CS054: Relations

The goal of this worksheet is to give you practice with relations, functions, and their properties. It's not for a grade—no need to turn it in! I'll post solutions, but you'll get the most out of it if you don't peek.

1. What's a descriptive name for the following relation $T \subseteq \texttt{bool} \times \texttt{bool}$?

$$T = \{(\top, \top), (\top, \bot), (\bot, \bot)\}$$

Answer:

Solution: $a\ T\ b$ when a is "at least as true as" b.

2. Construct a relation that is reflexive but not symmetric. It can be on any pair of sets you like.

Answer:

Solution: There's an infinity of possibilities here! The T relation above is one.

3. Construct a relation that is reflexive but not transitive. It can be on any pair of sets you like.

Answer:

Solution: Again, there are lots of ways to go here. On RPS, we might say

$$U = \{(\mathsf{rock}, \mathsf{rock}), (\mathsf{paper}, \mathsf{paper}), (\mathsf{scissors}, \mathsf{scissors}), (\mathsf{rock}, \mathsf{paper}), (\mathsf{paper}, \mathsf{scissors})\}.$$

Note that U is reflexive, but we don't have $\mathsf{rock}\ U\ \mathsf{scissors}$ as transitivity would demand.

4. Prove that the symmetric closure of a relation $R \subseteq A \times A$ is symmetric.

   Proof:

   Solution: We have:

$$
\begin{aligned}
R \cup R^{-1} &= \{(a,b) \mid (a,b) \in R \vee (a,b) \in R^{-1}\} \\
&= \{(a,b) \mid (a,b) \in R \vee (b,a) \in R\}
\end{aligned}
$$

   Let $(a,b) \in R \cup R^{-1}$ be given. Either $(a,b) \in R$—in which case $(b,a) \in R^{-1}$—or vice versa. QED

5. Write a relation $R \subseteq \mathbb{N} \times \mathbb{N}$ that is total but not deterministic.

   Answer:

   Solution: One example is less than or equal to: every natural number is less than or equal to some other natural number (total), but also to many others (not deterministic).

6. Write a relation $R \subseteq \mathbb{N} \times \mathbb{N}$ that is deterministic but not total.

   Answer:

   Solution: One example is true predecessor: every natural number has a unique predecessor... except for 0.

7. Prove that $\texttt{map}(f, \texttt{map}(g, l)) = \texttt{map}(f \circ g, l)$.

Proof:

Solution: By induction on l.

($l = \texttt{[]}$) Both sides yield `[]` immediately.
($l = x :: l'$) We must show that $\texttt{map}(f, \texttt{map}(g, x :: l')) = \texttt{map}(f \circ g, x :: l')$; we have as our IH that $\texttt{map}(f, \texttt{map}(g, l')) = \texttt{map}(f \circ g, l')$. We compute:

$$\begin{array}{rcll} \texttt{map}(f, \texttt{map}(g, x :: l')) & = & \texttt{map}(f, g(x) :: \texttt{map}(g, l')) & \\ & = & f(g(x)) :: \texttt{map}(f, \texttt{map}(g, l')) & \\ & = & (f \circ g)(x) :: \texttt{map}(f, \texttt{map}(g, l')) & \\ & = & (f \circ g)(x) :: \texttt{map}(f \circ g, l')) & \text{(IH)} \\ & = & \texttt{map}(f \circ g, x :: l')) & \end{array}$$

8. Prove that if $f : A \to B$ is a bijection, then $f^{-1} : B \to A$ and is also a bijection. (Some theorems from the book will help, but you'll learn the most if you do it all by hand.)

Proof:

Solution: Let bijective $f : A \to B$ be given. We have, by definition, $f^{-1} = \{(b, a) \mid f(a) = b\}$.

We must first show that f^{-1} is indeed a function; then we must show that it is injective and surjective, i.e., bijective.

First f^{-1} is total because f is surjective: $\forall b \in B, \exists a \in A, f(a) = b$, and so $(b, a) \in f^{-1}$. We can find at least one a for each b.

Next, f^{-1} is deterministic because f is injective: $\forall a_1, a_2 \in A, f(a_1) = f(a_2) \Rightarrow a_1 = a_2$. Since $(b, a_1) \in f^{-1}$ iff $(a_1, b) \in f$ (and similarly for a_2), we know that $(b, a_1), (b, a_2) \in f^{-1}$ implies that $a_1 = a_2$. So f^{-1} is deterministic.

Having concluded that f^{-1} is a function, we need to show that it's injective and surjective.

To see that f^{-1} is injective, recall that f is deterministic: $\forall a \in A, b_1, b_2 \in B, (a, b_1) \in f \wedge (a, b_2) \in f \Rightarrow b_1 = b_2$. So if (b_1, a) and $(b_2, a) \in f^{-1}$, then $b_1 = b_2$—i.e., $\forall b_1, b_2 \in B, f^{-1}(b_1) = f^{-1}(b_2) \Rightarrow b_1 = b_2$.

Finally, we find that f^{-1} is surjective from the fact that f is total: $\forall a \in A, \exists b \in B, (a, b) \in f$, i.e., for any a, we can find a $b \in B$. Since $(a, b) \in f$ implies $(b, a) in f^{-1}$, we know that $\forall a \in A, \exists b \in B, f^{-1}(b) = a$, i.e., f^{-1} is surjective. QED