Homework 5

Due midnight, Thursday, 2/23/2012

Please submit your homework solutions online at `http://www.dci.pomona.edu/tools-bin/cs081upload.php`. If you have more than one file to be turned in, please put it in a folder and zip it up before turning it in.

Problems from the texts are given in the form c.n where c is the chapter and n is the problem number. Thus problem 2.7 is problem 7 from Chapter 2.

1. (20 points) **Unique Readability**

   Consider the following function on symbols from the alphabet $\mathcal{A}$ of propositional logic.

   $K(() = -2$
   $K()) = 2$
   $K(p) = 1$, for a proposition letter p
   $K(\top) = K(\bot) = 1$
   $K(\neg) = 0$
   $K(\wedge) = K(\vee) = K(\rightarrow) = -1$

   We extend the function to strings of symbols, $K^* : \mathcal{A}^* \rightarrow N$ in the natural way: $K^*(\epsilon) = 0$ and $K^*(as) = K(a) + K^*(s)$. You may use without proof the fact that $K^*(st) = K^*(s) + K^*(t)$.

   (a) Prove by induction that if ϕ is a formula of propositional logic, then $K^*(\phi) = 1$.

   (b) A string s is a *proper prefix* of a formula u if there is a non-empty string t such that $st = u$. Prove that $K^*(s) < 1$ for any string s which is a proper prefix of some formula.

   (c) Conclude that a formula cannot be a proper prefix of another formula. Informally explain why this fact tells us that there is only one way to parse a formula. The latter property is called "unique readability."

   (d) Give an example of a strings s and u such that s is a proper prefix of u and $K^*(s) = K^*(u) = 1$ Why does your example not contradict the conclusions above?

2. (15 points) **Constructive Proofs** Write careful box and line proofs for the following inferences. Use only the constructive rules for parts a through c. For this problem (only), restrict yourself to the *basic* rules for natural deduction in Figure 1.2 in Huth and Ryan's book.

   (a) $\phi \rightarrow \psi, \psi \rightarrow \chi \vdash \phi \rightarrow \chi$

   (b) $\phi \rightarrow (\psi \rightarrow \chi) \vdash (\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \chi)$

   (c) $\neg(\phi \wedge \psi), \phi \vdash \neg\psi$

   (d) $\vdash (\phi \rightarrow \neg\phi) \rightarrow \neg\phi$

   (e) $\vdash (\neg\phi \rightarrow \phi) \rightarrow \phi$

3. (10 points) **True and False**

   Exactly three of the following formulas are constructively provable. Identify those three and give constructive proofs of them. Then derive $\bot$ from the remaining formula.

   (a) $\top \rightarrow \neg\bot$

(b) $\top \to \bot$

(c) $\bot \to \top$

(d) $\bot \to \neg\bot$

4. (15 points) **Classical Rules**

Show that the following rules of inference are all equivalent to one another, in the sense that adding one to the constructive rules gives a logical system that proves the same theorems as adding any other to the constructive rules.

(a) $\dfrac{\neg\neg\phi}{\phi}\,\neg\neg\text{e}$

(b) $\dfrac{}{\neg\phi \vee \phi}\,\text{LEM}$

(c) $\dfrac{\phi \to \psi \quad \neg\phi \to \psi}{\psi}\,\text{Cases}$

(d) $\dfrac{\phi \to \psi}{\neg\phi \vee \psi}\,\text{CDI}$

(e) PBC

Suggestion: For two rules, X and Y, show that X can be simulated by the constructive rules and Y. This will show that any theorem provable with X is also provable with Y. Create a cycle of five such proofs that run through all the rules. You may choose any order for the cycle, as long as there is a loop with all five rules.

LEM, which stands for "law of the excluded middle," is standard jargon. PBC is used by Huth and Ryan for "proof by contradiction." CDI stands for "classical definition of implication"; it is made up especially for this problem.

5. (5 points) **Contrapositive**

Recall that the *contrapositive* of $\phi \to \psi$ is $\neg\psi \to \neg\phi$. Often, a proof by contradiction can be recast as a proof by contraposition. The equivalence of an implication and its contrapositive is an important tool for mathematicians.

(a) Give a constructive proof of $\phi \to \psi \vdash \neg\psi \to \neg\phi$.

(b) Give a classical proof of $\neg\psi \to \neg\phi \vdash \phi \to \psi$.

(c) Show constructively that $(\neg\psi \to \neg\top) \to (\top \to \psi) \vdash \neg\neg\psi \to \psi$. This shows that there cannot be a constructive proof of the result in part b. Hint: Prove the formula $\neg\neg\psi \to (\neg\psi \to \neg\top)$.

6. (10 points) **De Morgan laws**

In class, we gave a constructive proof of one direction of a De Morgan law, $\phi \vee \psi \vdash \neg(\neg\phi \wedge \neg\psi)$. Is the converse, $\neg(\neg\phi \wedge \neg\psi) \vdash \phi \vee \psi$, constructively provable? If so, give a constructive proof. If not, give a classical proof and show how to derive the classical contradiction rule (or an equivalent rule from Problem 4) from it.