Countability and Uncountability

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Outline

Reminder

- If f is an injection from A to B, $|A| <= |B|$
  - We can pick a different output for each input...
  - so there are at least as many outputs as inputs
- If f is a surjection from A to B, $|A| >= |B|$
  - We can hit every output with some input...
  - so we have at least as many inputs as outputs
- If f is a bijection from A to B, $|A| = |B|$
  - Greater-or-equal $\wedge$ less-or-equal is just equal

Practice

- Claim: the left set has the same cardinality as the one on the right.
  - Prove it by finding an injection and a surjection (or a single bijection) to some third set (maybe the nats!), and use a transitivity argument
- |Perfect squares| = |powers of two|
- |Pairs of natural numbers| = |number of possible finite-length bitstrings|
- |Bool -> Bool -> Nat| ? |natural numbers|
  - Hint: How many possible pairs of inputs can this function take? What is this question really asking?

What isn't Countable?

- Sets are countable if their cardinality is $<=$ that of N
- Is any set bigger than the natural numbers?

The Real Numbers

- Review:
  - Natural numbers (N): "counting numbers"
  - Integers (Z), "Zahlen": positive and negative natural numbers
  - Rationals (Q): Ratio between two integers, as simplified as possible
    - These are a subset of the pairs of integers, so they're definitely countable
  - Irrationals (no fun letter): Numbers that can't be represented as ratios of integers, e.g. pi, e, $\sqrt{2}$, ...
- Reals (R): Rationals $\cup$ Irrationals
  - Numbers described as infinite sequences of digits

Are the reals countable?

- Are the reals countably infinite?
  - We could try to find a bijection with natural numbers...
  - spoiler: we can't.
- Let's use a proof by contradiction:
  - Suppose the reals are countable, i.e. $|R| <= |N|$ (or equivalently $|N| >= |R|$)
  - In fact, let's focus on the reals between 0 and 1, not including 1.
    - If that range is bigger than N, then surely all of R is also bigger than N.
  - Then there must be a surjection f : N -> R{0..1}, which enumerates every real between 0 and 1 without missing any.
  - We'll show that leads to a contradiction.

Cantor's Diagonal Argument

Here is an example of a surjection from N->R{0..1}.
This specific example is just to illustrate a gimmick. We don't actually care what f is, all functions f will have the same problem.

x		y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	...
0	0.	0	0	0	1	1	1	0	0	...
1	0.	1	1	0	1	1	0	1	1	...
2	0.	2	3	0	1	4	3	2	1	...
3	0.	0	0	1	1	2	5	5	2	...
4	0.	5	9	2	1	8	9	3	1	...
5	0.	9	8	2	4	5	4	1	0	...

Gimmick Time

Let's name a number g (g is for gimmick!). G is a real number between 0 and 1, and it's defined like this:

1. Its only digit before the decimal is 0.
2. Its first digit after the decimal (its "0th digit") is the first digit after the decimal of whatever number f(0) is, plus 1 (wrapping around to 0 if the result is 10), i.e. $f(0)_0 + 1 \mathrm{mod} 10$.
3. Its second decimal digit (its "1st digit") is the second decimal digit of f(1), plus one, mod 10.
4. And so on: $g_n = (f(n)_n + 1) \bmod 10$

The Contradiction

Since g is a real number, and f is a surjection, there must be some number k so that f(k) = g.
We know from the definition of g that g's kth digit must be different from f(k)'s kth digit.
But g=f(k)! This is a contradiction, so either g isn't a real number or f can't be a surjection.
g is definitely a real, so f must not be a surjection. That means that there is no surjection from N to our subset of R, so our subset must be strictly bigger than N.

Another diagonal proof

- Are infinite bitstrings countably infinite?
- Let's use a proof by contradiction:
  - Suppose that the set of infinite bitstrings is countable, i.e. |B| <= |N| (or equivalently |N| >= |B|)
  - Then there must be a surjection f : N -> B, which enumerates every bitstring without missing any.
  - We'll show that leads to a contradiction.

Here is an example of a surjection from N->B.

x	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	...
0	0	0	0	1	1	1	0	0	...
1	1	1	0	1	1	0	1	1	...
2	1	0	0	1	0	1	0	1	...
3	0	0	1	1	0	1	1	0	...
4	1	1	1	1	1	1	1	1	...
5	0	0	0	0	0	0	1	0	...

Gimmick Time

Let's name a number g (g is for gimmick!). G is an infinite bitstring defined so that g(n) = ...

(Remember: Our gimmick for reals was $g(n) = (f(n)_n + 1) \bmod 10$)

Gimmick Time

G is an infinite bitstring defined so that g(n) = 1 - $f(n)_n$
Which leads to a contradiction because. . .

The Contradiction

Which leads to a contradiction because there must be some k so that f(k) = g, since f is a surjection.
But then g(k) differs from itself at index k: $g_k = 1 - f(k)_k$, but $f(k) = g$, so $g_k = 1 - g_k$.
This is a contradiction; but g is definitely an infinite bitstring.
So our assumption that such a surjection f exists is impossible, so the set of infinite bitstrings must be uncountably infinite.

Final thoughts

Using the same techniques we saw earlier, we can prove lots of stuff:

- There are as many real numbers as there are reals between 0 and 1
- There are as many reals as there are pairs of reals
- The set of functions Bool -> N is countably infinite
- The set of functions N -> Bool is uncountably infinite
- . . . and more!
- Intuition: Infinitely large sets of infinitely described objects are uncountably infinite
  - Infinitely large sets of finitely described objects are countably infinite

Other weird stuff

- Rationals are dense: between any two rationals are infinitely many rationals
- Reals are also dense
- Between any two rationals are infinitely many reals
  - Sure, all rationals are also reals
- Between any two reals are infinitely many rationals
  - ...
  - Even though there are uncountably many reals and countably many rationals!
- Rationals form (countably) infinitely many points on the number line
  - but this doesn't give you a continuum of numbers!

Other countability techniques

- Instead of proving a function f exists from A -> N...
  - Find a function N -> A
  - Find a bijection or injection A -> B where we know B is countably infinite
- Instead of doing a diagonal proof for uncountability...
  - Find a function A -> R which is a bijection
  - Find a function R -> A which is an injection
    - "A is at least as big as R"