CS054: Choice

The goal of this worksheet is to give you practice with the "choice" operation $\binom{n}{k}$ and develop your combinatorics intuitions, particularly with regard to the Binomial Theorem. I hope you'll enjoy the pretty picture, and maybe even have your curiosity tweaked. It's not for a grade—no need to turn it in! I'll post solutions, but you'll get the most out of it if you don't peek.

1. Compute $\binom{5}{3}$, showing your work.

   Solution:

   $$\binom{5}{3} = \frac{5!}{3! \cdot (5-3)!} = \frac{5!}{3! \cdot 2!} = \frac{5 \cdot 4 \cdot 3}{3 \cdot 2 \cdot 1} = \frac{60}{6} = 10$$

2. Reduce $\binom{100}{17}$. No need to calculate an actual number, but write something simpler.

   Solution:

   $$\binom{100}{17} = \frac{100!}{83! \cdot 17!} = \frac{100 \cdot \cdots \cdot 84}{17!} = \frac{P(100,17)}{17!}$$

3. How many strings of zeros and ones (*bitstrings*) are there that contain exactly a ones and b zeros?

   Solution:

   $$\binom{a+b}{b} = \binom{a+b}{a}$$

   Briefly, what's the underlying idea/counting method?

   You have n+k slots. Choose k for ones; the rest are zeros.

4. How many bitstrings are there that contain a ones and b zeroes such that no two ones are adjacent?

   Solution:

   $$\binom{b+1}{a}$$

   Briefly, what's the underlying idea/counting method?

   Fix the zeros. There are $b+1$ "gaps" for a one; choose a of them.

5. There are $\binom{n}{k}$ ways to choose k objects from n objects without replacement, i.e., without ever choosing the same object twice. How many ways are there to choose k objects from n objects *with* replacement, i.e., you can choose the same thing multiple times? For example, to choose three things from the set $\{1, 2, 3, 4, 5\}$, one could choose $1, 2, 3$ or $1, 1, 5$ or $1, 1, 1$—to name a few.

   Write a formula for the number of ways to choose k from n with replacement.

   Solution:

   $$\binom{n+k-1}{k}$$

   In a few sentences, what's the underlying idea/counting method?

   Solution: In addition to the n objects, add $k-1$ slots as "possible duplicates". We want $k-1$ because having k of them wouldn't make sense: all duplicates of *what*?

   Why *not* $\binom{k \cdot n}{k}$? That would distinguish between different choices of replacement.

6. Please look at the following picture carefully. It's "Pascal's Triangle", named after Blaise Pascal (though this triangle and its identities were well known before Pascal and outside of Europe—Pascal popularized it in European mathematics).

Row 0								1						
Row 1							1		1					
Row 2						1		2		1				
Row 3					1		3		3		1			
Row 4				1		4		6		4		1		
Row 5			1		5		10		10		5		1	
Row 6		1		6		15		20		15		6		1

List five things you notice about this triangle. It might be something interesting or something banal—whatever you notice is good.

1. **Sides are all 1s**

2. **Inner triangle is row count**

3. **Symmetrical**

4. **Each number is the sum of the two above it**

5. **Row n sums to 2^n**

What do you think the next row holds?

1 7 21 35 35 21 7 1

Can you give a formula for the kth entry of the nth row?

Solution: $\binom{n}{k}$

In light of the formula above, can you take any of the things you noticed and come up with a related theorem or property?

Solution:

1. $\binom{n}{0} = \binom{n}{n} = 1$
2. $\binom{n}{1} = \binom{n}{n-1} = n$ when $n \geq 1$
3. $\binom{n}{k} = \binom{n}{n-k}$
4. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ for $n \geq 1$ and $k \leq n$
5. $\sum_{k=0}^{n} \binom{n}{k} = 2^n$