CS054: Sets

The goal of this worksheet is to give you practice with proofs about sets, using both the element-wise and algebraic approaches. It's not for a grade—no need to turn it in! I'll post solutions, but you'll get the most out of it if you don't peek.

1. Prove that $A \cap A = A$ both ways.

   Element-wise proof:

   Solution: First, let $x \in A \cap A$ be given—that is, $x \in A$. We merely need show $xinA$, which we have assumed.

   Next, suppose $x \in A$; we must show $x \in A \cap A$, i.e., $x \in A \wedge x \in A$—which we have immediately.

   Algebraic proof:

   Solution:

$$\begin{aligned} A \cap A &= (A \cap A) \cup \emptyset && \cup \text{ identity} \\ &= (A \cap A) \cup (A \cap \overline{A}) && \cap \text{ complement} \\ &= A \cap (A \cup \overline{A}) && \cap \text{ distributivity} \\ &= A \cap U && \cup \text{ complement} \\ &= A && \cap \text{ identity} \end{aligned}$$

2. Prove that $\overline{\emptyset} = U$ both ways.

   When doing proofs about complement, U represents the universe of discourse; you can assume $\forall x, x \in U$. (Try to avoid making such an assumption in proofs that don't involve set complement, though!) You can also assume that $U \neq \emptyset$, and, as a consequence, you can introduce arbitrary sets A into your proof as needed.

   Element-wise proof:

   Solution: First, let $x \in \overline{\emptyset}$ be given—that is, $x \notin \emptyset$. We need to show that $x \in U$, but we have that by the definition U as our universe.

   Next, suppose $x \in U$; we must show $x \notin \emptyset$... which is immediate. QED

   Algebraic proof:

   Solution: Fixing some arbitrary set A...

$$\begin{aligned} \overline{\emptyset} &= \overline{(A \cap \overline{A})} && \cap \text{ complement} \\ &= (\overline{A} \cup \overline{\overline{A}}) && \cap \text{ De Morgan's law} \\ &= U && \cup \text{ complement on } \overline{A} \end{aligned}$$

   Or:

$$\begin{aligned} \overline{\emptyset} &= \overline{\emptyset} \cup \emptyset && \cup \text{ identity} \\ &= \emptyset \cup \overline{\emptyset} && \cup \text{ commutativity} \\ &= U && \cup \text{ complement on } \emptyset \end{aligned}$$

3. Prove that $\overline{U} = \emptyset$ both ways.

   Element-wise proof:

   Solution: Let $x \in \overline{U}$ be given... we need to show $x \in \emptyset$. But we already have a contradiction—$x \in \overline{U}$ holds when $x \notin U$, but $x \in U$ holds for every x!

   Let $x \in \emptyset$ be given; we need to show $x \notin U$. But $x \in \emptyset$ is a contradiction!

   Algebraic proof:

   Solution: Fixing some arbitrary set A...

$$\begin{aligned} \overline{U} &= \overline{A \cup \overline{A}} && \cup \text{ complement} \\ &= \overline{A} \cap \overline{\overline{A}} && \cup \text{ De Morgan's law} \\ &= \emptyset && \cap \text{ complement on } \overline{A} \end{aligned}$$

   Can you find a solution that doesn't use De Morgan's law, like the second one above? Hint: your solution should be 'dual', i.e., swapping $\cup/\cap$ and $U/\emptyset$.

4. Prove that $\overline{\overline{A}} = A$ both ways. Your element-wise proof will need the law of the excluded middle, i.e., you may assume that $x \in A \vee x \notin A$ for all x and A—that is, you can use the law of the excluded middle.

Element-wise proof:

Solution: First, let $x \in \overline{\overline{A}}$ be given; we must show $x \in A$. By definition of set complement, $x \notin \overline{A} = \{y \mid y \notin A\}$. If x isn't in the set of things *not* in A, then x must be in A itself—as desired.

In the other direction, suppose $x \in A$; we must show $x \in \overline{\overline{A}}$, i.e., $x \notin \overline{A}$.

Suppose for a contradiction that $x \in \overline{A}$. In that case, we would have $x \notin A$... but we've assumed $x \in A$! So it must be the case that $x \notin \overline{A}$, and so $x \in \overline{\overline{A}}$.

Algebraic proof:

Solution:

$$
\begin{aligned}
\overline{\overline{A}} &= \overline{\overline{A}} \cap U && \cap \text{ identity} \\
&= U \cap \overline{\overline{A}} && \cap \text{ commutatvity} \\
&= (A \cup \overline{A}) \cap \overline{\overline{A}} && \cup \text{ complement} \\
&= (A \cap \overline{\overline{A}}) \cup (\overline{A} \cap \overline{\overline{A}}) && \cap \text{ distributivity} \\
&= (A \cap \overline{\overline{A}}) \cup \overline{(A \cup \overline{A})} && \cup \text{ De Morgan's law} \\
&= (A \cap \overline{\overline{A}}) \cup \overline{U} && \cup \text{ complement} \\
&= (A \cap \overline{\overline{A}}) \cup \emptyset && \text{question (3)} \\
&= (A \cap \overline{\overline{A}}) \cup (A \cap \overline{A}) && \cap \text{ complement} \\
&= A \cap (\overline{\overline{A}} \cup \overline{A}) && \cap \text{ distributivity} \\
&= A \cap (\overline{A} \cup \overline{\overline{A}}) && \cup \text{ commutativity} \\
&= A \cap U && \cup \text{ complement} \\
&= A && \cap \text{ identity}
\end{aligned}
$$