Countability and Uncountability

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Outline

The Story So Far

- We counted with our fingers when our sets were small
  - The most natural of natural numbers
- Now we want to count sets we don't want to enumerate
  - So today, we're counting with functions

Definitions

- Injectivity/on: $\forall xy, f(x) = f(y) \Rightarrow x = y$
  - "Every input has a distinct output"
  - "One to one"
- Surjectivity: $\forall y, \exists x, f(x) = y$
  - "Every output is reached by some input"
  - "Onto"

Counting with Functions

- If f is an injection from A to B, $|A| <= |B|$
  - We can pick a different output for each input...
  - so there are at least as many outputs as inputs
- If f is a surjection from A to B, $|A| >= |B|$
  - We can hit every output with some input...
  - so we have at least as many inputs as outputs
- If f is a bijection from A to B, $|A| = |B|$
  - Greater-or-equal $\wedge$ less-or-equal is just equal

Practice

- Which set is bigger?
  - Prove it by finding a function (either from A->B or B->A)...
  - and proving it is injective/surjective.
- $|\{T,F\}|$? $|\{1,2,3\}|$
- $|\text{Bool} \times \text{Bool}|$? $|\{1,2,3\}|$
- |Bitstrings of length 8| ? |Alphanumeric strings of length 1|
- |Bool -> Bool| ? $|\text{Bool} \times \{1,2,3\}|$
  - We can even count functions with functions

Comparing Arbitrary Sets

We can also handle sets whose contents we don't even know!
Imagine we have sets A and B. We know $|A| <= |B|$.
Define a function f to show that $|A - B| <= |B|$.
Hint: Use the fact that $|A| <= |B|$ to find a function g you can use in your definition of f!
Hint: You can also check whether the input argument is a member of A or B in a piecewise function definition.

Counting Sets

- We can compare cardinalities of arbitrary sets using functions
- Some sets are infinite
- But... sets are sets, right?

Principle

- Which set is bigger: the positive integers (1 and up) or the non-negative integers (0 and up)?
  - Well, Z^+ is a strict subset of Z_0^+. Case closed?

Example

- Observe: f(pos) = pos - 1
  - $f : Z^+ \rightarrow Z_0^+$
  - f is a bijection
    - f(x) = f(y) => x = y, since x-1=y-1 => x=y.
    - forall y, exists x f(x) = y; if x=y+1, f(y+1) = y+1-1 = y.
- So they're... the same cardinality!?
  - $Z^+ <= Z_0^+$, and:
  - $Z^+ >= Z_0^+$

"Countably Infinite"

- Any set S where $|S| = |N|$ is "countably infinite"
  - If it's just $<= |N|$ it's "countable"
- All countably infinite sets therefore have the same cardinality!
- Let's play with this a bit...

Practice

- Claim: the left set has the same cardinality as the one on the right.
  - Prove it by finding an injection A->B and a surjection A->B
  - or a single bijection A->B
  - or an injection A->B and an injection B->A (so A <= B and B <= A)
  - or a surjection A->B and a surjection B->A (so A >= B and B >= A)
  - or a bijection B->A
- $|\text{Natural numbers}| = |\text{even numbers}|$
- $|\text{negative integers}| = |\text{positive integers}|$

Remember

f : A -> B might give us an inequality between A and B:

- If it's injective, we get |A|<=|B|
- If it's surjective, we get |A|>=|B|

This works for g : B -> A too!

- If it's injective, we get |B|<=|A|
- If it's surjective, we get |B|>=|A|

Pick the functions with the properties that give you the inequalities you want!
Also, since we're looking at inequalities or equalities, we can use all the stuff we know already about reflexivity, transitivity, etc.