CS054: Truth tables

The goal of this worksheet is to give you practice with truth tables: what are truth tables and how do you construct them? It's not for a grade—no need to turn it in! I'll post solutions, but you'll get the most out of it if you don't peek.

For all of these questions, I'll use programmatic notation—`&&` for `andb` a/k/a `&&` in Coq, `||` for `orb` a/k/a `||` in Coq, and `!` for `negb`. We write $\mathfrak{t}$ for `true` and $\mathfrak{f}$ for `false`; you can also use T and F or ⊤ and ⊥. The questions are asking about the definitions we have in `Basics.v`.

I expect you to simply *know* the truth tables for `||`, `&&`, `!`, and ⇒.

1. **Sample:** Consider the boolean expression $!a \mathbin{||} b$ given booleans a and b.
   (a) How many rows will the truth table have? **Answer:** four, because there are two variables, a and b, and we must consider each value they consider.
   (b) How many columns will the truth table have and what are they? **Answer:** four; one each for a and b, one for $!a$, and one for the whole expression.
   (c) What is the truth table? **Answer:**

a	b	$!a$	$!a \mathbin{\vert\vert} b$
$\mathfrak{t}$	$\mathfrak{t}$	$\mathfrak{f}$	$\mathfrak{t}$
$\mathfrak{t}$	$\mathfrak{f}$	$\mathfrak{f}$	$\mathfrak{f}$
$\mathfrak{f}$	$\mathfrak{t}$	$\mathfrak{t}$	$\mathfrak{t}$
$\mathfrak{f}$	$\mathfrak{f}$	$\mathfrak{t}$	$\mathfrak{t}$

   (d) Can you rephrase this expression in terms of other boolean operators? That is, can you find a smaller boolean expression that has an equivalent truth table? **Answer:** The expression $a \Rightarrow b$ (i.e., `impb a b`) has the same truth table.

2. Consider the boolean expression $p \mathbin{||} (p \mathbin{\&\&} q)$.
   (a) How many rows will the truth table have?

   Solution: Four, because there are two variables.

   (b) How many columns will the truth table have?

   Solution: Four: one for each variable (2), one for $p \mathbin{\&\&} q$, and one for the outer expression.

   (c) What is the truth table?

   Solution:

p	q	$p \mathbin{\&\&} q$	$p \mathbin{\vert\vert} (p \mathbin{\&\&} q)$
$\mathfrak{t}$	$\mathfrak{t}$	$\mathfrak{t}$	$\mathfrak{t}$
$\mathfrak{t}$	$\mathfrak{f}$	$\mathfrak{f}$	$\mathfrak{t}$
$\mathfrak{f}$	$\mathfrak{t}$	$\mathfrak{f}$	$\mathfrak{f}$
$\mathfrak{f}$	$\mathfrak{f}$	$\mathfrak{f}$	$\mathfrak{f}$

   (d) Can you rephrase this expression in terms of other boolean operators? That is, can you find a smaller boolean expression that has an equivalent truth table?

Solution: The formulae p || (p && q) has the same truth table as the formula p itself.

3. Consider the boolean expression x && (x || y).
   (a) How many rows will the truth table have?

   Solution: Four, because there are two variables.

   (b) How many columns will the truth table have?

   Solution: Four: one for each variable (2), one for x || y, and one for the outer expression.

   (c) What is the truth table?

   Solution:

x	y	x || y	x && (x || y)
t	t	t	t
t	f	t	t
f	t	t	f
f	f	f	f

   (d) Can you rephrase this expression in terms of other boolean operators? That is, can you find a smaller boolean expression that has an equivalent truth table?

   Solution: The formula x && (x || y) has the same truth table as the formula x itself.

4. Consider the boolean expression (x && y) || (x && z).
   (a) How many rows will the truth table have?

   Solution: Eight, because there are three variables.

   (b) How many columns will the truth table have?

   Solution: Six: one for each variable (3), one for x && y, one for x && z, and one for the outer expression.

   (c) What is the truth table?

   Solution:

x	y	z	x && y	x && z	(x && y) || (x && z)
t	t	t	t	t	t
t	t	f	t	f	t
t	f	t	f	t	t
t	f	f	f	f	f
f	t	t	f	f	f
f	t	f	f	f	f
f	f	t	f	f	f
f	f	f	f	f	f

(d) Can you rephrase this expression in terms of other boolean operators? That is, can you find a smaller boolean expression that has an equivalent truth table?

> **Solution:** The formula $(x$ `&&` $y)$ `||` $(x$ `&&` $z)$ has the same truth table as the formula x `&&` $(y$ `||` $z)$. The latter formula has one fewer column in its truth table and one fewer function call, so it's "smaller".

5. Consider the boolean expression `!`x `&&` `!`y.

(a) How many rows will the truth table have?

> **Solution:** Four, because there are two variables.

(b) How many columns will the truth table have?

> **Solution:** Five: one for each variable (2), one for the negation of each variable, and one for the outer expression.

(c) What is the truth table?

> **Solution:**

x	y	`!`x	`!`y	`!`x `&&` `!`y
t	t	f	f	f
t	f	f	t	f
f	t	t	f	f
f	f	t	t	t

(d) Can you rephrase this expression in terms of other boolean operators? That is, can you find a smaller boolean expression that has an equivalent truth table?

> **Solution:** The formula `!`x `&&` `!`y has the same truth table as the formula `!`$(x$ `||` $y)$, which has one fewer column and one fewer function call, so it's "smaller".

6. Consider the boolean expression `!`a `||` `!`b.

(a) How many rows will the truth table have?

> **Solution:** Four, because there are two variables.

(b) How many columns will the truth table have?

> **Solution:** Five: one for each variable (2), one for the negation of each variable, and one for the outer expression.

(c) What is the truth table?

> **Solution:**

a	b	`!`a	`!`b	`!`a `||` `!`b
t	t	f	f	f
t	f	f	t	t
f	t	t	f	t
f	f	t	t	t

(d) Can you rephrase this expression in terms of other boolean operators? That is, can you find a smaller boolean expression that has an equivalent truth table?

Solution: The formula `!`a `&&` `!`b has the same truth table as the formula `!`$(a$ `&&` $b)$, which has one fewer column and one fewer function call, so it's "smaller".

7. Use a truth table to prove that `||` is commutative, i.e., p `||` q is the same as q `||` p.

Solution:

p	q	p `||` q	q `||` p
t	t	t	t
t	f	t	t
f	t	t	t
f	f	f	f

8. Use a truth table to prove that `||` is associative, i.e., p `||` $(q$ `||` $r)$ is the same as $(p$ `||` $q)$ `||` r.

Solution:

p	q	r	p `||` q	q `||` r	p `||` $(q$ `||` $r)$	$(p$ `||` $q)$ `||` r
t	t	t	t	t	t	t
t	t	f	t	t	t	t
t	f	t	t	t	t	t
t	f	f	t	f	t	t
f	t	t	t	t	t	t
f	t	f	t	t	t	t
f	f	t	f	t	t	t
f	f	f	f	f	f	f

9. Write a logical formula that's equivalent to the following truth table over the variables a and b.

a	b	???
t	t	f
t	f	f
f	t	f
f	f	t

What might you name this formula?

Solution: It is $\neg(a \vee b)$; it is called NOR, by analogy to NAND.

Other good practice exercises (for which no solutions will be provided):

- Use truth tables to prove that `&&` and $\otimes$ (a/k/a `xorb`) are commutative and associative.
- Use truth tables to prove that `!` is involutive, i.e., `!` (`!` b) is equivalent to b.
- Is $\Rightarrow$ (a/k/a `impb`) commutative or associative? Use truth tables to prove or disprove it.