Lecture 5: Lambda Calculus

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Pure Lambda Calculus

- Terms of pure lambda calculus
  - M ::= v | (M M) | λv. M
  - Impure versions add constants, but not necessary!
  - Turing-complete
- Left associative: M N P = (M N) P.
- Computation based on substituting actual parameter for formal parameters

Free Variables

- Substitution easy to mess up!
- Def: If M is a term, then FV(M), the collection of free variables of M, is defined as follows:
  - FV(x) = { x }
  - FV(M N) = FV(M) ∪ FV(N)
  - FV(λv. M) = FV(M) - {v}

Substitution

- Write [N/x] M to denote result of replacing all free occurrences of x by N in expression M.
  - [N/x] x = N,
  - [N/x] y = y, if y ≠ x,
  - [N/x] (L M) = ([N/x] L) ([N/x] M),
  - [N/x] (λy. M) = λy. ([N/x] M), if y ≠ x and y ∉ FV(N),
  - [N/x] (λx. M) = λx. M.

Computation Rules

- Reduction rules for lambda calculus:

(α) λx. M → λy. ([y/x] M), if y ∉ FV(M).

(β) (λx. M) N → [N/x] M.

(η) λx. (M x) → M. *Optional rule*

Why so complicated?

illegal substitution

$\underline{(\lambda f.\ \lambda z.\ f\ (f\ z))\ (\lambda x.\ x + z)} \to \lambda z.\ (\lambda x.\ x + z)(\underline{(\lambda x.\ x + z)\ z})$

$\to \lambda z.\ \underline{(\lambda x.\ x + z)(z + z)}$

$\to \lambda z.\ (z + z) + z = \lambda z.\ 3z.$

- *rather than the correct*

$\underline{(\lambda f.\ \lambda y.\ f\ (f\ y))\ (\lambda x.\ x + z))} \to \lambda y.\ (\lambda x.\ x + z)(\underline{(\lambda x.\ x + z)\ y})$

$\to \lambda y.\ \underline{(\lambda x.\ x + z)(y + z)}$

$\to \lambda y.\ (y + z) + z = \lambda y.\ y + 2z.$

Normal Forms

- A term M is in normal form if no reduction rules apply, even after applications of α.
- Not all terms have normal forms
  - $\Omega = (\lambda x.\ (x\ x))(\lambda x.\ (x\ x))$

How to evaluate

- Many strategies:
  - $(\lambda x.\ x + 32)(\underline{(\lambda y.\ y * 3)\ 5}) \rightarrow \underline{(\lambda x.\ x + 32)\ 15} \rightarrow 47$ *Inside-out*
  - versus
  - $\underline{(\lambda x.\ x + 32)((\lambda y.\ y * 3)\ 5)} \rightarrow (\underline{(\lambda y.\ y * 3)\ 5}) + 32 \rightarrow 47$ *Outside-in*
- Confluence: If M can be reduced to a normal form, then there is only one such normal form.
- However, not all strategies give a normal form:
  - $(\lambda x.\ 47)\ \Omega$

Computability

- Can encode all computable functions in pure untyped lambda calculus.
  - $\underline{\text{true}} = \lambda\ u.\ \lambda\ v.\ u$
  - $\underline{\text{false}} = \lambda\ u.\ \lambda\ v.\ v$
  - $\underline{\text{cond}} = \lambda\ u.\ \lambda\ v.\ \lambda\ w.\ u\ v\ w$

Lambda Encoding

- Pairing:
  - $\underline{\text{Pair}} = \lambda\ m.\ \lambda\ n.\ \lambda\ b.\ \underline{\text{cond}}\ b\ m\ n.$
  - $\underline{\text{fst}} = \lambda p.\ p\ \underline{\text{true}}$
  - $\underline{\text{snd}} = \lambda p.\ p\ \underline{\text{false}}$

Encoding Integers

- Integers:
  - $\underline{0} = \lambda\ s.\ \lambda\ z.\ z.$
  - $\underline{1} = \lambda\ s.\ \lambda\ z.\ s\ z.$
  - $\underline{2} = \lambda\ s.\ \lambda\ z.\ s\ (s\ z).$
- Integers encode repetition:
  - $\underline{2}\ f\ x = f\ (f\ x)$
  - $\underline{n}\ f\ x = f^{(n)}\ (x)$