Insertion Sort

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Outline

Sorted Lists

- Next semester, you'll take CS62
- You'll talk a lot about algorithms and data structures
  - Including sorting lists
- Today we'll get a preview in the functional setting
- So what's a sorted list?

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  2. It is a *permutation* of the input list

Ascending Order

- We can write this different ways for a list l:
  - forall x, if x appears at n in l, and n is not the end of the list, then `nth l (n+1)` is at least as big as x.
  - forall n, if n $<$ `length l` then `nth l n` $\leq$ `nth l (n+1)`
  - "An empty list and a one element list are sorted; prepending an element x to a list is sorted if x $\leq$ the first element of the list, if any" (an inductive definition!)
  - `sorted(l)` = `True` where:

```
sorted (x:y:l) = x <= y && sorted (y:l)
sorted _ = True
```

Permutations

- We need this property too!
  - Otherwise `[]` is a perfect output for any "sorting function"
- The new list needs the same elements as the old list, but possibly in a different order

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- Once we've inserted every element, we're done!

Insert

```
insertion_sort [] = []
insertion_sort (x:l) = insert x (insertion_sort l)

insert _x [] = [x]
insert x (y:l)
  | x <= y = x:y:l
  | otherwise = y:(insert x l)
```

- Try it on these lists: `[2, 1, 3]`, `[1, 2, 3]`, `[3, 2, 1]`.

Preservation Properties

- We often want to prove that applying some function doesn't lose us some property
  - We call these "preservation properties"
  - E.g., "map preserves length" is the property that mapping over a list doesn't change its length
  - E.g., "filter preserves order" states that the order of elements in a list won't change through filter
- Preservation properties are an easy way to build up proofs about a whole procedure
  - If each step of the procedure preserves the thing we care about, then the whole procedure will too

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  - WTP `sorted (y:l')` = `True` implies forall z, `sorted (insert z (y:l'))` = `True`.

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  - Assume `sorted (y:l')` = `True`. Let z be given.

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    - Because we know y < z in this case and we assumed `sorted (y:l')`.
  - So `sorted (y:(insert z l'))` is true exactly when `sorted (insert z l')` is true.
  - We could have `sorted (insert z l')` by our IH, if we could prove `sorted (y:l')`; and we already know `sorted (y:l')` from our assumption.

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  - `insertion_sort (y:l')` is just `insert y (insertion_sort l')`
  - Since our IH states that `insertion_sort l'` is sorted, and we know that insert preserves sortedness, we know `insert y (insertion_sort l')` must also be sorted.

Using Sortedness

- Consider: forall l f, `filter f (insertion_sort l) = insertion_sort (filter f l)`
  1. Why is this an interesting property?
  2. Prove it!