Lecture 20: Soundness & Completeness of Predicate Logic

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Soundness & Completeness

- Soundness Theorem.
  - If $\Gamma \vdash \psi$, then $\Gamma \vDash \psi$.
  - If Γ is satisfiable, then Γ is consistent.
- Completeness Theorem.
  - If $\Gamma \vDash \psi$, then $\Gamma \vdash \psi$.
  - If Γ is consistent, then Γ is satisfiable.
  - *This is different meaning of complete from that of theory!*

Useful Lemmas

- Lemma: For all terms t and u, models $\mathcal{M}$, and lookup tables ℓ such that ℓ provides meanings for all free variables in t and u, then $t[u/x]^{\mathcal{M}}{}_{\ell} = t^{\mathcal{M}}{}_{\ell'}$ where $\ell' = \ell[x \to u^{\mathcal{M}}{}_{\ell}]$
- Lemma: For all terms u, wffs ϕ, models $\mathcal{M}$, and lookup tables ℓ such that ℓ provides meanings for all free variables in u and ϕ, then $\mathcal{M},\ell \vDash \phi[u/x]$ iff $\mathcal{M},\ell' \vDash \phi$ where $\ell' = \ell[x \to u^{\mathcal{M}}{}_{\ell}]$

Soundness

- Suppose $\Gamma \vdash \psi$. Show for all $\mathcal{M},\ell$ if $\mathcal{M},\ell \vDash \Gamma$ then $\mathcal{M},\ell \vDash \psi$.
- Proof (sketch!) by induction on length n of proof
  - n = 1, then step is premise -- easy.
  - Suppose true for proofs of length k < n, prove for n
  - Cases depending on rules -- skip prop. logic rules
  - =i, easy

Proof of Soundness

- =e. Suppose $t = u$, $\phi[t/x]$ are earlier steps of proof. By induction, $\mathcal{M}, \ell \vDash t = u, \phi[t/x]$.
  - Thus $t_\ell^\mathcal{M} = u_\ell^\mathcal{M}$ and $\mathcal{M}, \ell' \vDash \phi$ for $\ell' = \ell$ except that $\ell'(x) = t_\ell^\mathcal{M}$.
    - By the lemma, $\mathcal{M}, \ell \vDash \phi[t/x]$ iff $\mathcal{M}, \ell' \vDash \phi$ by induction on size of ϕ
  - But since $t_\ell^\mathcal{M} = u_\ell^\mathcal{M}$ it follows that $\mathcal{M}, \ell \vDash \phi[u/x]$
- $\forall$e, $\exists$i, easy

More Soundness Proof

- $\exists$e, $\forall$i similar. Do $\exists$e.
  - Suppose have proof of χ using $\exists$e. Then $\Gamma \vdash \exists x\, \phi$ and $\Gamma \cup \{\phi(x_o)\} \vdash \chi$ where x_o is new. By induction, $\mathcal{M}, \ell \vDash \exists x\, \phi$
  - Thus there is d s.t. $\mathcal{M}, \ell' \vDash \phi$ for $\ell' = \ell$ except that $\ell'(x) = d$.
  - Now let $\ell'' = \ell$ except that $\ell''(x_o) = d$. Thus $\mathcal{M}, \ell'' \vDash \phi(x_o)$
  - But $\mathcal{M}, \ell'' \vDash \Gamma \cup \{\phi(x_o)\}$ & $\Gamma \cup \{\phi(x_o)\} \vdash \chi$ in shorter proof.
  - By induction, $\mathcal{M}, \ell'' \vDash \chi$ and hence $\mathcal{M}, \ell \vDash \chi$ since χ not have x_o free.

Completeness

- Rather than proving if $\Gamma \vDash \psi$, then $\Gamma \vdash \psi$, instead prove if Γ’ is consistent, then Γ’ is satisfiable.
  - Spose Γ’ is consistent implies Γ’ is satisfiable.
  - If $\Gamma \vDash \psi$ then $\Gamma \cup \{\neg\psi\}$ is unsatisfiable.
  - Hence, by above, $\Gamma \cup \{\neg\psi\}$ is inconsistent
  - By ¬i and ¬¬e, $\Gamma \vdash \psi$

Completeness

- Show if Γ is consistent, then Γ is satisfiable.
- Proof sketch: Add an infinite number of constants to language and “Skolem functions” to provide witnesses for all existentials. Show that can define a model that satisfies everything in Γ.
- Originally proved by Kurt Gödel in 1929 Ph.D. dissertation. Nicer proof by Henkin in 1950 Ph.D. dissertation

Undecidability

- Validity hard! To show $\Gamma \vDash \phi$, must show all models of Γ also satisfy ϕ.
- Natural deduction gives semi-decidability by soundness and completeness.

Compactness

- Let Γ be a (possibly infinite) set of sentences of predicate logic. If all finite subsets of Γ are satisfiable, the Γ itself is satisfiable.
  - Proof: Suppose Γ not satisfiable. Therefore Γ not consistent. Write proof of $\bot$.
    Formal proof uses only finite set $\Gamma_0 \subseteq \Gamma$.
    Therefore Γ_0 not consistent, and hence not satisfiable.
- Many important applications

Applications of compactness

- Theorem (Löwenheim-Skolem Theorem):
  Let ψ be a sentence of predicate logic such that for any natural number $n \geq 1$ there is a model of ψ with at least n elements. Then ψ has a model with infinitely many elements.