csci54 – discrete math & functional programming
propositional logic

Simplify each of the following Haskell expressions:

(a) `a && not a`

(b) `a || (not a && b)`

(c)
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(not a || b) && (not b || c) &&
  (not c || not a) && (not c || not b)
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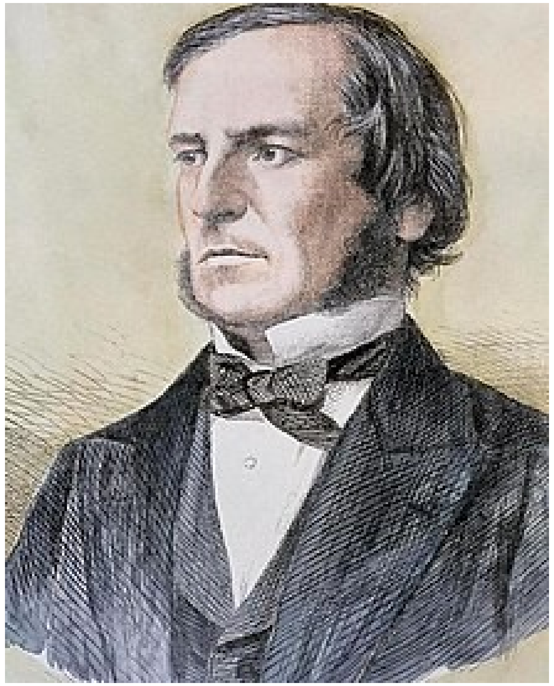

George
Boole
1815-1864

On "True" and "False"

- logic is the study of valid reasoning

- The starting point:

  A proposition is a statement that is either True or False.

- What are examples of propositions that are True? False? Unknown?

On propositional logic

- the study of propositions: how to formulate, evaluate, manipulate
- atomic proposition: a proposition that is conceptually indivisible
- compound proposition: a proposition that is build up out of conceptually simpler propositions
  - How?

Creating compound propositions

- We can create more complex propositional statements using <u>logical connectives</u>
  - negation (not, ¬, ~)
  - conjunction (and, ∧)
  - disjunction (or, ∨)
  - implication (implies, ⇒, →)

Precedence rules:

- negation binds most tightly
- then conjunction
- then disjunction
- then implication

implication is right-associative

- In particular, a well-formed propositional logic formula is defined as:

$$\phi ::= T | F | (\neg \phi) | (\phi \wedge \phi) | (\phi \vee \phi) | (\phi \Rightarrow \phi)$$

Evaluating compound propositional statements

- Convenient to use a truth table to display the relationships between truth values of different propositions

- Truth table for negation:

p	¬p
T	F
F	T

- For conjunction (and) and disjunction (or):

p	q	p ∧ q	p ∨ q
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

$$\phi ::= T|F|(\neg\phi)|(\phi \wedge \phi)|(\phi \vee \phi)|(\phi \Rightarrow \phi)$$

Implication

- What does it mean to say "p implies q"?
  - p => q is true if q is true or p is false

p	q	p ⇒ q
T	T	T
T	F	F
F	T	T
F	F	T

- What is the truth value of each of the following statements?
  - 1 + 1 = 2 implies that 2 + 3 = 5
  - 1 + 1 = 2 implies that 2 + 3 = 6
  - 1 + 1 = 3 implies that 2 + 3 = 5
  - 1 + 1 = 3 implies that 2 + 3 = 6

A little more on implications

- p => q
  - “if p, then q”
  - “p implies q”
  - “p only if q”
  - “q whenever p”
  - “q, if p"
  - “q is necessary for p”
  - “p is sufficient for q”

- Bidirectional implication p <=> q
  - "p if and only if q", "p iff q"
  - True only when p and q have same truth value: either both true or both false.

Example

- "Since Sandra is wearing a soccer jersey, she must be a soccer player."

- This compound proposition is composed of 2 atomic propositions:
  - (1) = Sandra is wearing a soccer jersey
  - (2) = Sandra is a soccer player

- The compound proposition can written as:
  - (1) ⇔ (2)

inspired by:
https://philosophy.lander.edu/logic/diagram_quiz.html

Passwords

- "A password is valid only if it is at least 8 characters long, is not one that you have used previously, and contains at least 2 of the following: a number, a lowercase character, an uppercase character."
- This is a compound proposition that is composed of how many atomic propositions?
- What are the 6 atomic propositions?
- How can you write the compound proposition in terms of the atomic propositions?

categorizing well-formed formulas (wff)

- A formula in propositional logic is one of:
  - tautology (valid): if it evalutes to T in all cases
  - satisfiable: evaluates to T in some cases
  - contingency (falsifiable): evaluates to F in some cases
  - contradiction (unsatisfiable): evaluates to F in all cases

- Consider the following formula:

$$(p \vee q) \Rightarrow (\neg p \wedge \neg q)$$

- Which of the following describes the formula: tautology, satisfiable, contingency, contradiction? Why?

a collection of tautologies

$(p \Rightarrow q) \land p \Rightarrow q$	Modus Ponens
$(p \Rightarrow q) \land \neg q \Rightarrow \neg p$	Modus Tollens
$p \lor \neg p$	Law of the Excluded Middle
$p \Leftrightarrow \neg\neg p$	Double Negation
$p \Leftrightarrow p$	
$p \Rightarrow p \lor q$	
$p \land q \Rightarrow p$	

$$(p \lor q) \land \neg p \Rightarrow q$$
$$(p \Rightarrow q) \land (\neg p \Rightarrow q) \Rightarrow q$$
$$(p \Rightarrow q) \land (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$$

$$(p \Rightarrow q) \land (p \Rightarrow r) \Leftrightarrow p \Rightarrow q \land r$$
$$(p \Rightarrow q) \lor (p \Rightarrow r) \Leftrightarrow p \Rightarrow q \lor r$$
$$p \land (q \lor r) \Leftrightarrow (p \land q) \lor (p \land r)$$
$$p \Rightarrow (q \Rightarrow r) \Leftrightarrow p \land q \Rightarrow r$$

logical equivalence

- Two propositions are logically equivalent (written) if they have exactly identical truth tables (i.e. their truth values are the same under every truth assignment)

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some logically equivalent propositions

Commutativity	$p \vee q \equiv q \vee p$
	$p \wedge q \equiv q \wedge p$
	$p \oplus q \equiv q \oplus p$
	$p \Leftrightarrow q \equiv q \Leftrightarrow p$
Associativity	$p \vee (q \vee r) \equiv (p \vee q) \vee r$
	$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$
	$p \oplus (q \oplus r) \equiv (p \oplus q) \oplus r$
	$p \Leftrightarrow (q \Leftrightarrow r) \equiv (p \Leftrightarrow q) \Leftrightarrow r$
Idempotence	$p \vee p \equiv p$
	$p \wedge p \equiv p$
Distribution of $\wedge$ over $\vee$	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
Distribution of $\vee$ over $\wedge$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
Contrapositive	$p \Rightarrow q \equiv \neg q \Rightarrow \neg p$
	$p \Rightarrow q \equiv \neg p \vee q$
	$p \Rightarrow (q \Rightarrow r) \equiv p \wedge q \Rightarrow r$
	$p \Leftrightarrow q \equiv \neg p \Leftrightarrow \neg q$
Mutual Implication	$(p \Rightarrow q) \wedge (q \Rightarrow p) \equiv p \Leftrightarrow q$
De Morgan's Laws	$\neg(p \wedge q) \equiv \neg p \vee \neg q$
	$\neg(p \vee q) \equiv \neg p \wedge \neg q$

$$(\neg a \vee b) \wedge (\neg b \vee c) \wedge (\neg c \vee \neg a) \wedge (\neg c \vee \neg b)$$