Countability and Uncountability

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Outline

Different format today

How to Count (Part 3)

Counting Infinities

The Uncountable

Beamer

- I really dislike slideshow programs
  - (Yes, even your favorite one)
- So we're going to party like it's 1999

The Story So Far

- We counted with our fingers when our sets were small
  - The most natural of natural numbers
- Then we used multiplication and addition to count really big sets
  - How many license plates, etc
- Today, we're counting with functions

Definitions

- Injectivity: $\forall xy, f(x) = f(y) \Rightarrow x = y$
  - "Every input has a distinct output"
  - "One to one"
- Surjectivity: $\forall y, \exists x, f(x) = y$
  - "Every output is reached by some input"
  - "Onto"

Counting with Functions

- If f is an injection from A to B, |A| <= |B|
  - We can pick a different output for each input...
  - so there are at least as many outputs as inputs
- If f is a surjection from A to B, |A| >= |B|
  - We can hit every output with some input...
  - so we have at least as many inputs as outputs
- If f is a bijection from A to B, |A| = |B|
  - Greater-or-equal and less-or-equal is just equal

Practice

- Which set is bigger?
  - Prove it by finding a function (either from A->B or B->A)...
  - and proving it is injective/surjective.
- $|\{T,F\}|$? $|\{1,2,3\}|$
- $|\text{Bool} \times \text{Bool}|$? $|\{1,2,3\}|$
- |Bitstrings of length 8| ? |Alphanumeric strings of length 1|
- |Bool -> Bool| ? $|\text{Bool} \times \{1,2,3\}|$
  - We can even count functions with functions

Comparing Arbitrary Sets

We can also handle sets whose contents we don't even know!
Imagine we have sets A, B, and C. We know:

1. $|A| <= |B|$
2. $|A| <= |C|$

Define a function f to show that $|B \cup C| >= |A|$.
Hint: Use (1) and (2) to find functions g and h to use in your definition of f!
Hint: You can also check whether the input argument is a member of B or C in a piecewise function definition.

Counting Sets

- We can compare cardinalities of arbitrary sets using functions
- Some sets are infinite
- But. . . sets are sets, right?

Principle

- Which set is bigger: the positive integers (1 and up) or the non-negative integers (0 and up)?
  - Well, Z^+ is a strict subset of Z_0^+. Case closed?
- Observe: f(pos) = pos - 1
  - $f : Z^+ \rightarrow Z_0^+$
  - f is a bijection
    - f(x) = f(y) => x = y, since x-1=y-1 => x=y.
    - forall y, exists x f(x) = y; if x=y+1, f(y+1) = y+1-1 = y.
- So they're... the same cardinality!?
  - $Z^+ <= Z_0^+$, and:
  - $Z^+ >= Z_0^+$

"Countably Infinite"

- Any set S where $|S| = |N|$ is "countably infinite"
- All countably infinite sets therefore have the same cardinality!
- Let's play with this a bit...

Practice

- Claim: the left set has the same cardinality as the one on the right.
  - Prove it by finding an injection A->B and a surjection A->B
  - or a single bijection A->B
  - or an injection A->B and an injection B->A (so A <= B and B <= A)
  - or a surjection A->B and a surjection B->A (so A >= B and B >= A)
  - or a bijection B->A
- $|\text{Natural numbers}| = |\text{even numbers}|$
- $|\text{negative integers}| = |\text{positive integers}|$

Remember

f : A -> B might give us an inequality between A and B:

- If it's injective, we get |A|<=|B|
- If it's surjective, we get |A|>=|B|

This works for g : B -> A too!

- If it's injective, we get |B|<=|A|
- If it's surjective, we get |B|>=|A|

Pick the functions with the properties that give you the inequalities you want!
Also, since we're looking at inequalities or equalities, we can use all the stuff we know already about reflexivity, transitivity, etc.

More Practice

- Claim: the left set has the same cardinality as the one on the right.
  - Prove it by finding an injection and a surjection (or a single bijection) to some third set (maybe the nats!), and use a transitivity argument
- $|\text{Perfect squares}| = |\text{powers of two}|$
- $|\text{Pairs of numbers}| = |\text{number of possible bitstrings}|$
- |Bool -> Bool -> Nat| ? |natural numbers|
  - Hint: How many possible pairs of inputs can this function take? What is this question really asking?

What isn't Countable?

- Sets are countable if their cardinality is $<=$ that of N
- Is any set bigger than the natural numbers?

The Real Numbers

- Review:
  - Natural numbers (N): "counting numbers"
  - Integers (Z): positive and negative natural numbers
  - Rationals (Q): Ratio between two integers, as simplified as possible
    - These are a subset of the pairs of integers, so they're definitely countable
  - Irrationals (no fun letter): Numbers that can't be represented as ratios, e.g. pi, e, $\sqrt{2}$, ...
- Reals (R): Rationals $\cup$ Irrationals
- Numbers described as infinite sequences of digits

Are the reals countable?

- Are the reals countably infinite?
  - We could try to find a bijection with natural numbers...
  - spoiler: we can't.
- Let's use a proof by contradiction:
  - Suppose the reals are countable, i.e. $|R| <= |N|$ (or equivalently $|N| >= |R|$)
  - In fact, let's focus on the reals between 0 and 1, not including 1.
    - If that range is bigger than N, then surely all of R is also bigger than N.
  - Then there must be a surjection f : N -> R{0..1}, which enumerates every real between 0 and 1 without missing any.
  - We'll show that leads to a contradiction.

Cantor's Diagonal Argument

Here is an example of a surjection from N->R{0..1}.
This is just to illustrate a gimmick. We don't actually care what f is, all functions f will have the same problem.

x	z	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	...
0	0.	0	0	0	1	1	1	0	0	...
1	0.	1	1	0	1	1	0	1	1	...
2	0.	2	3	0	1	4	3	2	1	...
3	0.	0	0	1	1	2	5	5	2	...
4	0.	5	9	2	1	8	9	3	1	...
5	0.	9	8	2	4	5	4	1	0	...

Gimmick Time

Let's name a number g (g is for gimmick!). G is a real number between 0 and 1, and it's defined like this:

1. Its only digit before the decimal is 0.
2. Its first digit after the decimal is the first digit after the decimal of whatever number f(0) is, plus 1 (wrapping around to 0 if the result is 10).
3. Its second decimal digit is the second decimal digit of f(1), plus one, mod 10.
4. And so on: $g_n = (f(n)_n + 1)\mathrm{mod}10$

The Contradiction

Since g is a real number, and f is a surjection, there must be some number k so that $f(k) = g$.
We know from the definition of g that g's kth digit must be different from f(k)'s kth digit.
But $g=f(k)$! This is a contradiction, so either g isn't a real number or f can't be a surjection.
g is definitely a real, so f must not be a surjection. That means that there is no surjection from N to our subset of R, so our subset must be strictly bigger than N.

Final thoughts

Using the same techniques we saw earlier, we can prove lots of stuff:

- There are as many real numbers as there are reals between 0 and 1
- There are as many reals as there are pairs of reals
- The set of functions Bool -> N is countably infinite
- The set of functions N -> Bool is uncountably infinite (whaaaaaaaa?)
- . . . and more!

Other weird stuff

- Rationals are dense: between any two rationals are infinitely many rationals
- Reals are also dense
- Between any two rationals are infinitely many reals
  - Sure, all rationals are also reals
- Between any two reals are infinitely many rationals
  - ...
  - Even though there are uncountably many reals and countably many rationals!
- Rationals form (countably) infinitely many points on the number line
  - but this doesn't give you a continuum of numbers!