csci54 – discrete math & functional programming induction

this week: continuing with proofs

- logic
 - propositional
 - predicate
- proof techniques
 - direct proofs
 - proof of the contrapositive
 - proof by example / disproof by counterexample
 - proof by contradiction
 - using cases
 - induction!



some definitions (recap)

- an integer k is <u>even</u> if and only if there exists an integer r such that k=2r
- ▶ an integer k is <u>odd</u> if and only if there exists an integer r such that k=2r+1
- k|m if and only if there exists an integer r such that m=kr. This is equivalent to saying that "m mod k = 0" or that "k evenly divides m".
- ▶ an integer k>1 is <u>prime</u> if the only positive integers that evenly divide k are 1 and k itself.
- ▶ an integer k>1 is <u>composite</u> if it is not prime.
- an integer k is a <u>perfect square</u> if and only if there exists an integer r such that k=r²

What about . . .

claim: given any non-negative integer n, the sum of integers up to n is n*(n+1)/2

- techniques we know:
 - direct proofs
 - proof of the contrapositive
 - proof by example / disproof by counterexample
 - proof by contradiction
 - using cases



(on summation notation)

claim: given any non-negative integer n, the sum of the integers from 1 up to n is n*(n+1)/2

could also write using summation notation:

$$\sum_{i=1}^{n} i = n (n+1)/2$$



What about . . .

observations:

- want to prove something is true for all elements of a set (the non-negative integers)
- the set is ordered in the sense that we can talk about the smallest/first element, then the next one, then the next one, ... (0, 1, 2, 3, ...)



Proofs by induction

Definition 5.1: Proof by mathematical induction.

Suppose that we want to prove that P(n) holds for all $n \in \mathbb{Z}^{\geq 0}$. To give a *proof by mathematical induction* of $\forall n \in \mathbb{Z}^{\geq 0} : P(n)$, we prove two things:

- 1 the base case: prove P(0).
- **2** the *inductive case*: for every $n \ge 1$, prove $P(n-1) \Rightarrow P(n)$.

Structure of a proof by induction

- claim: for all x, P(x)
- we prove the claim using a proof by induction on: x
- base case: P(x*) holds for the smallest x*
- ▶ inductive step: $P(x') \rightarrow P(x)$
 - If we assume P(x') for some x' (inductive hypothesis)
 - We must show that for every way we can grow x' into some $x, P(x') \rightarrow P(x)$
- ▶ therefore by the principle of mathematical induction: for all x, P(x)



Structure of a proof by induction

- claim: for all natural numbers n, P(n)
- we prove the claim using a proof by induction on: n
- base case: P(0)
- ▶ inductive step: $P(n) \rightarrow P(n+1)$
 - If we assume P(n) for some n (inductive hypothesis)
 - We must show that for $P(n) \rightarrow P(n+1)$
- therefore by the principle of mathematical induction: for all n, P(n)
- We will never miss a natural number with this induction scheme



Notes on writing proofs by induction

- we prove the claim using a proof by induction <...>
 - unless it's a direct proof should state the proof technique.
- base case
 - show true on the smallest element of the set
- inductive hypothesis (IHOP)
 - assume true for some value
- inductive step
 - wts: if IHOP is on n, then prove for n+1. if IHOP is on n-1, then prove for n.
 - some step in this part must refer back to the IHOP. otherwise it's definitely not a proof by induction (and may not be a proof at all)
- therefore by the principle of mathematical induction <...>
- have a concluding line

For every **positive** integer, $n + 1 \le n * 2$

- we prove the claim using a proof by induction on n:
- base case: 1 + 1 <= 1 * 2</p>
- inductive step:
 - inductive hypothesis (IH): n' + 1 <= n' * 2
 - Wts: (n'+1)+1 <= (n'+1)*2
 - $-(n'+1)+1 \le 2 * n' + 2 * 1$
 - We know $n'+1 \le 2*n'$ by the IH, so it suffices to show that $1 \le 2$
- therefore by the principle of mathematical induction:
- For all positive integers n, n + 1 <= n * 2</p>

For every list and function, map f I has the same length as I

- we prove the claim using a proof by induction on I:
- base case: map f [] = [] has the same length as []
- inductive step:
 - inductive hypothesis (IH): length (map f l') = length l'
 - Wts: length (map f(x:l')) = length (x:l')
 - map f(x:l') = (fx):(map f l') (second case of map)
 - length (x:l') = 1 + length l' (second case of length)
 - length ((f x):(map f l')) = 1 + length (map f l') (same)
 - So we have: 1 + length I' = 1 + length I' (by IH)
- Therefore, by induction, forall f and I, length (map f I) = length I

For every positive integer n, the sum from 1 up to n is equal to n*(n+1)/2.

- we prove the claim using a proof by induction on n:
- base case: for n=1, ...
- inductive step: (for all n', if P(n') then P(n'+1))
 - inductive hypothesis (IH): for n=n'...
 - Wts: for n=n'+1, ...
 - ...
- therefore by the principle of mathematical induction:
 - For all positive integers n, the sum from 1 up to n is n*(n+1)/2.

- ▶ Identify the smallest positive integer p such that for all n >= p, n! > 2ⁿ
- Prove that your choice of p is correct
 - statement needs to be true for all n
 - ▶ if p > 1, need to show that the statement is not true for p 1

n! =
0! = 1! = 1
n! is read "n
factorial"