csci54 – discrete math & functional programming
more logic, introduction to proofs

last time

- propositional logic:
  - practice with logical equivalence

- introduction to predicate logic:
  - definition of a predicate
  - quantifiers: forall, exists
  - theorems in predicate logic

from last time

- Exactly one of the following two propositions is a theorem. Which one?

(1) $[\forall x \in S : P(x) \vee Q(x)] \Leftrightarrow [\forall x \in S : P(x)] \vee [\forall x \in S : Q(x)]$

(2) $[\exists x \in S : P(x) \vee Q(x)] \Leftrightarrow [\exists x \in S : P(x)] \vee [\exists x \in S : Q(x)]$

- (2) is the theorem.

- Prove that your answer is correct.
  - What is a proof?
  - A convincing argument that something is true.

Solution. Claim (B) is a theorem. To prove it, we'll show that the left-hand side implies the right-hand side, and vice versa. (That is, we're proving $p \Leftrightarrow q$ by proving both $p \Rightarrow q$ and $q \Rightarrow p$, which is a legitimate proof because $p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)$.) Both proofs will use the technique of assuming the antecedent.

First, let's prove that $[\exists x \in S : P(x) \vee Q(x)]$ implies $[\exists x \in S : P(x)] \vee [\exists x \in S : Q(x)]$:

Suppose that $[\exists x \in S : P(x) \vee Q(x)]$ is true. Then there is some particular $x^* \in S$ for which either $P(x^*)$ or $Q(x^*)$. But in either case, we're done: if $P(x^*)$ then $\exists x \in S : P(x)$ because x^* satisfies the condition; if $Q(x^*)$ then $\exists x \in S : Q(x)$, again because x^* satisfies the condition.

Second, let's prove that $[\exists x \in S : P(x)] \vee [\exists x \in S : Q(x)]$ implies $[\exists x \in S : P(x) \vee Q(x)]$:

Suppose that $[\exists x \in S : P(x)] \vee [\exists x \in S : Q(x)]$ is true. Thus either there's an $x^* \in S$ such that $P(x^*)$ or an $x^* \in S$ such that $Q(x^*)$. That x^* suffices to make the left-hand side of (B) true.

- What makes something "a convincing argument"?

some definitions

- an integer k is <u>even</u> if and only if there exists an integer r such that k=2r
- an integer k is <u>odd</u> if and only if there exists an integer r such that k=2r+1
- k|m if and only if there exists an integer r such that m=kr. This is equivalent to saying that "m mod k = 0" or that "k evenly divides m".
- an integer k>1 is <u>prime</u> if the only positive integers that evenly divide k are 1 and k itself.
- an integer k>1 is <u>composite</u> if it is not prime.
- an integer k is a <u>perfect square</u> if and only if there exists an integer r such that $k=r^2$

example 1

- Consider the statement "for all positive integers n, $2n=n^2$"
  - Why isn't this true?
    - Consider n = 3
  - Why is this a valid justification?

- How would you write this as a statement in predicate logic?

$$\forall n \in \mathbb{Z}^+ : 2n = n^2$$

  - Showing that this statement is not true is the same as showing that its negation is true.

negating quantifiers

- The following are both theorems

$$\neg\left[\forall x \in S : P(x)\right] \Leftrightarrow \left[\exists x \in S : \neg P(x)\right]$$

$$\neg\left[\exists x \in S : P(x)\right] \Leftrightarrow \left[\forall x \in S : \neg P(x)\right]$$

- practice: what is the negation of the following? simplify as much as possible. $\exists x \in S : P(x) \vee Q(x)$

from Figure 3.21 in CDMCS

example 1 - revisited

- Consider the statement "for all positive integers n, $2n=n^2$ "

- How would you prove that this statement is false?
  - Consider the following counterexample. If $n=3$, then $2n=6$ and $n^2=9$.
  - Since there exists a positive integer such that $2n \neq n^2$, the original statement is false.

example 2

- Claim: let x be any integer. if x is a perfect square, then 4x is a perfect square

- How could you write the claim as a statement in predicate logic?

- How would you prove the claim is true?

- Why is this justification valid?

assuming the antecedent, modus ponens

- assuming the antecedent.
  - to show "if a then b", only need to show that if a is true, then b is true.

- two tautologies that are used repeatedly in proofs through a chain of reasoning.

$$(p \Rightarrow q) \wedge p \Rightarrow q \qquad \text{Modus Ponens}$$

$$(p \Rightarrow q) \wedge \neg q \Rightarrow \neg p \qquad \text{Modus Tollens}$$

example 2 - revisited

- Claim: let x be any integer. if x is a perfect square, then 4x is a perfect square

- How would you prove the claim is true?
  - assume x is a perfect square (assuming the antecedent)
  - then there exists an integer r such that $x = r^2$ (definition of perfect square, modus ponens)
  - then $4x = 4r^2 = (2r)^2$ (algebra)
  - therefore 4x is a perfect square (definition of perfect square)
  - in conclusion, for any integer x, if x is a perfect square then 4x is a perfect square.

Nested quantifiers

- Let A be an array of n integers with 1-based indexing. What is the following asserting?

$$\forall i \in \{1, 2, \ldots, n\} : [\exists j \in \{1, 2, \ldots, n\} : (i \neq j) \wedge (A[i] = A[j])]$$

- How could you write the following using nested quantifiers?

Every program that was turned in failed at least one test case.

Nested quantifiers - questions

- What are the rules with nested quantifiers?
- Can you flip the order of nested quantifiers?
- What happens if you negate a nested quantifier?

Nested quantifiers – order sometimes matters

- Exactly one of the following is true. Which? Why?

$$\exists y \in \mathbb{R} : \forall x \in \mathbb{R} : x < y$$

$$\forall x \in \mathbb{R} : \exists y \in \mathbb{R} : x < y$$

- However, we can swap the same type of quantifier:

$$\forall x \in S : \forall y \in T : P(x,y) \quad \Leftrightarrow \quad \forall y \in T : \forall x \in S : P(x,y)$$

$$\exists x \in S : \exists y \in T : P(x,y) \quad \Leftrightarrow \quad \exists y \in T : \exists x \in S : P(x,y)$$

Negating nested quantifiers

- Consider the following statement:

$$\forall i \in \{1, 2, \ldots, n\} : [\exists j \in \{1, 2, \ldots, n\} : (i \neq j) \wedge (A[i] = A[j])]$$

- Simplify the negation:
- $\neg \forall i \in \{1, 2, \ldots, n\} : [\exists j \in \{1, 2, \ldots, n\} : (i \neq j) \wedge (A[i] = A[j])]$