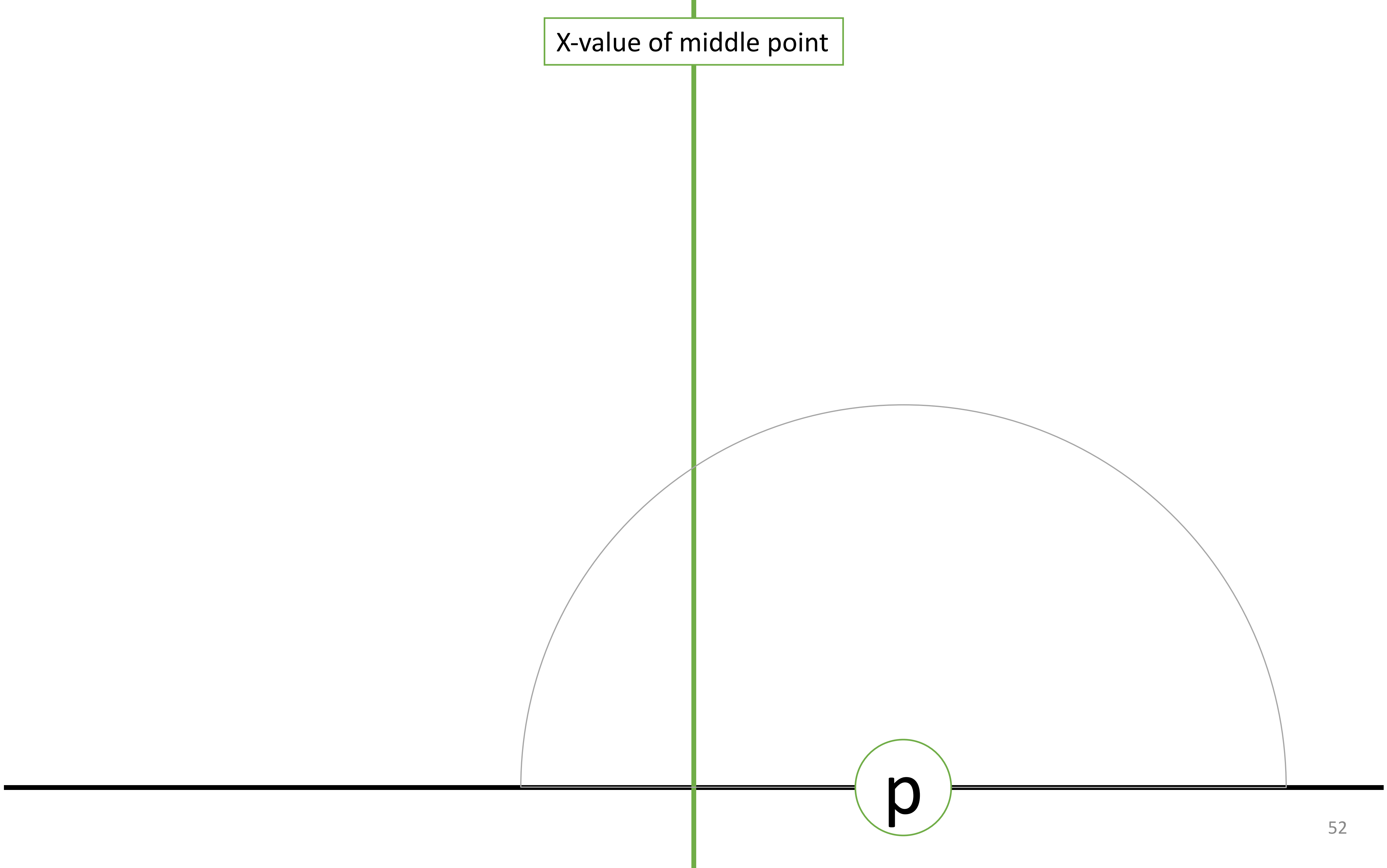
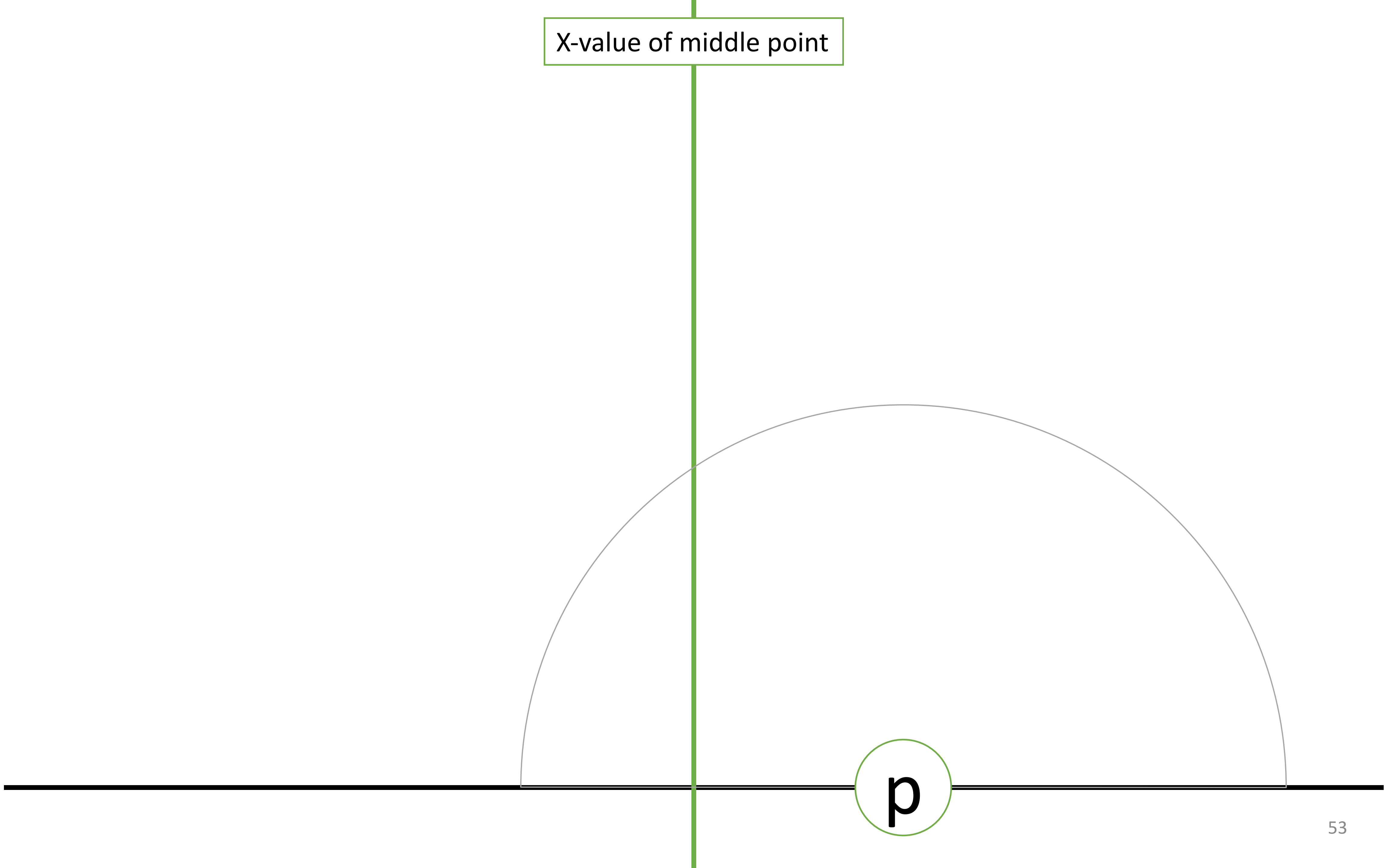

X-value of middle point
p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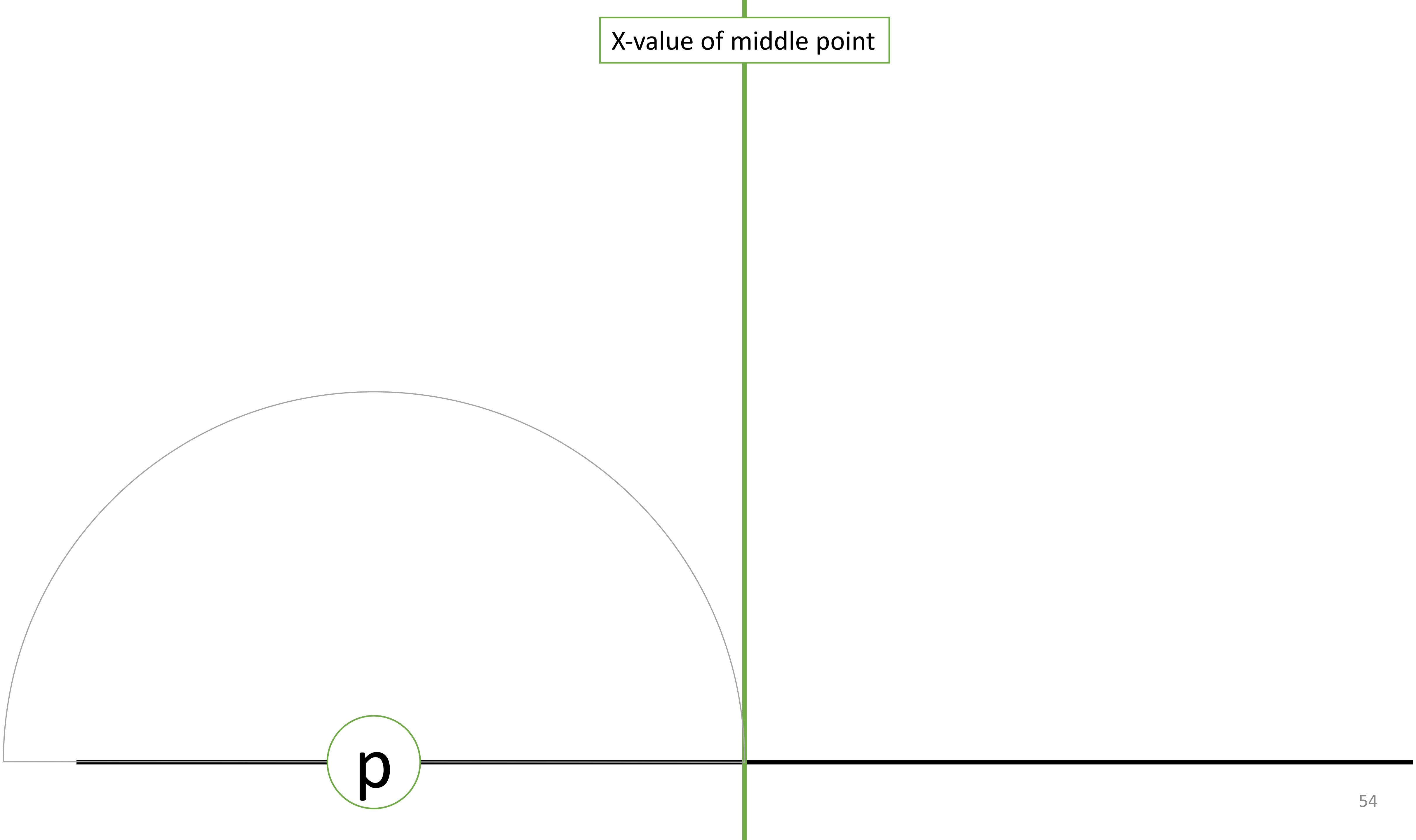
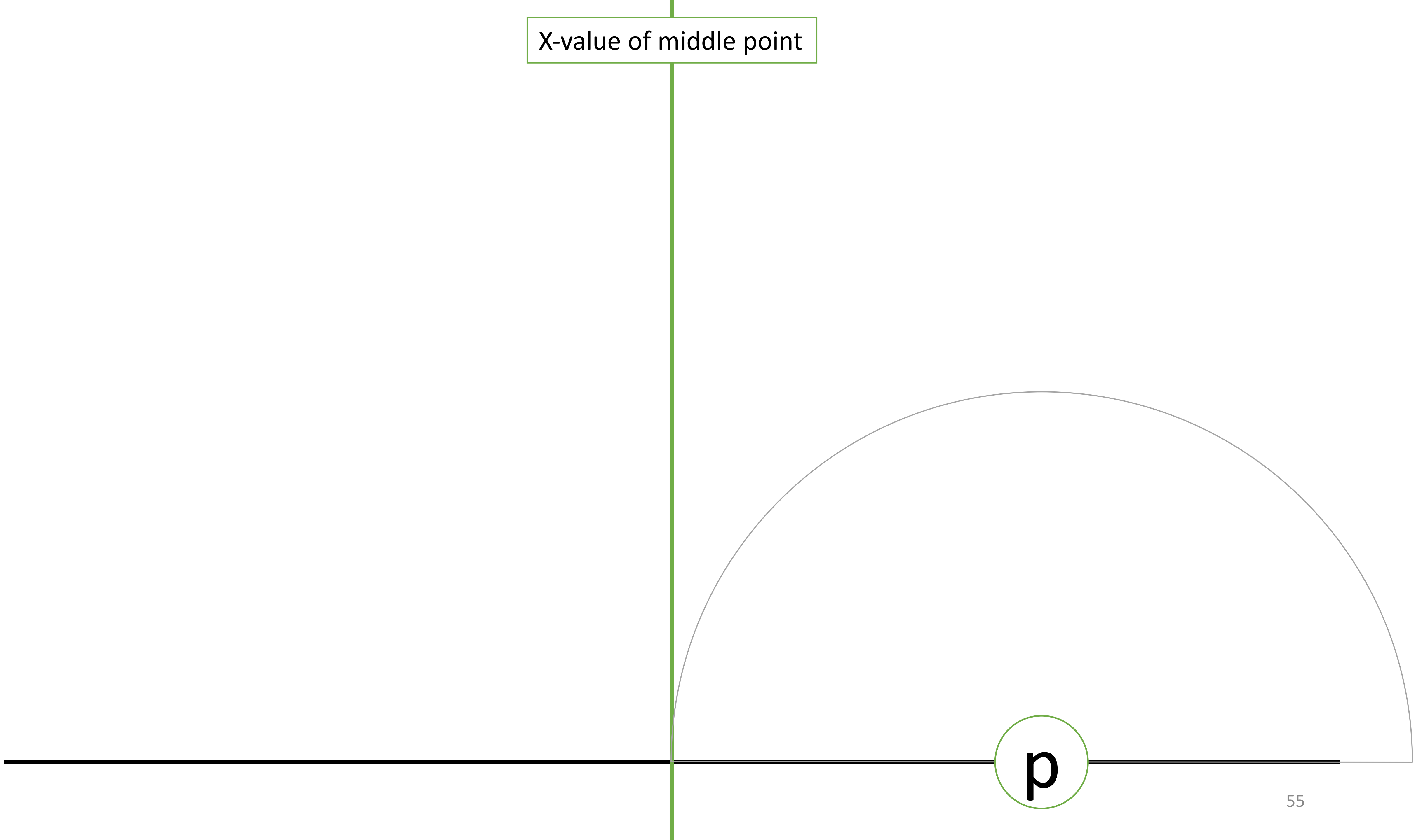
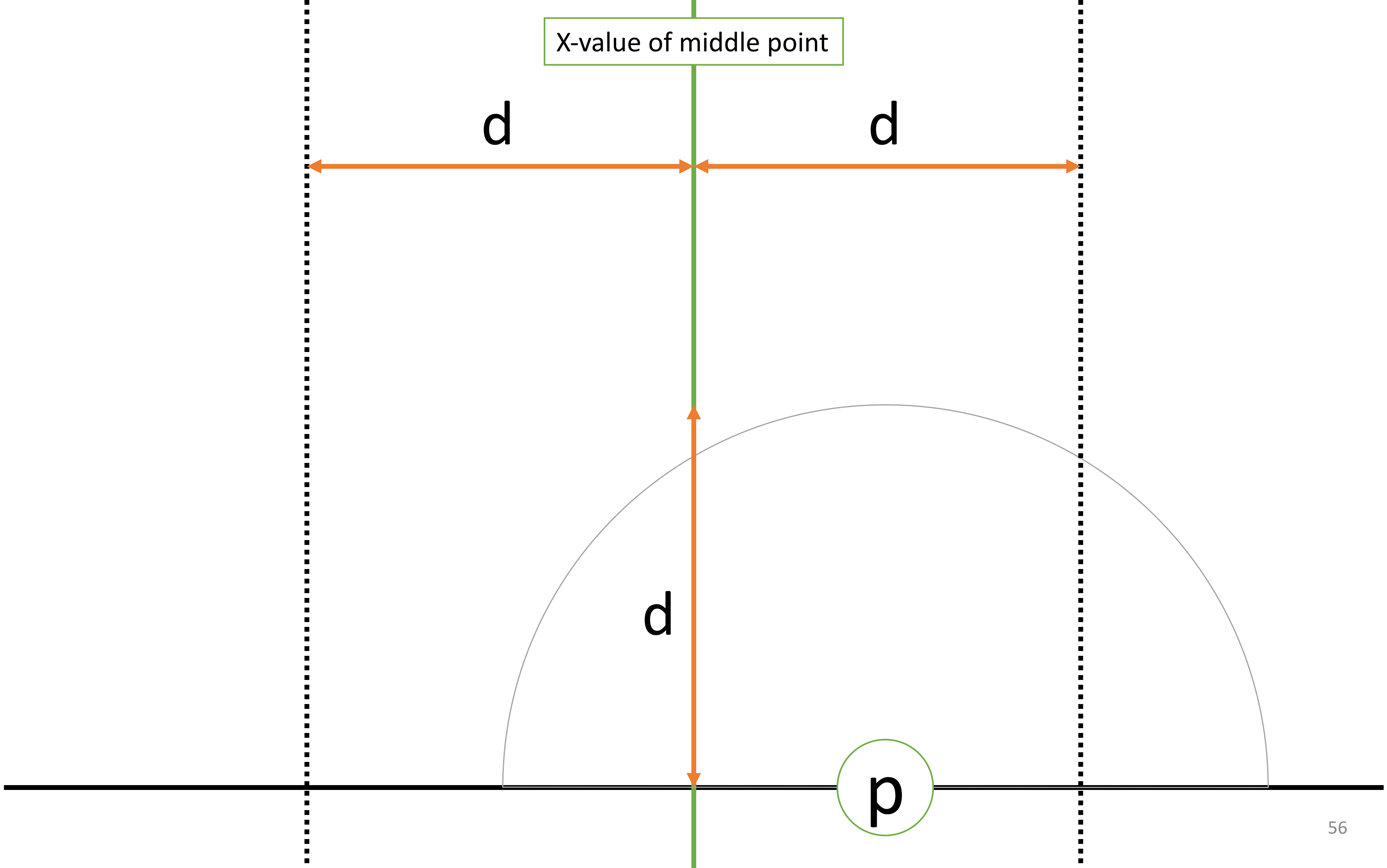
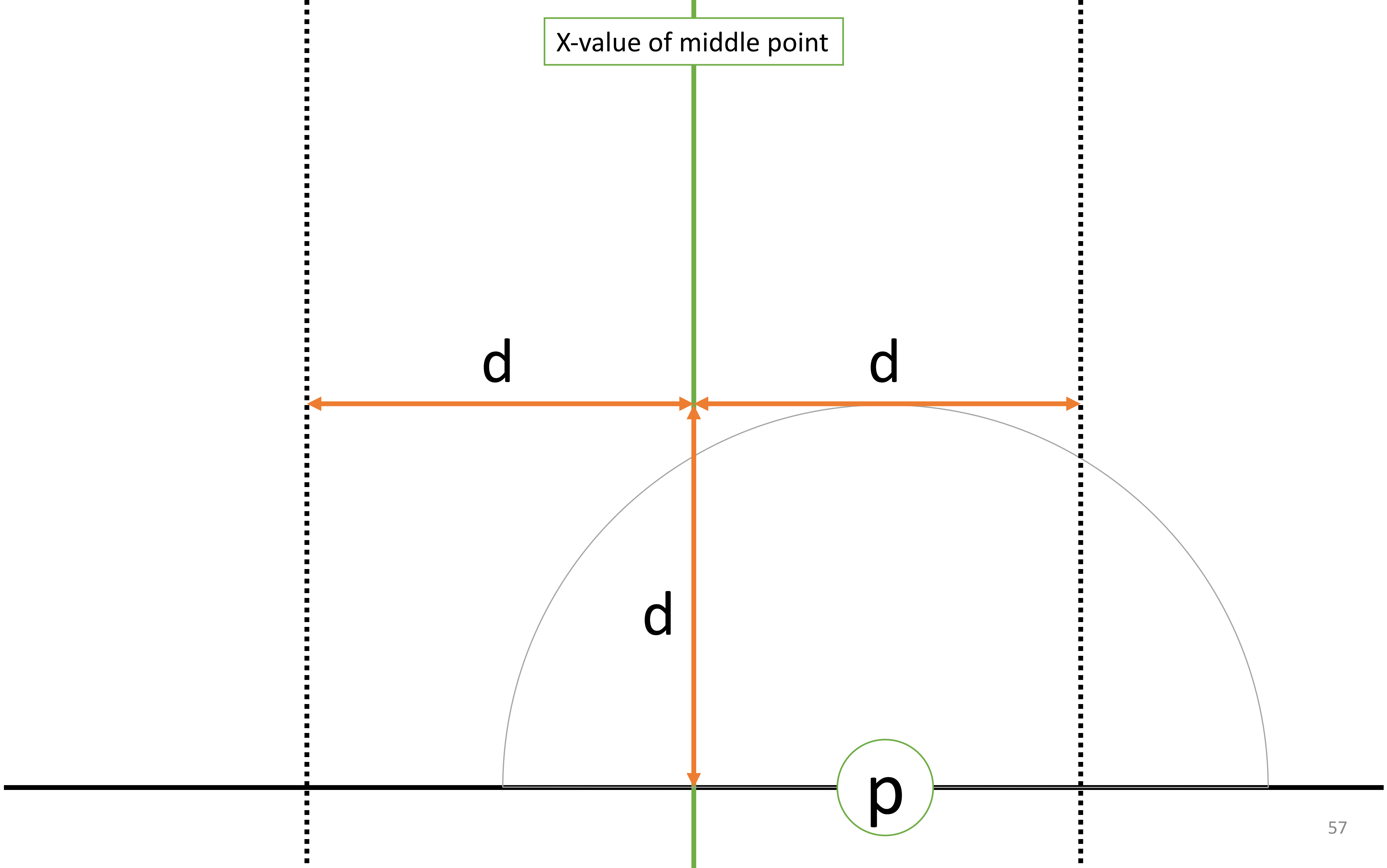
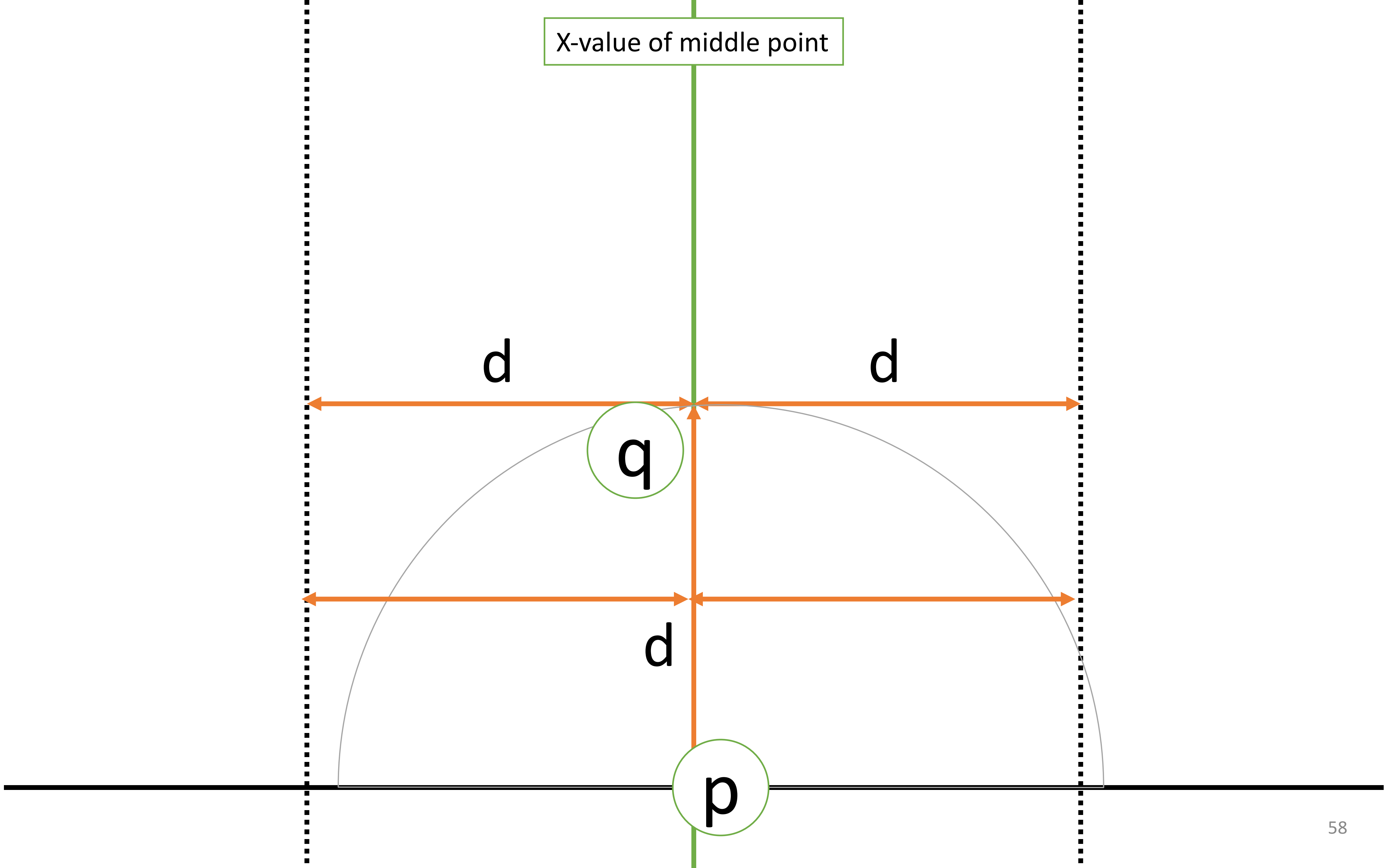

X-value of middle point
p


X-value of middle point
p

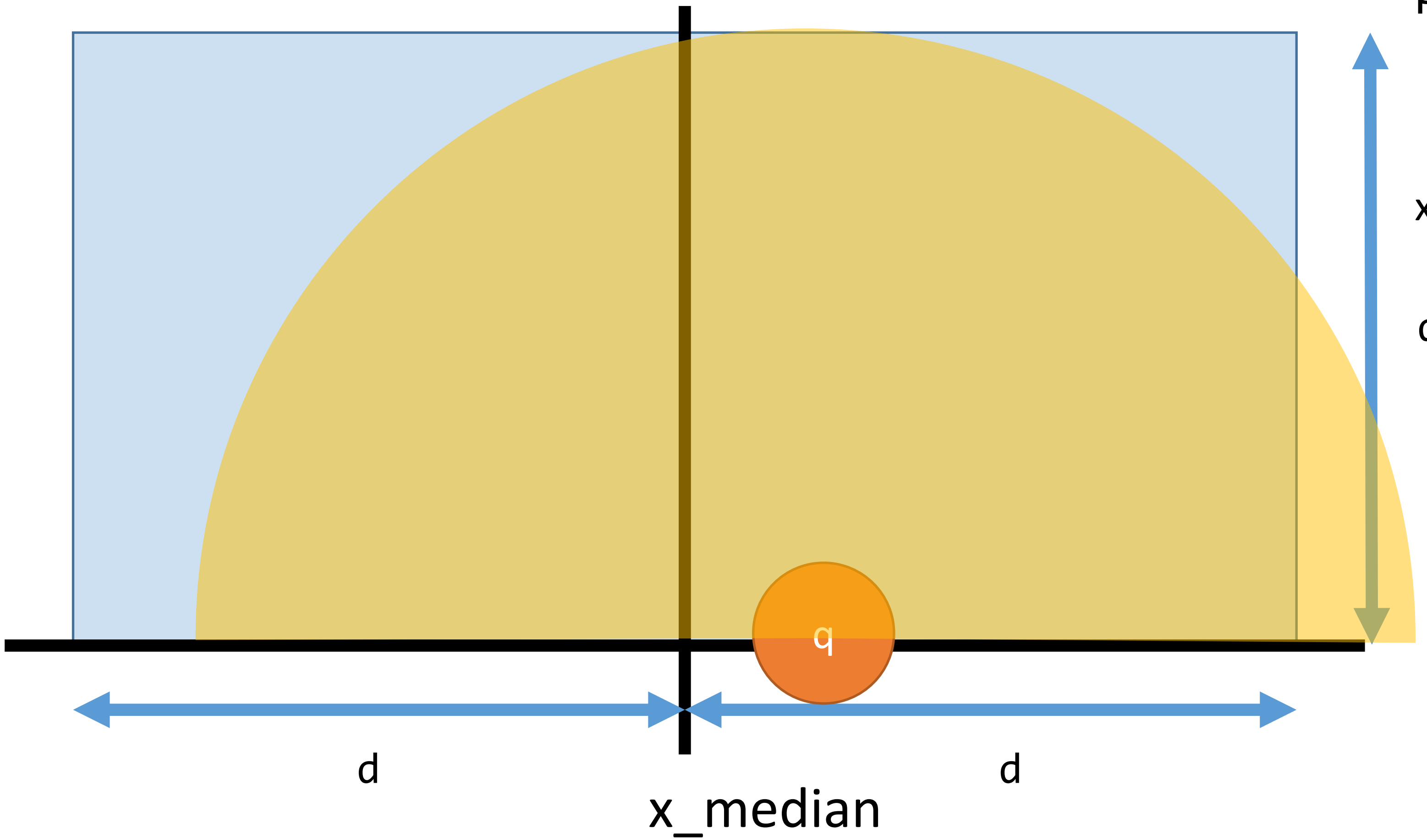

X-value of middle point
d
d
d
p

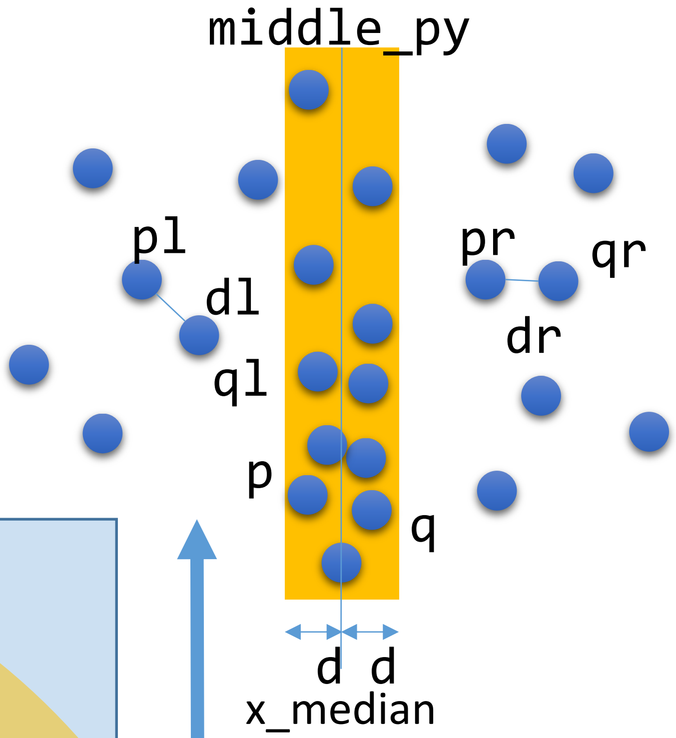

X-value of middle point
d
d
d
p


X-value of middle point
d
d
q
d
p

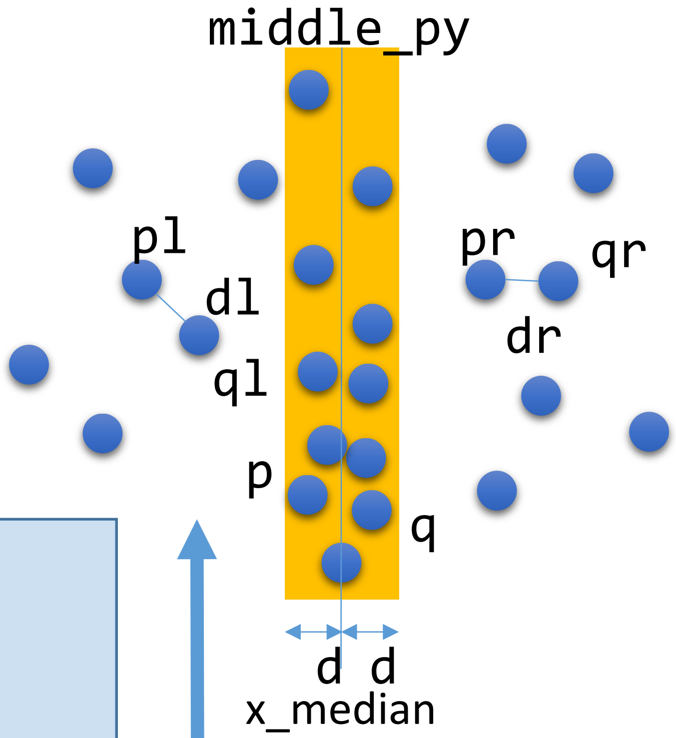
Proof—Part B

p and q are at most **7** positions apart in `middle_py`



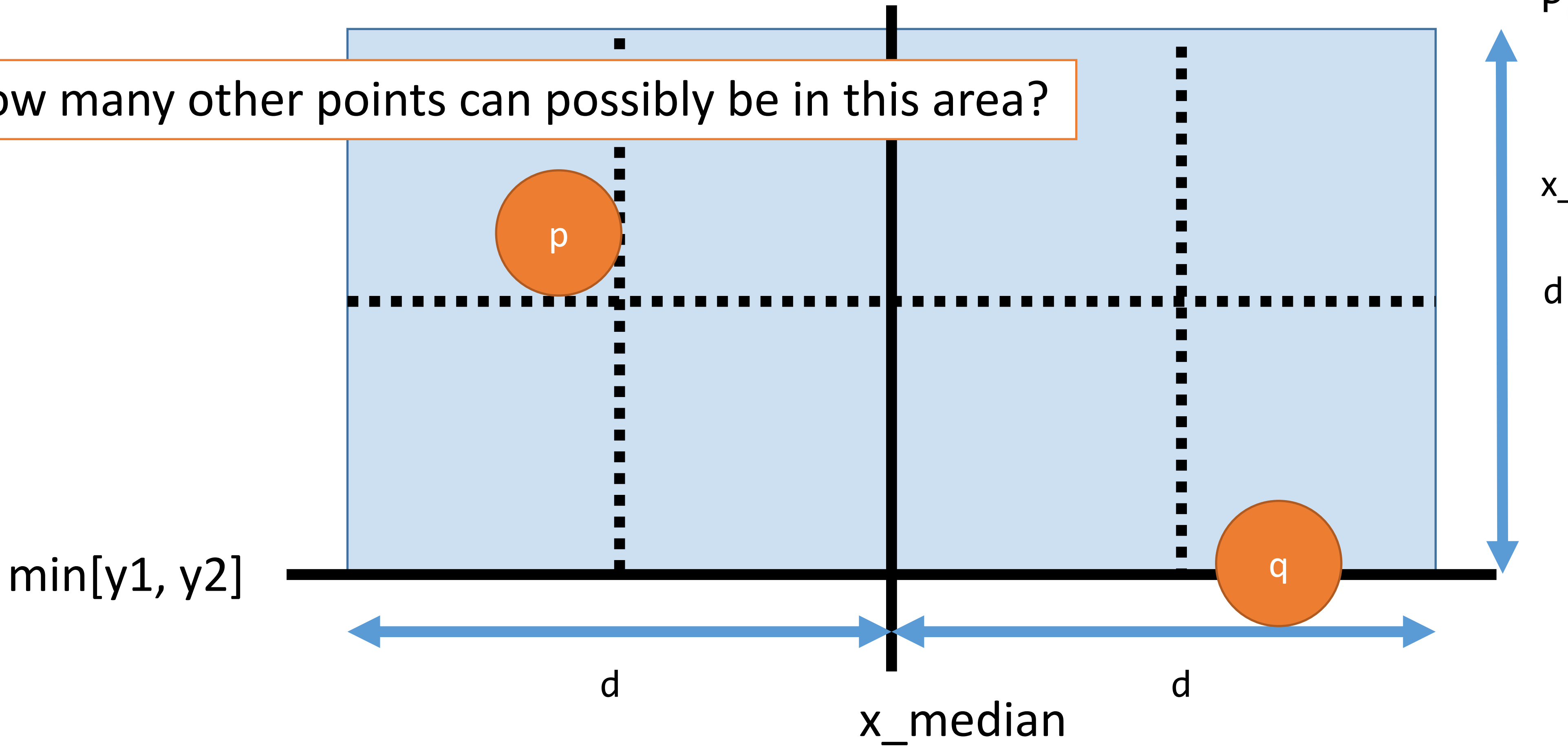


Proof—Part B



p and q are at most 7 positions apart in `middle_py`

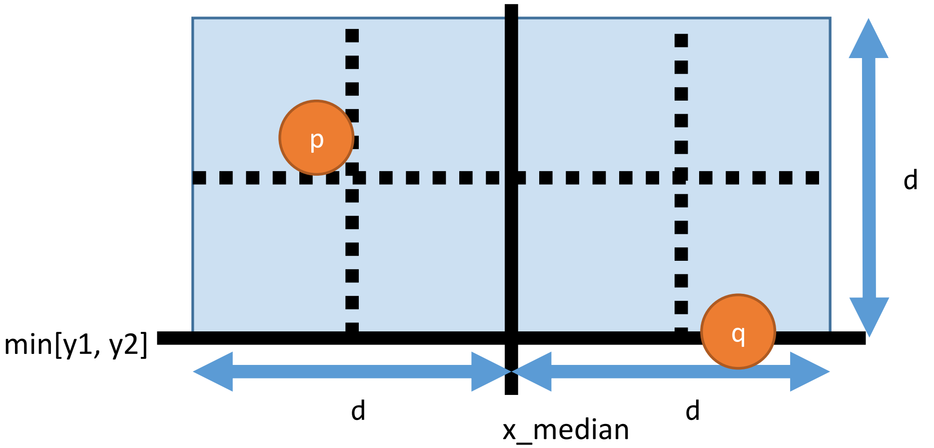
How many other points can possibly be in this area?



Proof—Part B

p and q are at most 7 positions apart

in `middle_py`

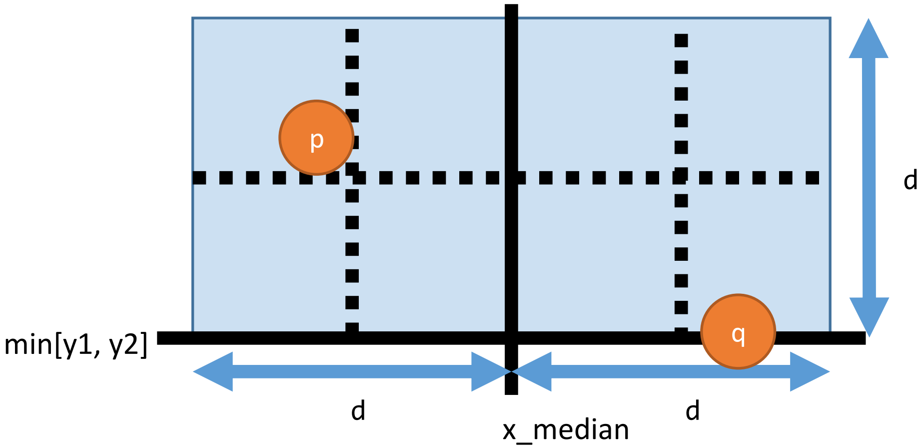


Lemma 1: All points of middle_py with a y-coordinate between those of p and q lie within those 8 boxes.

Proof:

1. First, recall that the y-coordinate of p, q differs by less than d.
2. Second, by definition of middle_py, all have an x-coordinate between x_median += d.

Proof—Part B



p and q are at most 7 positions apart in `middle_py`

Lemma 1: All points of middle_py with a y-coordinate between those of p and q lie within those 8 boxes.

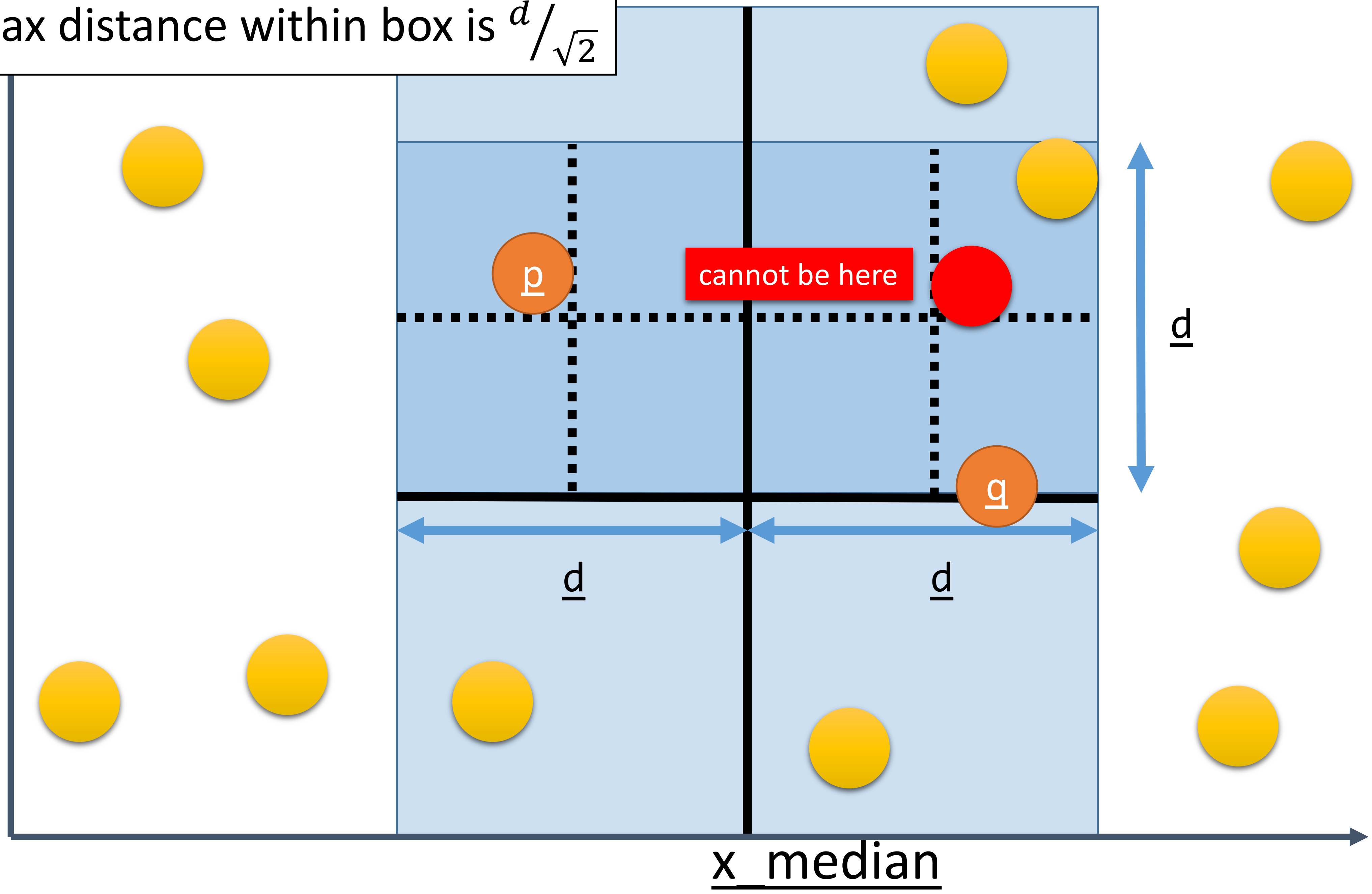
Lemma 2: At most one point of P can be in each box.

Proof: By contradiction. Suppose points a and b lie in the same box. Then

1. a and b are either both in L or both in R
2. d(a, b) <= `d/2 sqrt(2) < d`

This is a contradiction! How did we define d?

Max distance within box is $d/\sqrt{2}$



`ClosestPair` finds the closest pair

Let p `∈ left, q ∈ right` be a split pair with `d(p, q) < d`

Then

A. p and q ∈ `middle_py`, and

B. p and q are at most **7** positions apart in `middle_py`

If the claim is true:

Corollary 1: If the closest pair of P is in a split pair, then our `ClosestSplitPair` procedure finds it.

Corollary 2: `ClosestPair` is correct and runs in O(n lg n) since it has the same recursion tree as merge sort

Closest Pair

1. Copy P and <u>sort</u> one copy by x and the other copy by y in O(n lg n)
2. Divide P into a left and right in O(n)
3. Conquer by recursively searching left and right
4. Look for the closest pair in middle_py in O(n)
   - Must filter by x
   - And scan through middle_py by looking at adjacent points

$$T(n) = 2\,T(n/2) + O(n) = O(n \lg n)$$

```
T(n)    FUNCTION ClosestPair(px, py)
O(1)      n = px.length
O(1)      IF n == 2
O(1)          RETURN px[0], px[1], dist(px[0], px[1])

O(n)      left_px = px[0 ..< n//2]
O(n)      left_py = [p FOR p IN py IF p.x < px[n//2].x]
T(n/2)    pl, ql, dl = ClosestPair(left_px, left_py)

O(n)      right_px = px[n//2 ..< n]
O(n)      right_py = [p FOR p IN py IF p.x ≥ px[n//2].x]
T(n/2)    pr, qr, dr = ClosestPair(right_px, right_py)

O(1)      d = min(dl, dr)
O(n)      ps, qs, ds = ClosestSplitPair(px, py, d)

O(1)      RETURN Closest(pl, ql, dl, pr, qr, dr, ps, qs, ds)
```

T(n)

```
FUNCTION MergeSort(array)
    n = array.length
    IF n == 1
        RETURN array

    left_sorted = MergeSort(array[0 ..< n//2])
    right_sorted = MergeSort(array[n//2 ..< n])

    array_sorted = Merge(left_sorted, right_sorted)

    RETURN array_sorted
```

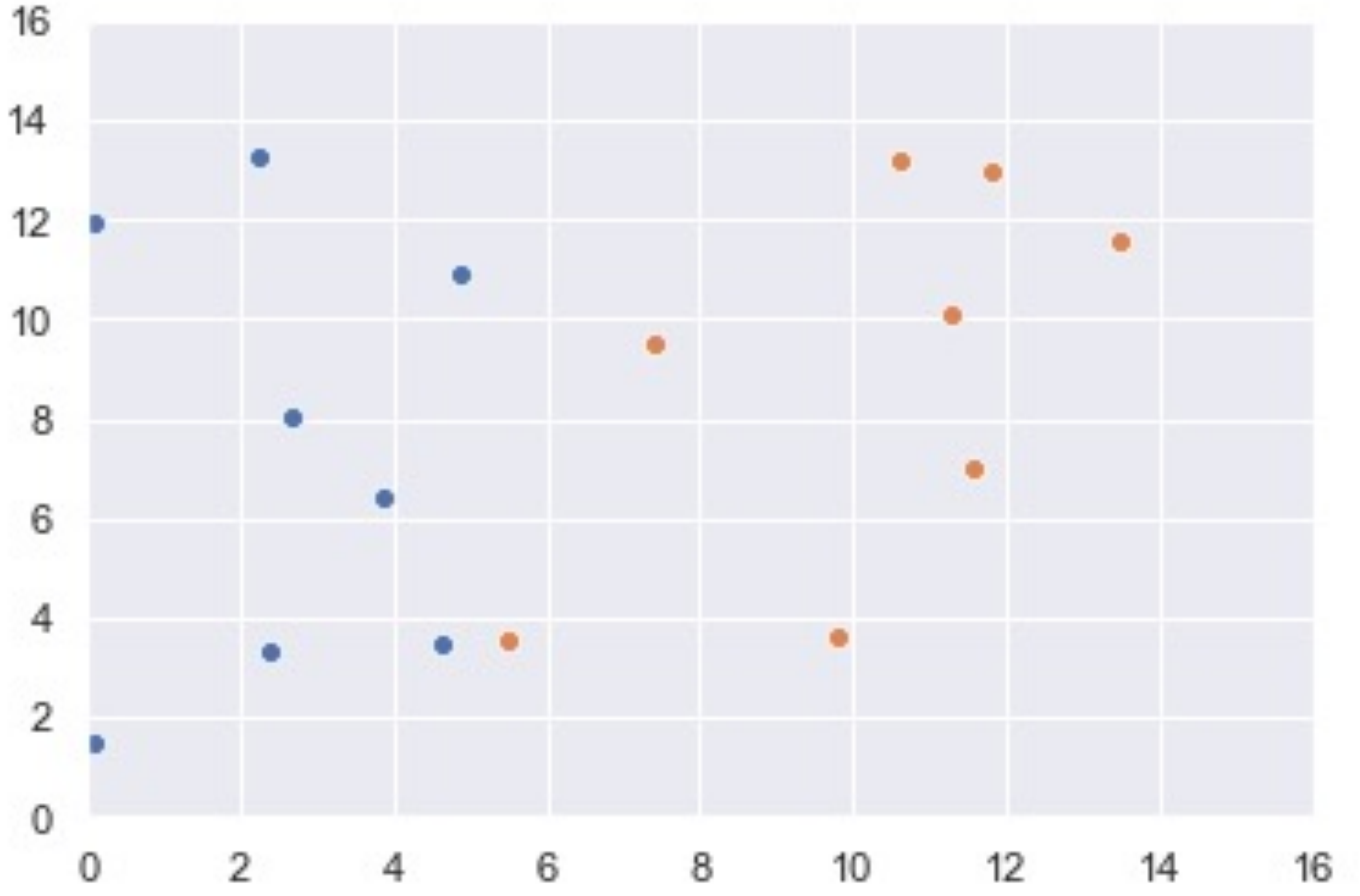
O(1) O(1) O(1) T(n/2) T(n/2) O(n) O(1)

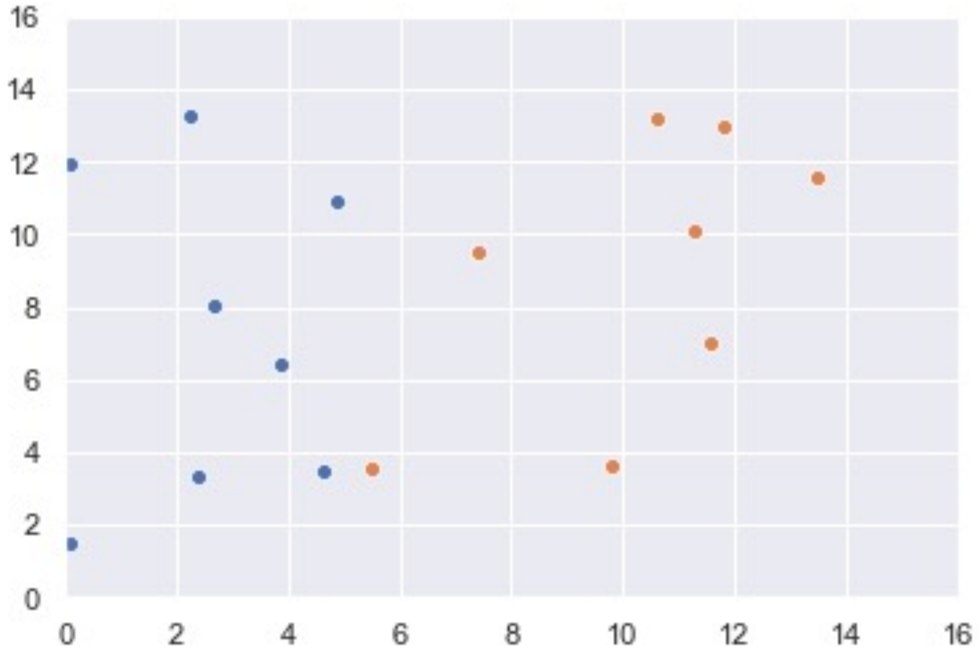
$$T(n) = 2\,T(n/2) + O(n) = O(n \lg n)$$

$$
\begin{aligned}
T(n) &= 2\,T(n/2) + O(n) \\
&= O(n \lg n)
\end{aligned}
$$

T(n)	`FUNCTION RecursiveFunction(some_input)`
O(1)	`    IF base_case:`
	`        # Usually O(1)`
O(1)	`        RETURN base_case_work(some_input)`
	`    # Two recursive calls, each with half the data`
T(n/2)	`    one = RecursiveFunction(some_input.first_half)`
T(n/2)	`    two = RecursiveFunction(some_input.second_half)`
	`    # Combine results from recursive calls (usually O(n))`
O(n)	`    one_and_two = Combine(one, two)`
O(1)	`    RETURN one_and_two`

Supplementary slides showing an example execution.

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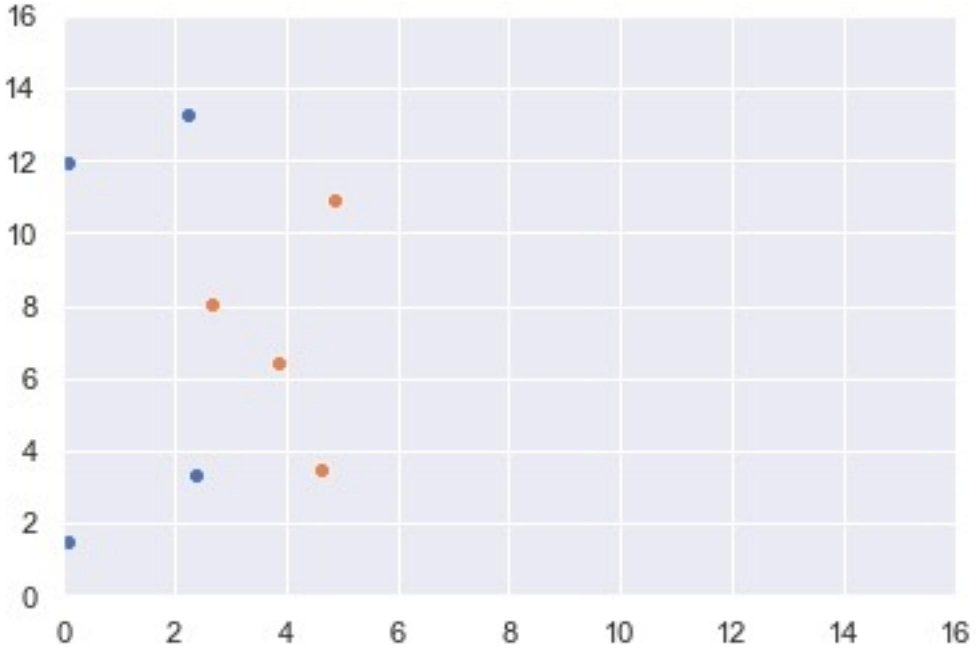
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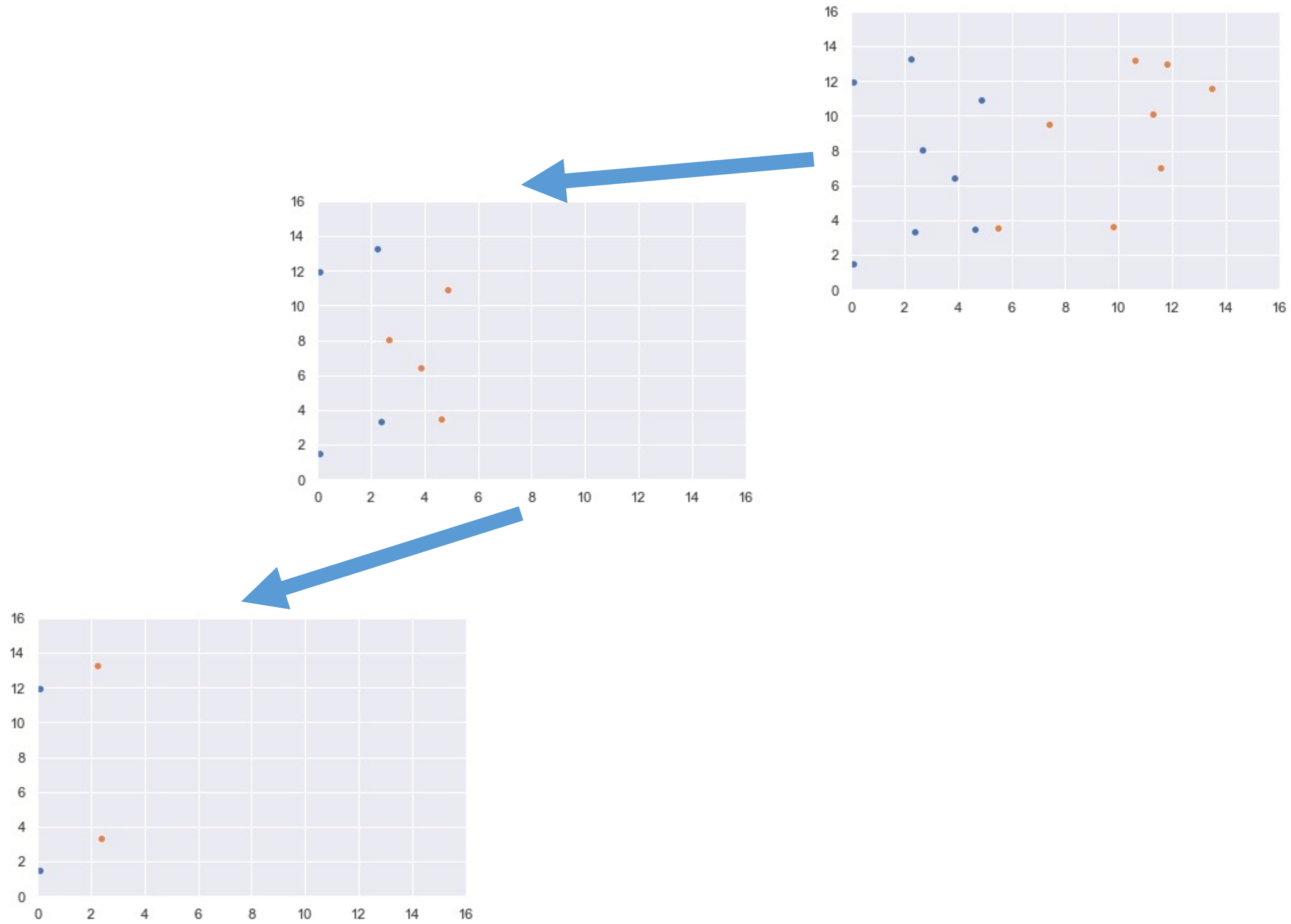
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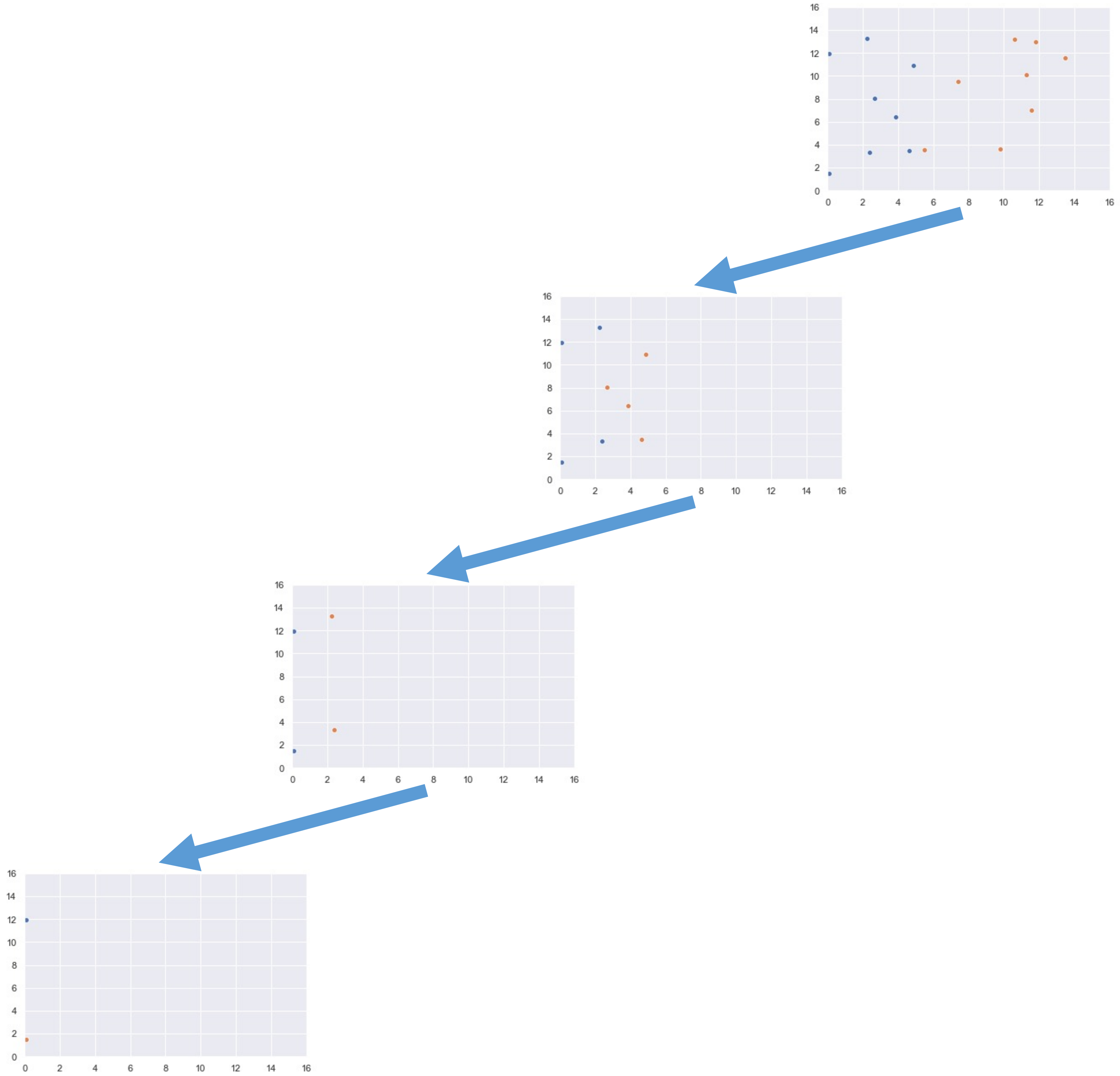
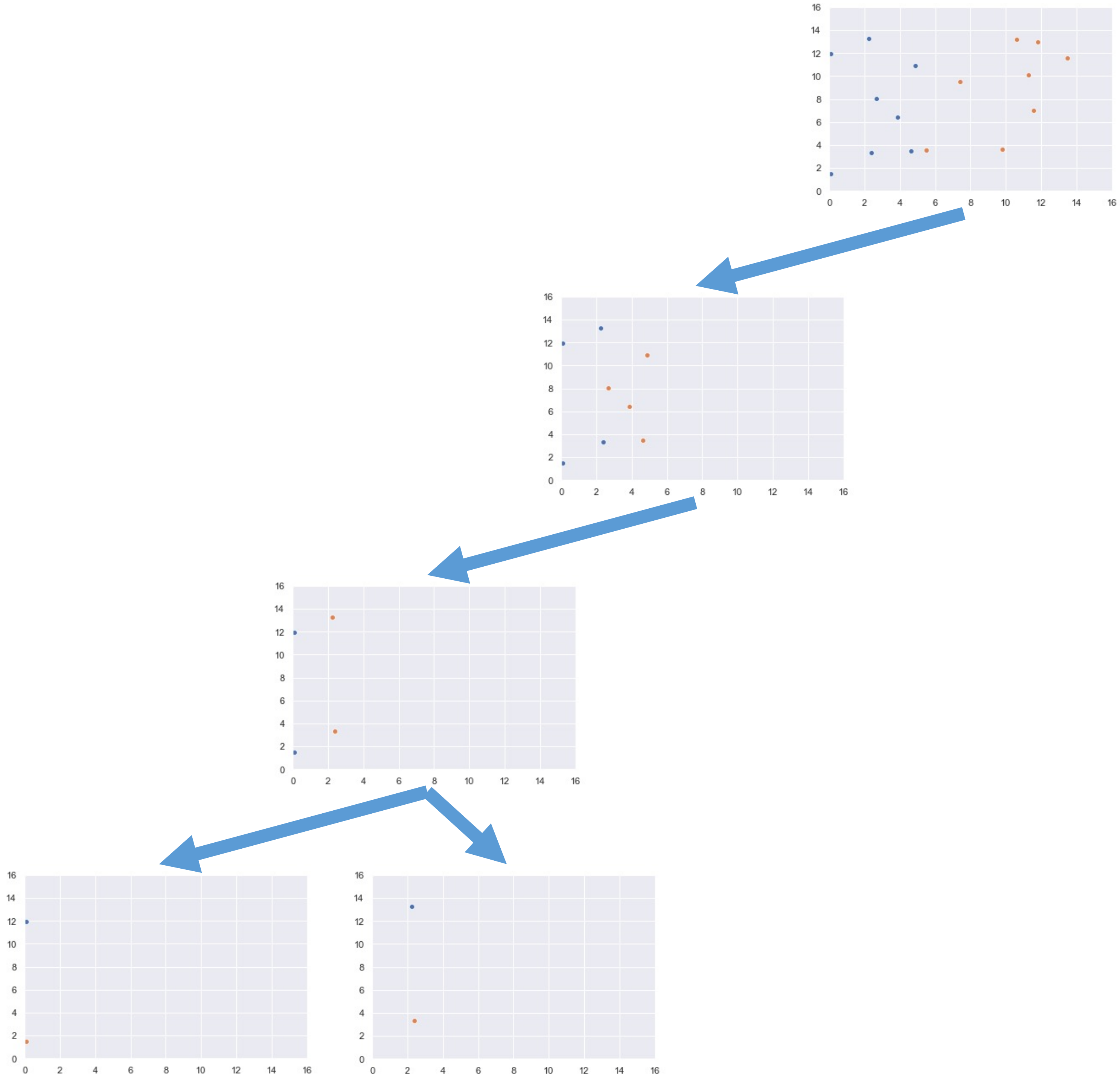
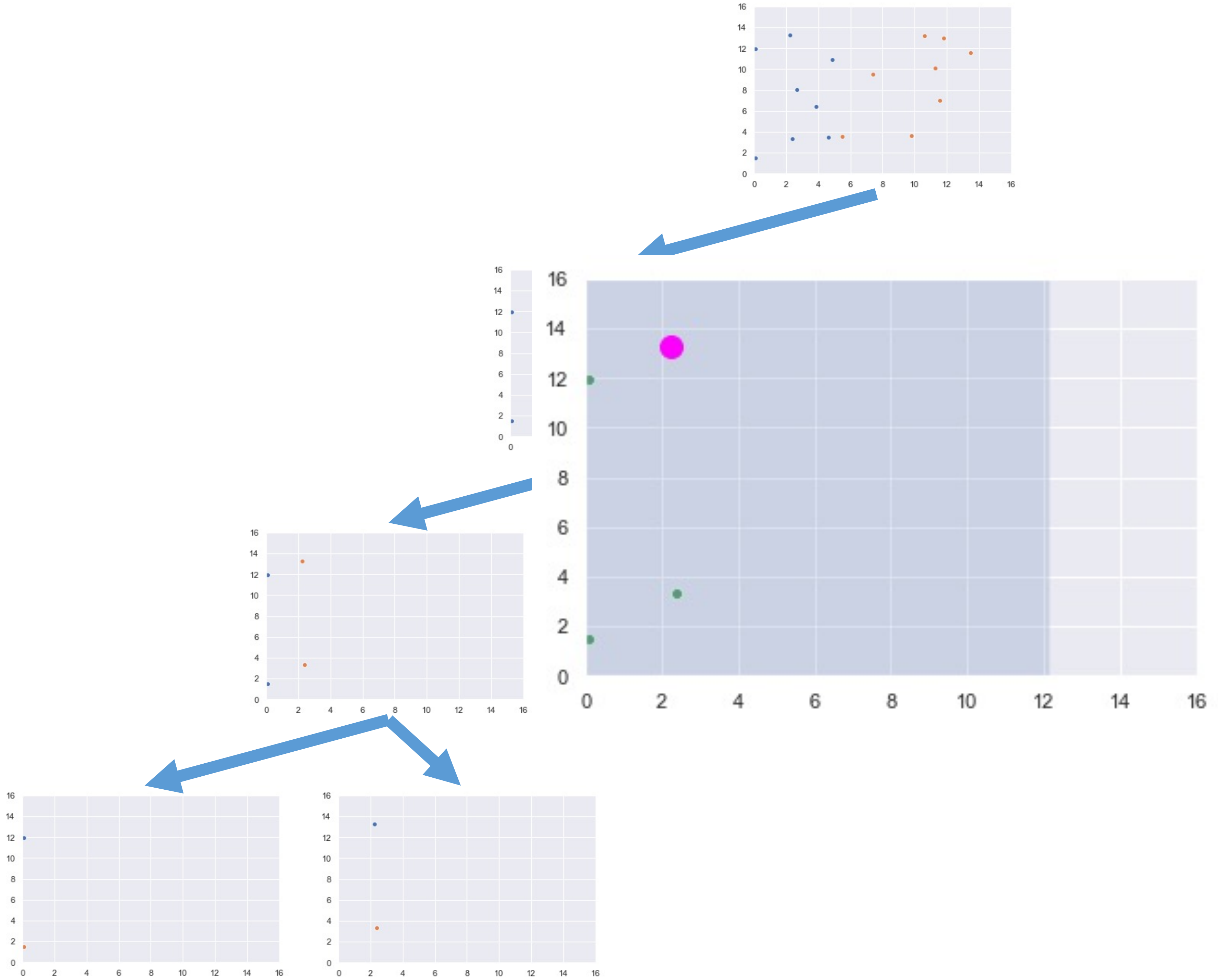
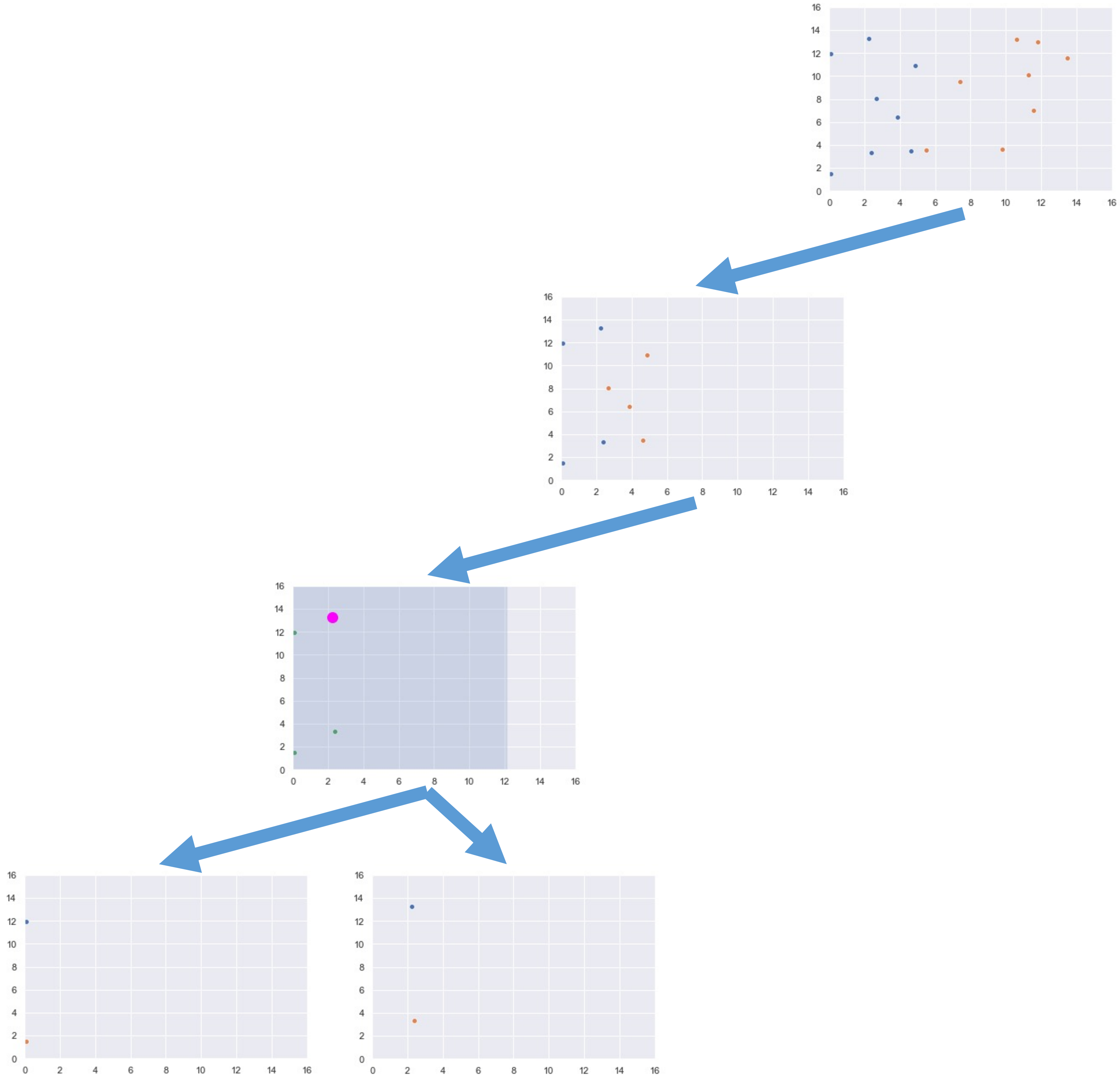






Closest Split Pair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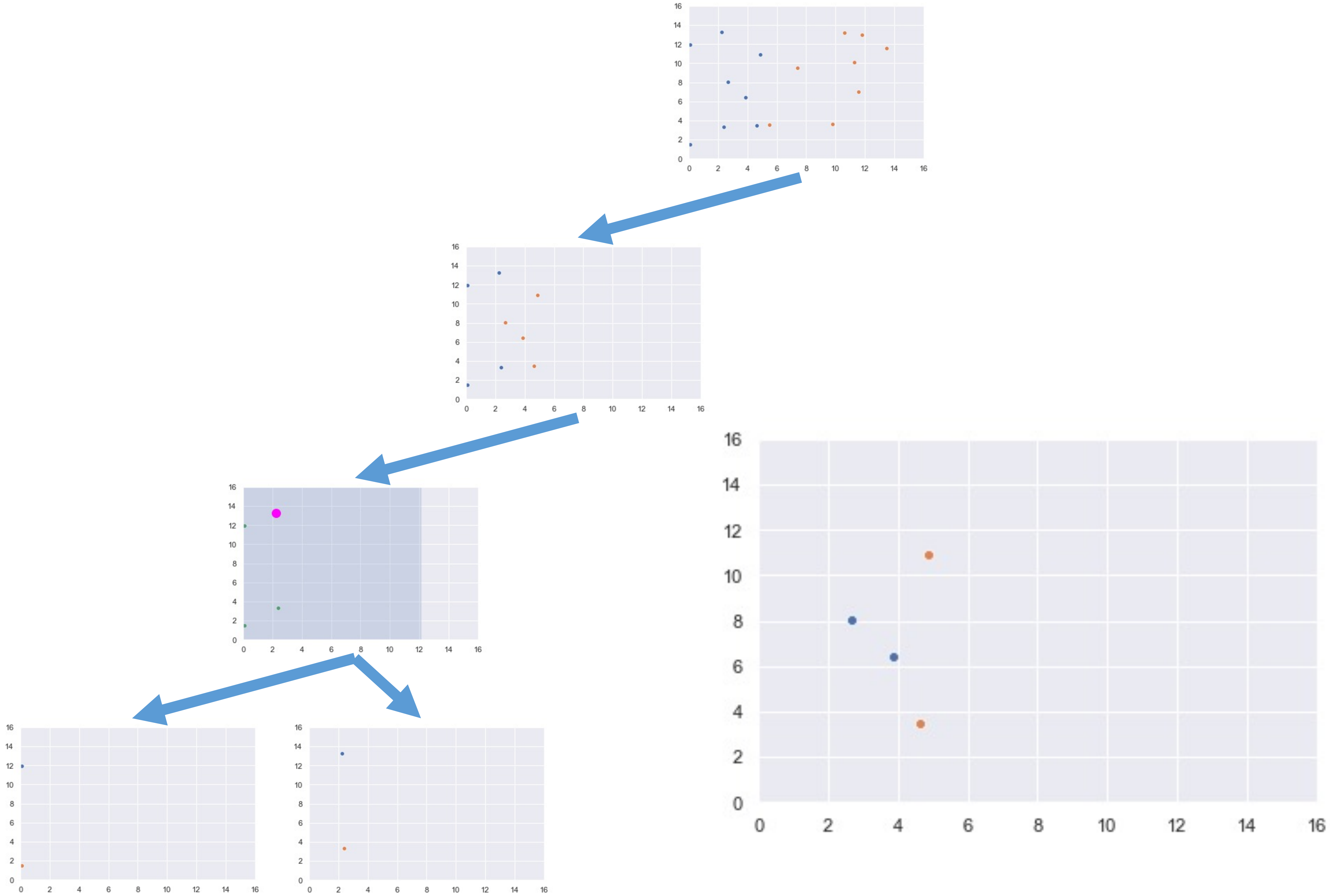
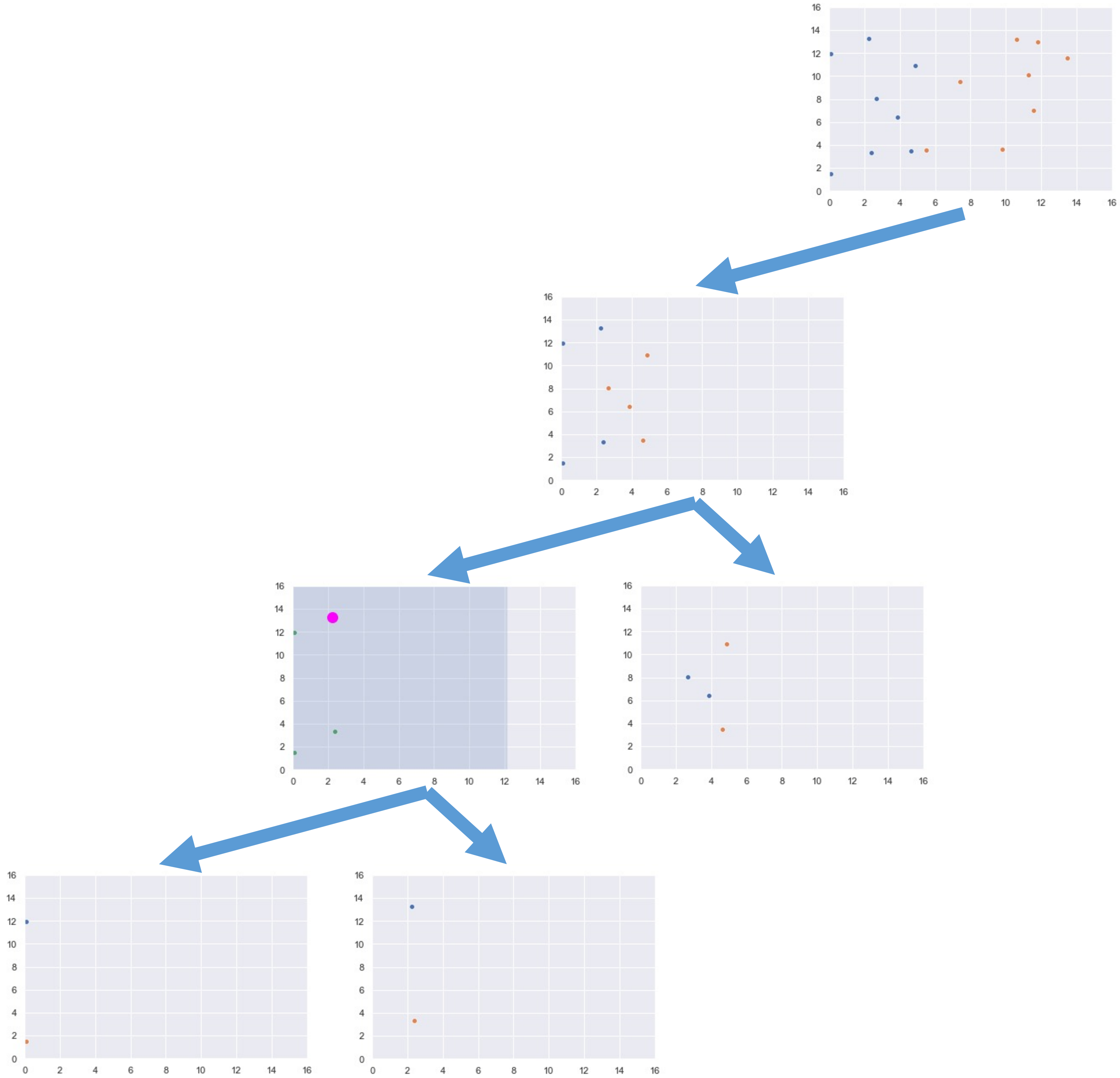
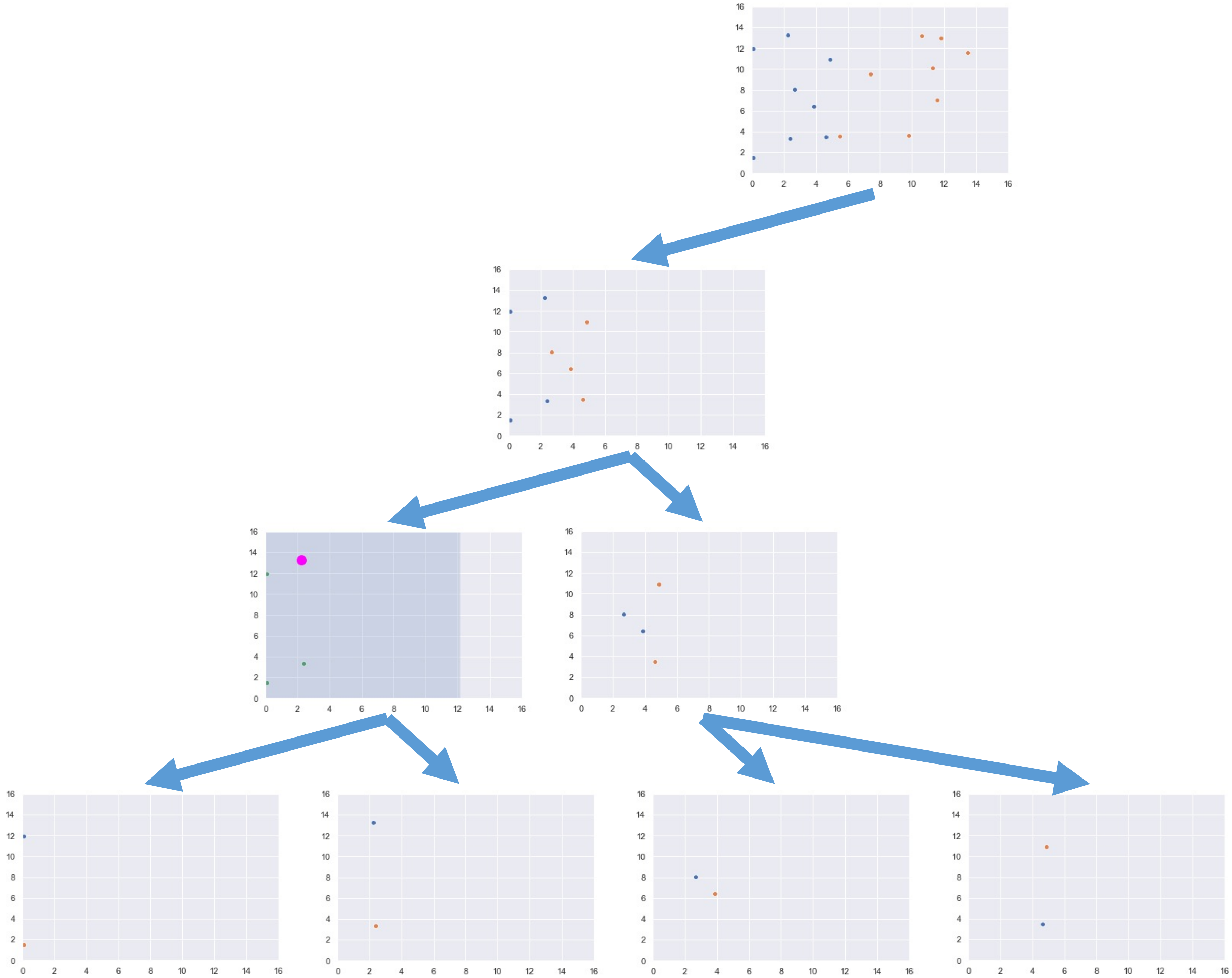
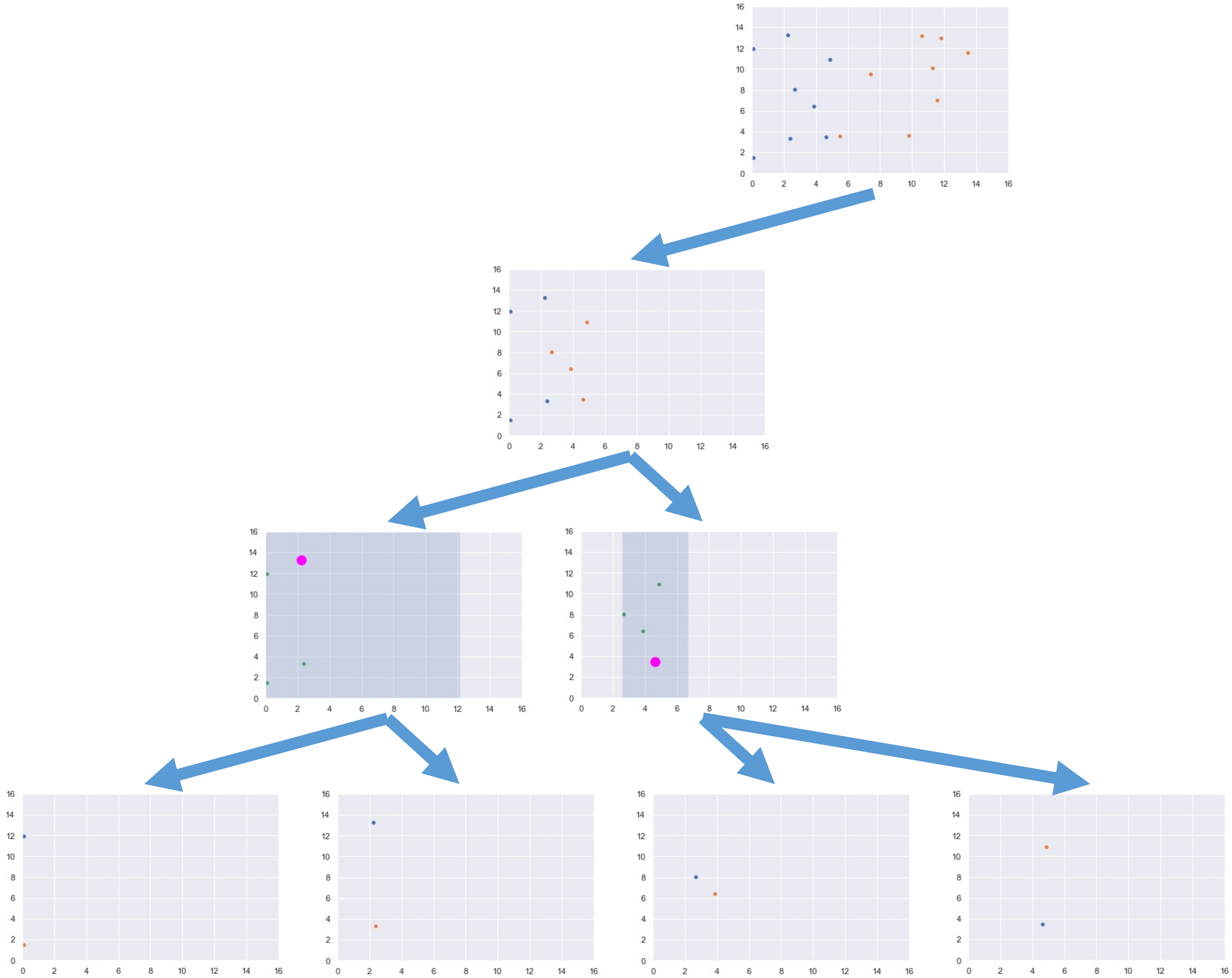
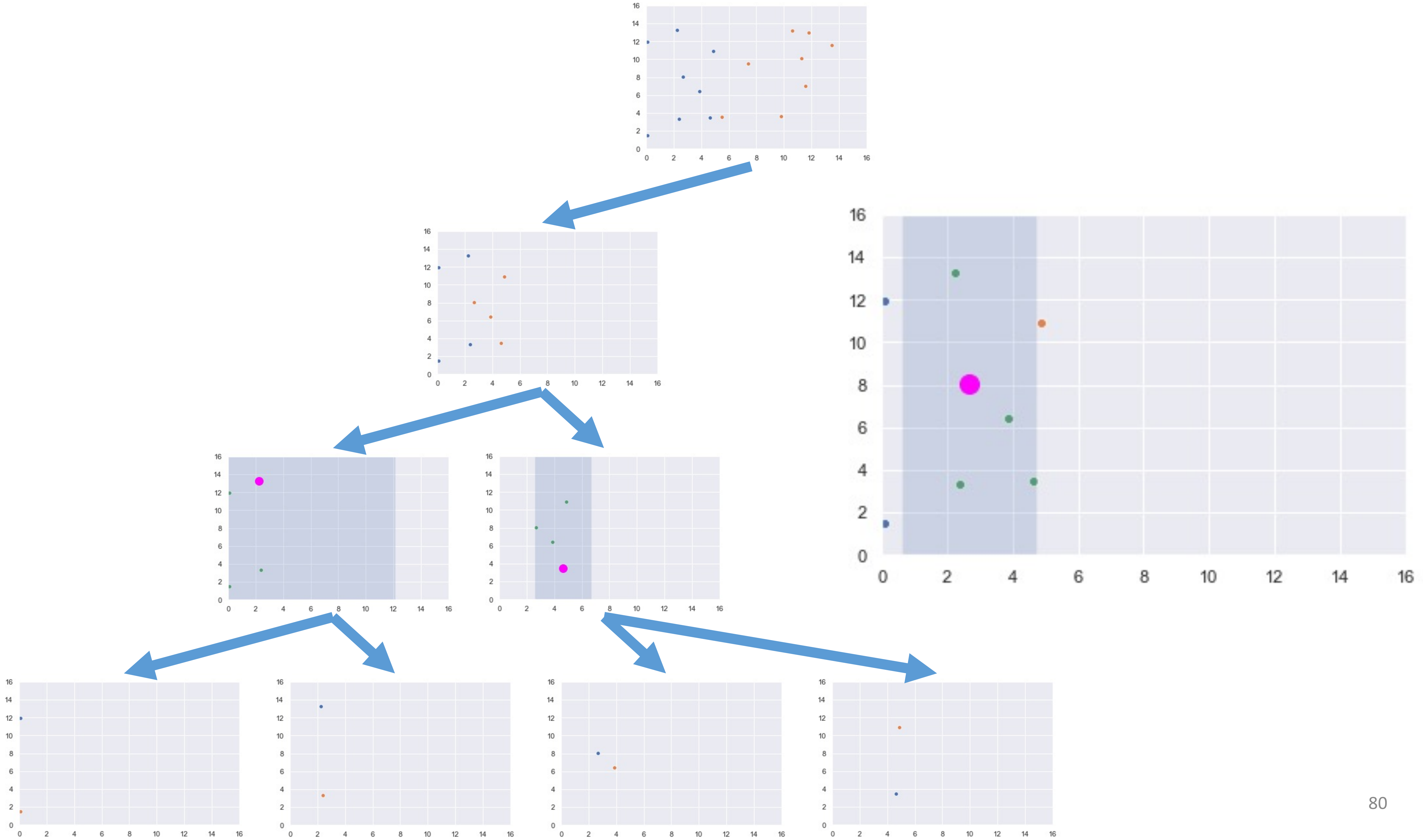
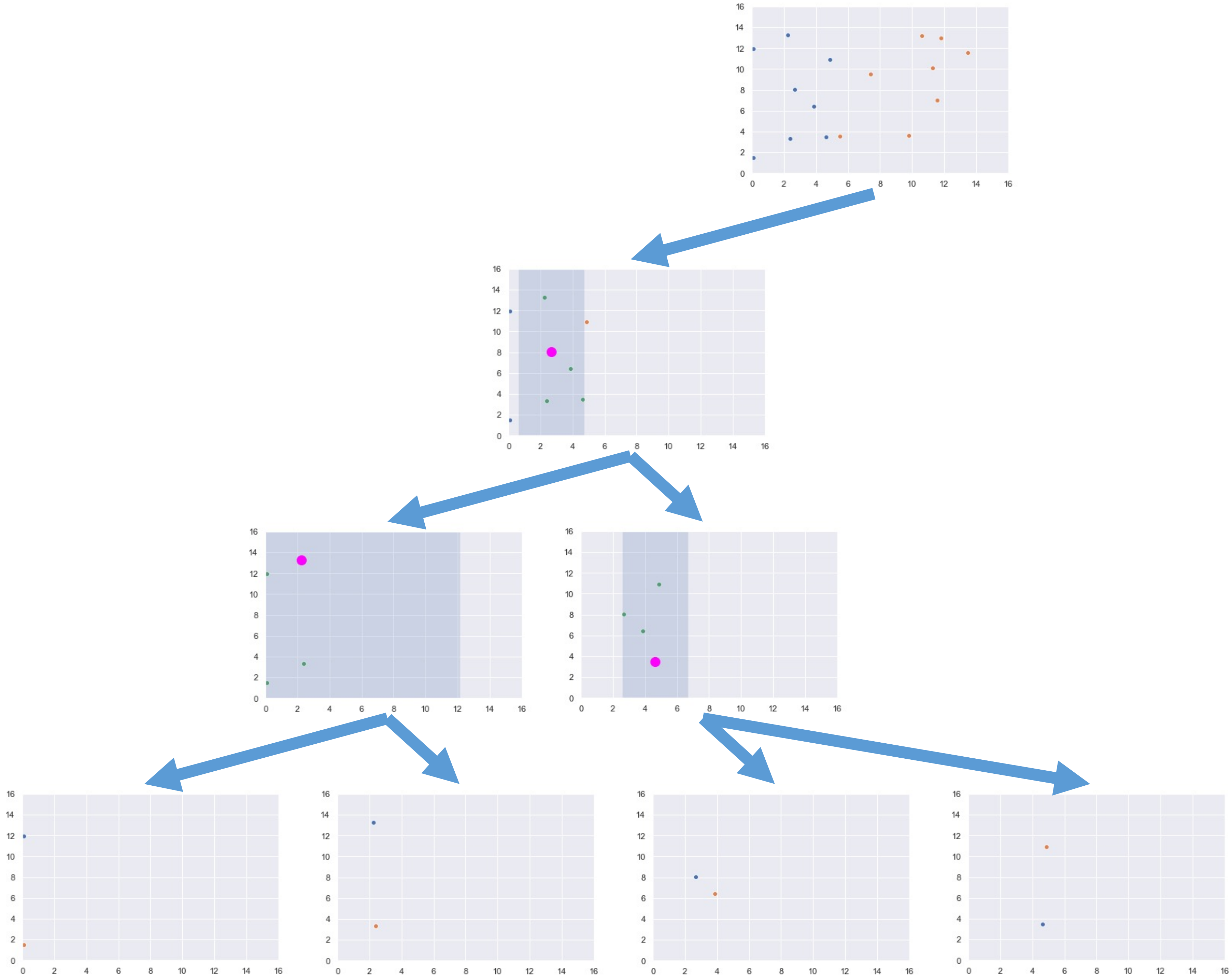


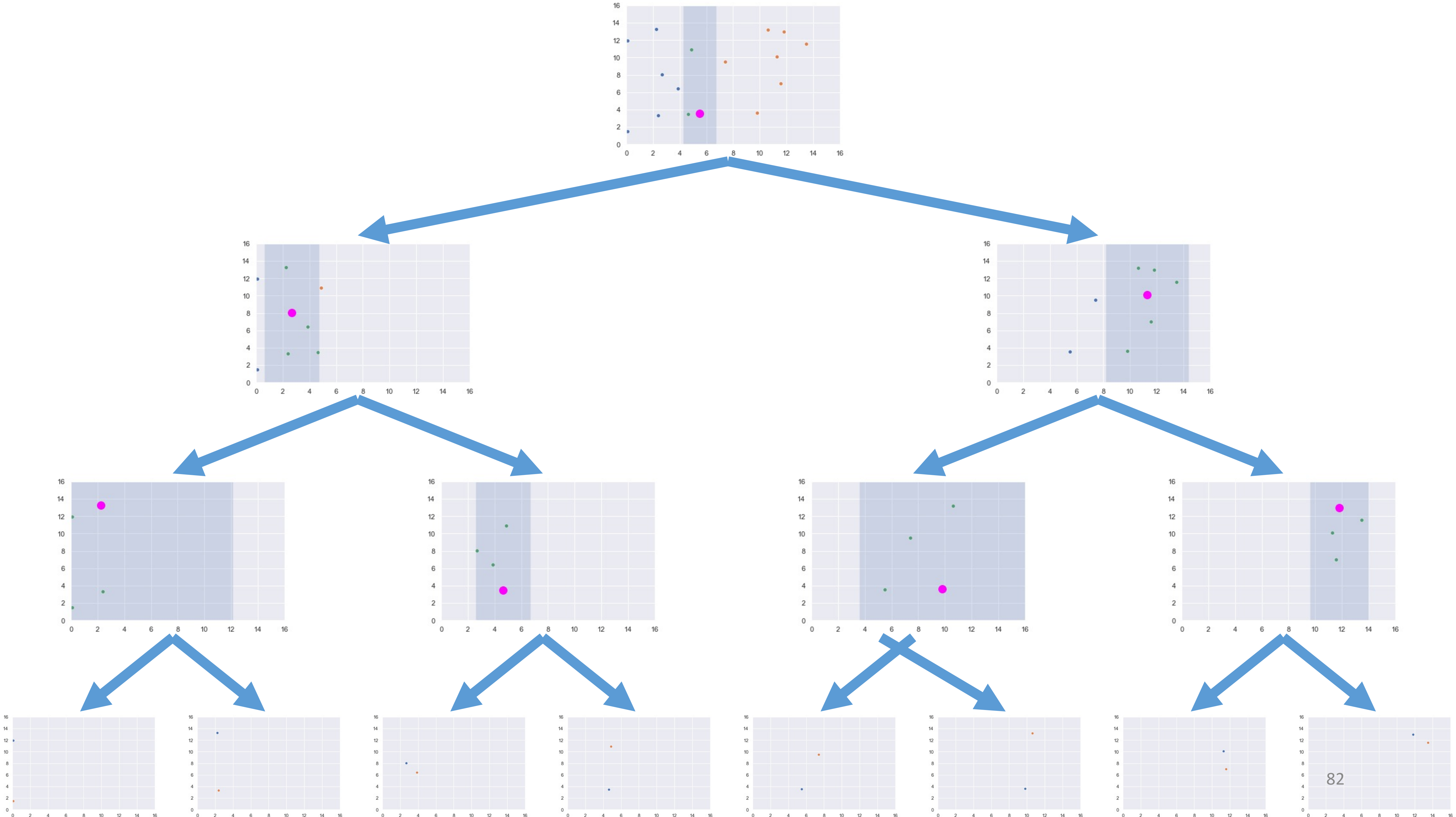




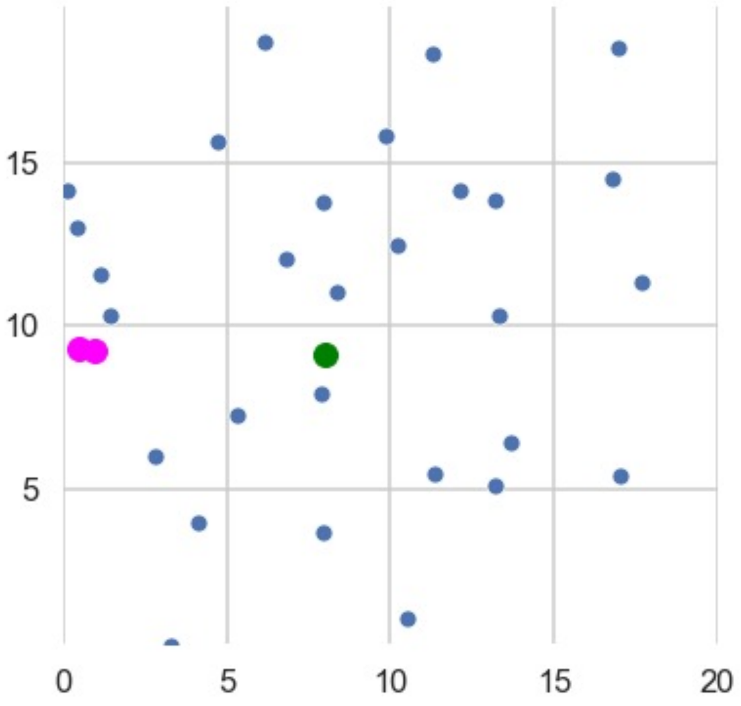




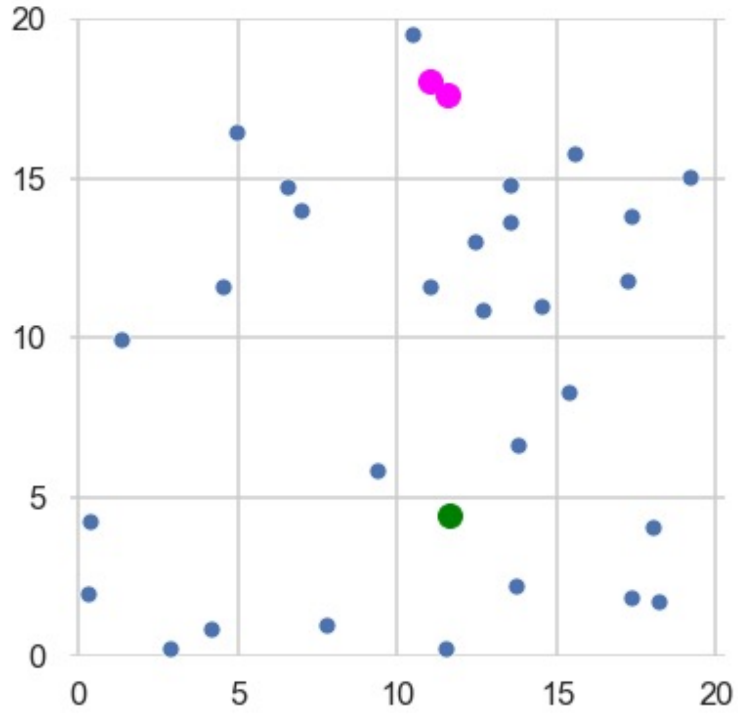




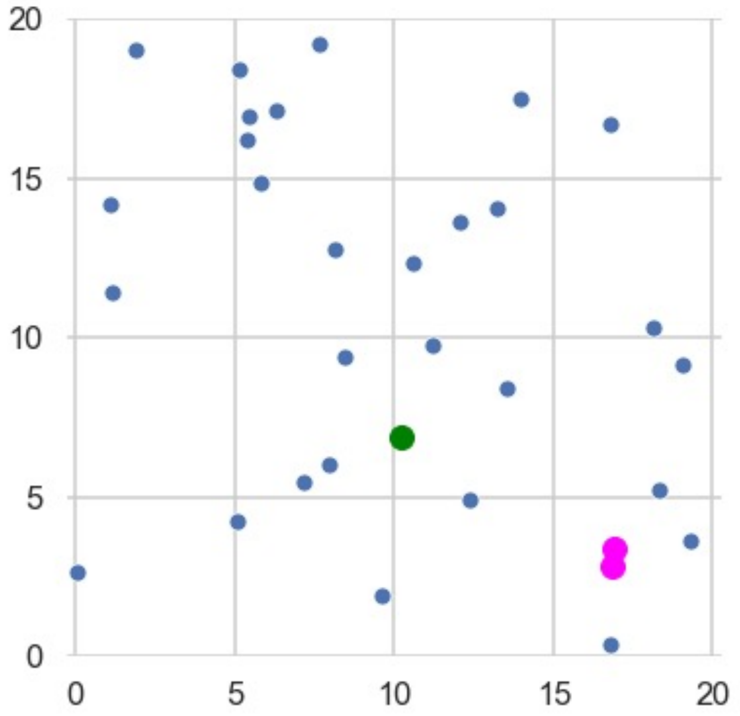
Closest on Left



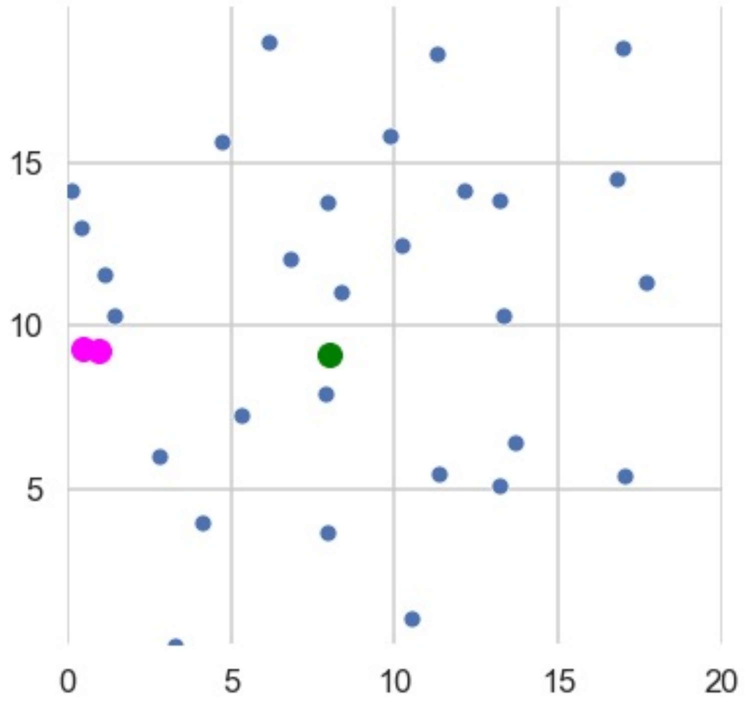
Closest is Split



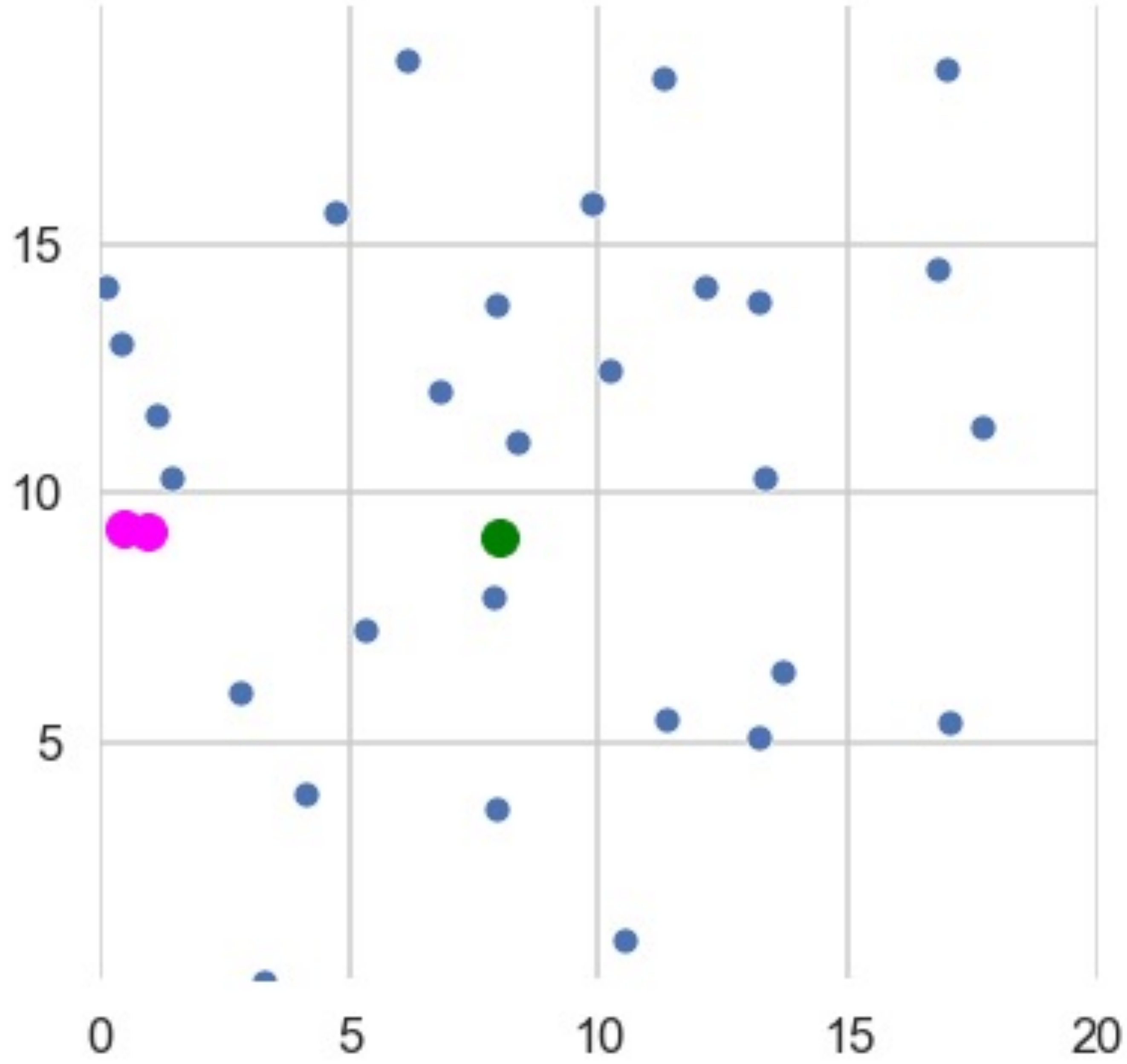
Closest on Right



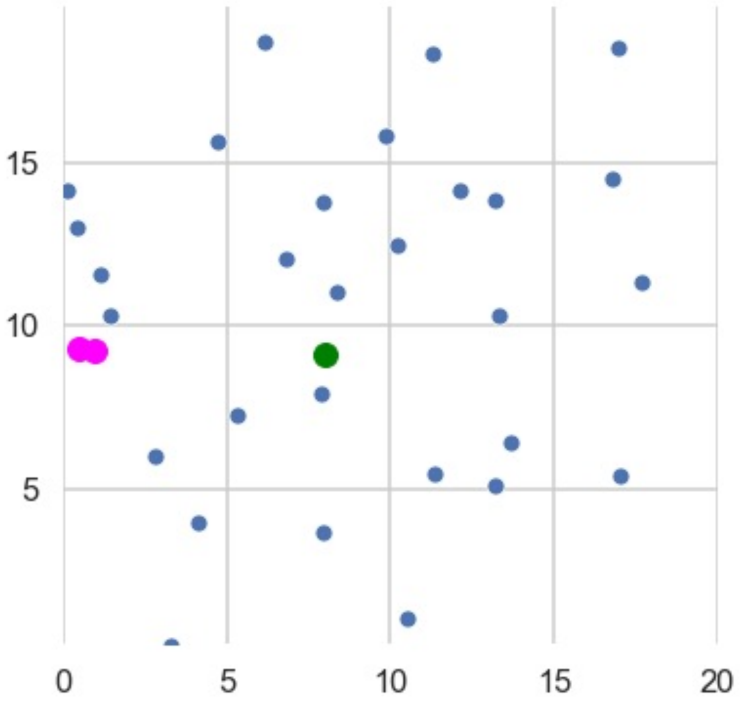
Closest on Left



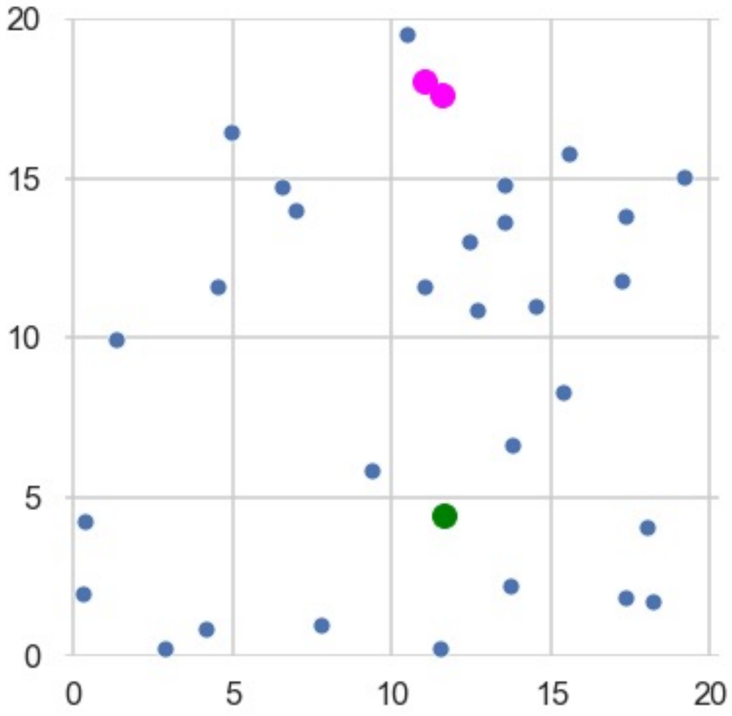
Closest on Left



Closest on Left



Closest is Split



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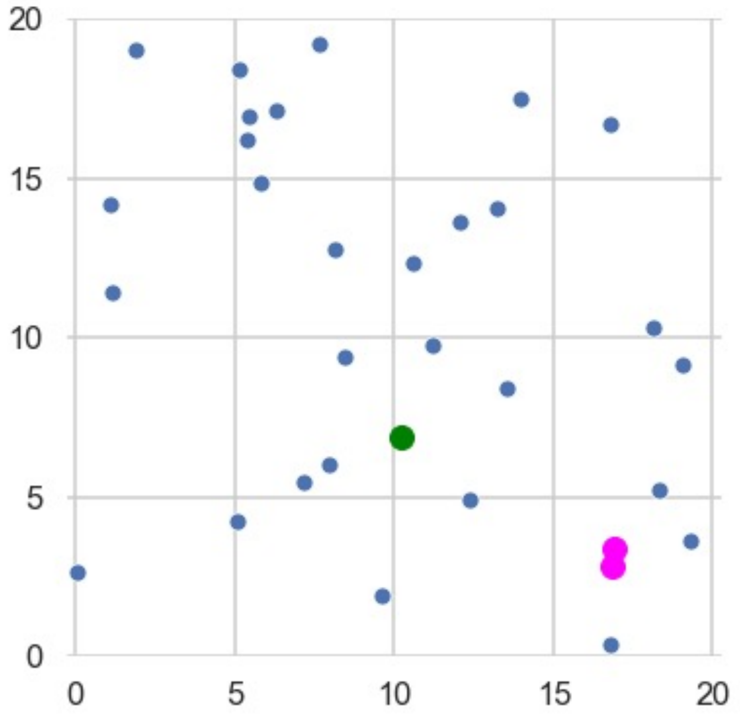
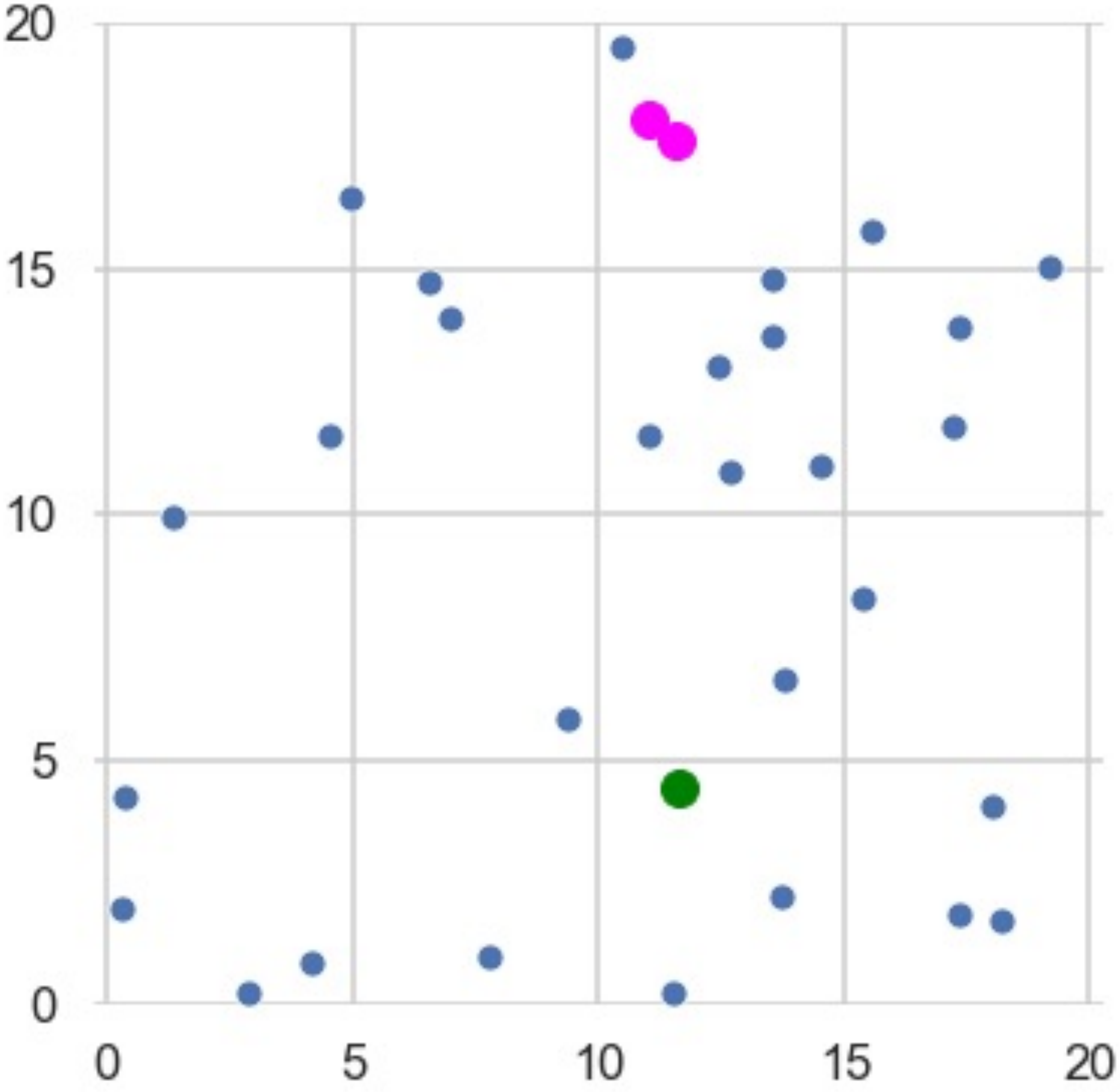


Closest is Split



Closest is Split