Homework 10

Due midnight, Thursday, 4/12/2012

Please submit your homework solutions online at `http://www.dci.pomona.edu/tools-bin/cs081upload.php`. If you have more than one file to be turned in, please put it in a folder and zip it up before turning it in.

1. (5 points) **Variants of TM's**

   Another useful variant of a Turing machine has a single tape, but two reading heads. That is, the two reading heads can be looking at two cells of the tape at any time. Thus one of these turing machines makes a transition based on the current state and the contents under each of the two read heads. It then writes something under each read head and moves each of the heads.

   (a) Give a formal definition of a two head Turing machine, define a configuration of this machine, and give the definition of $\vdash_M$ for this machine.

   (b) Describe informally a two-head Turing machine that decides the language $L = \{ww \mid w \in \{a,b\}^*\}$.

2. (10) **Closure Properties of TMs**

   Show that the decidable languages are closed under intersection and set difference. I.e., if L_1 and L_2 are decidable, then so are $L_1 \cap L_2$ and $L_1 - L_2$

3. (20) **Partial Recursive Functions**

   In lecture 20, we defined partial recursive functions (due to Kleene in 1936). The primitive recursive functions are those functions definable from the primitive functions using composition and primitive recursion. That is, you can use anything but minimization.

   A simple example of a primitive recursive function is the sum of two numbers:

   $plus(0, n) = n$
   $plus(m + 1, n) = S(plus(m, n))$

   where S is the successor function.

   We were a bit sloppy in the above definition. A more careful definition conforming to the rules for primitive recursive functions would have been:

   $plus(0, n) = P^1_1(n)$
   $plus(m + 1, n) = h(m, plus(m, n), n)$

   where $h(x, y, z) = S(P^3_2(x, y, z))$, which is itself defined by composition from S and $(P^3_2$.

   However, I'll allow you to be sloppy as in the first definition of `plus` in your solution to this problem.

   (a) Please give a primitive recursive definition of multiplication. You can use the above definition for *plus* in the definition.

   (b) Give a proof by induction that all primitive recursive functions are total. I.e., if f is an k-ary primitive recursive function and $n_1, \ldots, n_k$ are natural numbers, then $(f(n_1, \ldots, n_k)$ converges to a value.

   Hint: When doing the case for primitive recursion, do another inductive proof on the values of the first argument.

(c) Explain briefly why the set of all primitive recursive functions is countable. A very informal proof is fine.

(d) Use a diagonalization argument (like that which shows that the set of reals is uncountable) to show that there is a computable unary function from the natural numbers to natural numbers that is *not* primitive recursive.

Hint: First define the function using diagonalization, then explain informally why it is computable. You don't need to define a TM, just give an informal algorithm for computing values. Finally, show that it is different from all of the primitive recursive functions.

(e) What properties of the primitive recursive functions did you need to show the previous part of this exercise? State as general a theorem as you can that generalizes the previous part (i.e., your theorem shouldn't mention primitive recursive functions, just the properties you used in your proof).