
csci54 – discrete math & functional programming
recurrence relations, (strong) induction

proofs

- ▶ logic
- ▶ proof techniques so far
 - ▶ direct proofs
 - ▶ proof of the contrapositive
 - ▶ proof by example / disproof by counterexample
 - ▶ using cases
 - ▶ induction
- ▶ today:
 - ▶ more induction, including strong induction (for when regular induction is not enough)



Proofs by induction

Definition 5.1: Proof by mathematical induction.

Suppose that we want to prove that $P(n)$ holds for all $n \in \mathbb{Z}^{\geq 0}$. To give a *proof by mathematical induction* of $\forall n \in \mathbb{Z}^{\geq 0} : P(n)$, we prove two things:

- 1 the *base case*: prove $P(0)$.
- 2 the *inductive case*: for every $n \geq 1$, prove $P(n - 1) \Rightarrow P(n)$.

- ▶ we prove the claim using a proof by induction on:
 - ▶ base case:
 - ▶ inductive hypothesis (IHOP):
 - ▶ inductive step:
- ▶ therefore by the principle of mathematical induction:





Recurrence relations

- ▶ Consider Sierpinski's triangle.



- ▶ Let $T(n)$ be the number of filled triangles in a Sierpinski's triangle after n iterations, where $T(0)$ is a single filled triangle.
- ▶ Observation: $T(n) = 3T(n-1)$ where $T(0)=1$
- ▶ Claim: $T(n) = 3^n$



Recurrence relations

- ▶ A function that is defined in terms of itself
- ▶ How would you prove that $A(n)$ is odd for all N for the following recurrence rela

$$A(n) = A(n - 1) + A(n - 2) + A(n - 3)$$

$$A(0) = 1$$

$$A(1) = 1$$

$$A(2) = 3$$

- base case: $A(0)$ is odd
- IHOP: $A(n)$ is odd
- inductive step: wts $A(n+1)$ is odd
 - $A(n+1) = A(n) + A(n-1) + A(n-2)$
 - $A(n)$ is odd
 - now what?

Proofs by strong induction

Definition 5.1: Proof by mathematical induction.

Suppose that we want to prove that $P(n)$ holds for all $n \in \mathbb{Z}^{\geq 0}$. To give a *proof by mathematical induction* of $\forall n \in \mathbb{Z}^{\geq 0} : P(n)$, we prove two things:

- 1 the *base case*: prove $P(0)$.
- 2 the *inductive case*: for every $n \geq 1$, prove $P(n - 1) \Rightarrow P(n)$.

Definition 5.10: Proof by strong induction.

Suppose that we want to prove that $P(n)$ holds for all $n \in \mathbb{Z}^{\geq 0}$. To give a *proof by strong induction* of $\forall n \in \mathbb{Z}^{\geq 0} : P(n)$, we prove the following:

- 1 the *base case*: prove $P(0)$.
- 2 the *inductive case*: for every $n \geq 1$, prove $[P(0) \wedge P(1) \wedge \dots \wedge P(n - 1)] \Rightarrow P(n)$.



Proofs by strong induction

Definition 5.10: Proof by strong induction.

Suppose that we want to prove that $P(n)$ holds for all $n \in \mathbb{Z}^{\geq 0}$. To give a *proof by strong induction* of $\forall n \in \mathbb{Z}^{\geq 0} : P(n)$, we prove the following:

- 1 the *base case*: prove $P(0)$.
- 2 the *inductive case*: for every $n \geq 1$, prove $[P(0) \wedge P(1) \wedge \dots \wedge P(n-1)] \Rightarrow P(n)$.

▶ we prove the claim using a proof by strong induction on:

▶ base case(s):

▶ inductive hypothesis (IHOP):

▶ inductive step:

▶ wts:

▶ therefore by the principle of mathematical induction:

may need more than one base case;
need for every n where
inductive step doesn't hold

IHOP: assume true for all values up to $n-1$

inductive step: wts true for n



Proofs by strong induction

$$A(n) = A(n - 1) + A(n - 2) + A(n - 3)$$

$$A(0) = 1$$

$$A(1) = 1$$

$$A(2) = 3$$

claim: $A(n)$ is odd for all N

- ▶ we prove the claim using a proof by strong induction on:
- ▶ base case(s):
- ▶ inductive hypothesis (IHOP):
- ▶ inductive step:
 - ▶ wts:
- ▶ therefore by the principle of mathematical induction:



Proofs by strong induction

$$A(n) = A(n - 1) + A(n - 2) + A(n - 3)$$

$$A(0) = 1$$

$$A(1) = 1$$

$$A(2) = 3$$

Claim: $A(n)$ is odd for all N

- ▶ we prove the claim using a proof by strong induction on n
- ▶ base case(s): $A(0)$, $A(1)$, and $A(2)$ are all odd.
- ▶ inductive hypothesis (IHOP): $A(x)$ is odd for all $x < y$
- ▶ inductive step: we want to show that $A(y)$ is odd
 - ▶ by the IHOP we know that $A(y-1)$, $A(y-2)$, $A(y-3)$ are all odd, so there exist integers a, b, c such that ...
 - ▶ this means $a+b+c$ is odd and therefore $A(y)$ is also odd.
- ▶ therefore by the principle of mathematical induction: $A(n)$ is odd



Strong vs. regular (weak) induction

- ▶ Anything that can be proven using regular induction can also be shown using strong induction.
- ▶ However, if you can prove something using regular induction, you should.

