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csci54 – discrete math & functional programming  
recurrence relations, (strong) induction

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# proofs

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- ▶ logic
- ▶ proof techniques so far
  - ▶ direct proofs
  - ▶ proof of the contrapositive
  - ▶ proof by example / disproof by counterexample
  - ▶ using cases
  - ▶ induction
- ▶ today:
  - ▶ more induction, including strong induction (for when regular induction is not enough)



# Proofs by induction

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**Definition 5.1: Proof by mathematical induction.**

Suppose that we want to prove that  $P(n)$  holds for all  $n \in \mathbb{Z}^{\geq 0}$ . To give a *proof by mathematical induction* of  $\forall n \in \mathbb{Z}^{\geq 0} : P(n)$ , we prove two things:

- 1 the *base case*: prove  $P(0)$ .
- 2 the *inductive case*: for every  $n \geq 1$ , prove  $P(n - 1) \Rightarrow P(n)$ .

- ▶ we prove the claim using a proof by induction on:
- ▶ base case:
- ▶ inductive hypothesis (IHOP):
- ▶ inductive step:
- ▶ therefore by the principle of mathematical induction:



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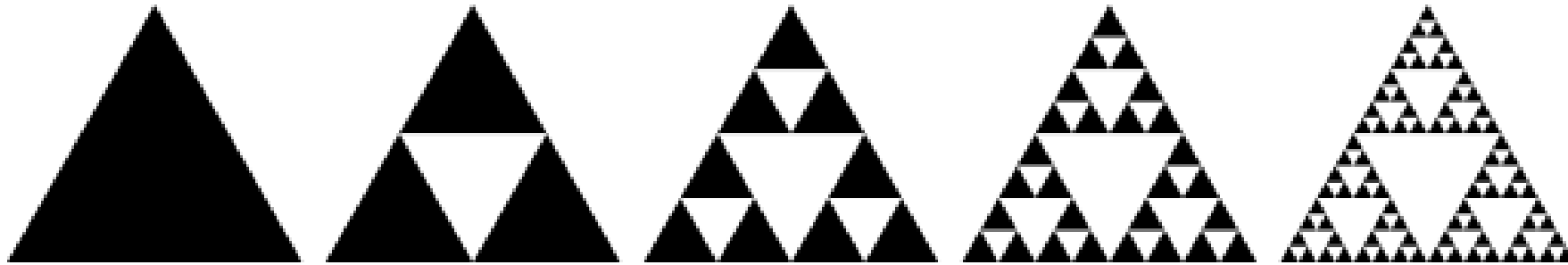
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# Recurrence relations

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- ▶ Consider Sierpinski's triangle.



- ▶ Let  $T(n)$  be the number of filled triangles in a Sierpinski's triangle after  $n$  iterations, where  $T(0)$  is a single filled triangle.
- ▶ Observation:  $T(n) = 3T(n-1)$  where  $T(0)=1$
- ▶ Claim:  $T(n) = 3^n$



# Recurrence relations

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- ▶ A function that is defined in terms of itself
- ▶ How would you prove that  $A(n)$  is odd for all  $N$  for the following recurrence rela

$$A(n) = A(n - 1) + A(n - 2) + A(n - 3)$$

$$A(0) = 1$$

$$A(1) = 1$$

$$A(2) = 3$$

- base case:  $A(0)$  is odd
- IHOP:  $A(n)$  is odd
- inductive step: wts  $A(n+1)$  is odd
  - $A(n+1) = A(n) + A(n-1) + A(n-2)$
  - $A(n)$  is odd
  - now what?

# Proofs by strong induction

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**Definition 5.1: Proof by mathematical induction.**

Suppose that we want to prove that  $P(n)$  holds for all  $n \in \mathbb{Z}^{\geq 0}$ . To give a *proof by mathematical induction* of  $\forall n \in \mathbb{Z}^{\geq 0} : P(n)$ , we prove two things:

- 1 the *base case*: prove  $P(0)$ .
- 2 the *inductive case*: for every  $n \geq 1$ , prove  $P(n - 1) \Rightarrow P(n)$ .

**Definition 5.10: Proof by strong induction.**

Suppose that we want to prove that  $P(n)$  holds for all  $n \in \mathbb{Z}^{\geq 0}$ . To give a *proof by strong induction* of  $\forall n \in \mathbb{Z}^{\geq 0} : P(n)$ , we prove the following:

- 1 the *base case*: prove  $P(0)$ .
- 2 the *inductive case*: for every  $n \geq 1$ , prove  $[P(0) \wedge P(1) \wedge \dots \wedge P(n - 1)] \Rightarrow P(n)$ .



# Proofs by strong induction

## Definition 5.10: Proof by strong induction.

Suppose that we want to prove that  $P(n)$  holds for all  $n \in \mathbb{Z}^{\geq 0}$ . To give a *proof by strong induction* of  $\forall n \in \mathbb{Z}^{\geq 0} : P(n)$ , we prove the following:

- 1 the *base case*: prove  $P(0)$ .
- 2 the *inductive case*: for every  $n \geq 1$ , prove  $[P(0) \wedge P(1) \wedge \dots \wedge P(n-1)] \Rightarrow P(n)$ .

► we prove the claim using a proof by strong induction on:

► base case(s):

may need more than one base case;  
need for every  $n$  where  
inductive step doesn't hold

► inductive hypothesis (IHOP):

► inductive step:

IHOP: assume true for all values up to  $n-1$

► wts:

► therefore by the principle of mathematical induction:

inductive step: wts true for  $n$

# Proofs by strong induction

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$$A(n) = A(n-1) + A(n-2) + A(n-3)$$

$$A(0) = 1$$

$$A(1) = 1$$

$$A(2) = 3$$

claim:  $A(n)$  is odd for all  $N$

- ▶ we prove the claim using a proof by strong induction on:
- ▶ base case(s):
- ▶ inductive hypothesis (IHOP):
- ▶ inductive step:
  - ▶ wts:
- ▶ therefore by the principle of mathematical induction:



# Proofs by strong induction

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$$A(n) = A(n-1) + A(n-2) + A(n-3)$$

$$A(0) = 1$$

$$A(1) = 1$$

$$A(2) = 3$$

Claim:  $A(n)$  is odd for all  $N$

- ▶ we prove the claim using a proof by strong induction on  $n$
- ▶ base case(s):  $A(0)$ ,  $A(1)$ , and  $A(2)$  are all odd.
- ▶ inductive hypothesis (IHOP):  $A(x)$  is odd for all  $x < y$
- ▶ inductive step: we want to show that  $A(y)$  is odd
  - ▶ by the IHOP we know that  $A(y-1)$ ,  $A(y-2)$ ,  $A(y-3)$  are all odd, so there exist integers  $a, b, c$  such that ...
  - ▶ this means  $a+b+c$  is odd and therefore  $A(y)$  is also odd.
- ▶ therefore by the principle of mathematical induction:  $A(n)$  is odd



## Strong vs. regular (weak) induction

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- ▶ Anything that can be proven using regular induction can also be shown using strong induction.
- ▶ However, if you can prove something using regular induction, you should.

