42: Minimum Spanning Trees
Lecture 42: Minimum Spanning Trees

- Introduction
- Kruskal’s Algorithm
- Prim’s Algorithm
Spanning Trees

- Given an edge weighted graph $G$ (not digraph!), a **spanning tree** of $G$ is a subgraph $T$ that is:
  - A tree: connected and acyclic.
  - Spanning: includes all of the vertices of $G$. 

[Graph and spanning trees image]
Properties

- A connected graph $G$ can have more than one spanning tree.
- All possible spanning trees of $G$, have the same number of vertices and edges.
- A spanning tree has $|V| - 1$ edges.
- A spanning tree by definition cannot have any cycle.
- Adding one edge to the spanning tree would create a cycle (i.e. spanning trees are maximally acyclic).
- Removing one edge from the spanning tree would make the graph disconnected (i.e. spanning trees are minimally connected).
Minimum spanning tree problem

- Given a connected edge-weighted undirected graph find a spanning tree of minimum weight.

An edge-weighted graph and its MST
Minimum spanning applications

- Network design
- Cluster analysis
- Cancer imaging
- Cosmology
- Weather data interpretation
- Many others

- https://personal.utdallas.edu/~besp/teaching/mst-applications.pdf
Lecture 42: Minimum Spanning Trees

- Introduction
- Kruskal’s Algorithm
- Prim’s Algorithm
Kruskal’s algorithm

- Sort edges in ascending order of weight.
- Starting from the one with the smallest weight, add it to the MST $T$ unless doing so would create a cycle.
- Uses a data structure called Union-Find (Chapter 1.5 in book).
- Running time of $|E| \log |V|$ in worst case.
Kruskal's Algorithm Demo
Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree $T$ unless doing so would create a cycle.

an edge-weighted graph

graph edges sorted by weight

0-7 0.16
2-3 0.17
1-7 0.19
0-2 0.26
5-7 0.28
1-3 0.29
1-5 0.32
2-7 0.34
4-5 0.35
1-2 0.36
4-7 0.37
0-4 0.38
6-2 0.40
3-6 0.52
6-0 0.58
6-4 0.93
Consider edges in ascending order of weight.

- Add next edge to tree $T$ unless doing so would create a cycle.

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Kruskal's algorithm demo

in MST 0-7 0.16

does not create a cycle
Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree $T$ unless doing so would create a cycle.
Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree $T$ unless doing so would create a cycle.

```plaintext
0-7  0.16
2-3  0.17
1-7  0.19
```

The edge 1-7 does not create a cycle in the MST.
Kruskal's algorithm demo

Consider edges in ascending order of weight.
- Add next edge to tree $T$ unless doing so would create a cycle.

<table>
<thead>
<tr>
<th>Edge</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-7</td>
<td>0.16</td>
</tr>
<tr>
<td>2-3</td>
<td>0.17</td>
</tr>
<tr>
<td>1-7</td>
<td>0.19</td>
</tr>
<tr>
<td>0-2</td>
<td>0.26</td>
</tr>
</tbody>
</table>

The edge 0-2 is added to the minimum spanning tree as it does not create a cycle.
Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree $T$ unless doing so would create a cycle.
Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree $T$ unless doing so would create a cycle.
Kruskal's algorithm demo

Consider edges in ascending order of weight.
- Add next edge to tree $T$ unless doing so would create a cycle.
Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree $T$ unless doing so would create a cycle.

0–7 0.16
2–3 0.17
1–7 0.19
0–2 0.26
5–7 0.28
1–3 0.29
1–5 0.32
2–7 0.34
Kruskal's algorithm demo

Consider edges in ascending order of weight.
  • Add next edge to tree $T$ unless doing so would create a cycle.
Consider edges in ascending order of weight.
- Add next edge to tree $T$ unless doing so would create a cycle.

Kruskal's algorithm demo

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</thead>
<tbody>
<tr>
<td>0–7</td>
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<td>0.19</td>
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</tr>
<tr>
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<td>0.29</td>
</tr>
<tr>
<td>1–5</td>
<td>0.32</td>
</tr>
<tr>
<td>2–7</td>
<td>0.34</td>
</tr>
<tr>
<td>4–5</td>
<td>0.35</td>
</tr>
<tr>
<td>1–2</td>
<td>0.36</td>
</tr>
</tbody>
</table>
Consider edges in ascending order of weight.

- Add next edge to tree $T$ unless doing so would create a cycle.
Consider edges in ascending order of weight.

- Add next edge to tree $T$ unless doing so would create a cycle.

```
0-7  0.16
2-3  0.17
1-7  0.19
0-2  0.26
5-7  0.28
1-3  0.29
1-5  0.32
2-7  0.34
4-5  0.35
1-2  0.36
4-7  0.37
0-4  0.38
```
Consider edges in ascending order of weight.

- Add next edge to tree $T$ unless doing so would create a cycle.
Consider edges in ascending order of weight.

- Add next edge to tree $T$ unless doing so would create a cycle.
Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree $T$ unless doing so would create a cycle.
Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree $T$ unless doing so would create a cycle.
Consider edges in ascending order of weight.

- Add next edge to tree $T$ unless doing so would create a cycle.

A minimum spanning tree
Practice Time
Answer
Lecture 42: Minimum Spanning Trees

- Introduction
- Kruskal’s Algorithm
- Prim’s Algorithm
Prim’s algorithm

- Start with a random vertex (here, 0) and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $|V| - 1$ edges.

- Two versions, lazy and eager. We will see lazy, here...
- Uses min-priority queue.
- Running time of $|E| \log |V|$ in worst case, as well.
Prim's algorithm demo

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V - 1$ edges.

An edge-weighted graph
Prim's algorithm demo

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V - 1$ edges.
Prime's algorithm demo

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V-1$ edges.
Prim's algorithm demo

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V - 1$ edges.

**MST edges**

0–7
Prim's algorithm demo

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V - 1$ edges.

MST edges

0-7

Edges in MST:
- 1-7 0.19
- 0-2 0.26
- 5-7 0.28
- 2-7 0.34
- 4-7 0.37
- 0-4 0.38
- 6-0 0.58

Edges with exactly one endpoint in T (sorted by weight):
- Min weight edge with exactly one endpoint in T
Prim's algorithm demo

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V - 1$ edges.

MST edges

0–7  1–7
Prim's algorithm demo

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V - 1$ edges.

MST edges

0–7  1–7
Prim's algorithm demo

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V - 1$ edges.

MST edges

0–7  1–7  0–2
Prim's algorithm demo

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V - 1$ edges.

MST edges:
0–7  1–7  0–2

Edges with exactly one endpoint in $T$ (sorted by weight):

<table>
<thead>
<tr>
<th>Edge</th>
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</tr>
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<tbody>
<tr>
<td>2–3</td>
<td>0.17</td>
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<tr>
<td>6–0</td>
<td>0.58</td>
</tr>
</tbody>
</table>

Min weight edge with exactly one endpoint in $T$: 0–7
**Prim's algorithm demo**

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V - 1$ edges.

MST edges

0–7  1–7  0–2  2–3
Prim's algorithm demo

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V - 1$ edges.

**MST edges**

0–7  1–7  0–2  2–3
Prim's algorithm demo

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V - 1$ edges.

MST edges

0–7 1–7 0–2 2–3 5–7
Prim's algorithm demo

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V - 1$ edges.

MST edges

$0-7$  $1-7$  $0-2$  $2-3$  $5-7$
Prim's algorithm demo

• Start with vertex 0 and greedily grow tree $T$.
• Add to $T$ the min weight edge with exactly one endpoint in $T$.
• Repeat until $V - 1$ edges.

MST edges

0-7   1-7   0-2   2-3   5-7   4-5
Prim's algorithm demo

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V - 1$ edges.

**MST edges**

- 0–7
- 1–7
- 0–2
- 2–3
- 5–7
- 4–5

**Edges with exactly one endpoint in T (sorted by weight)**

- 6–2 0.40
- 3–6 0.52
- 6–0 0.58
- 6–4 0.93
Prim's algorithm demo

- Start with vertex 0 and greedily grow tree $T$.
- Add to $T$ the min weight edge with exactly one endpoint in $T$.
- Repeat until $V - 1$ edges.

MST edges

0–7  1–7  0–2  2–3  5–7  4–5  6–2
Practice Time
Answer...
Readings:

- Textbook: Chapter 4.3 (Pages 604-629)
- Website: 
  - https://algs4.cs.princeton.edu/43mst/

Practice Problems:

https://visualgo.net/en/mst