csci54 – discrete math & functional programming
recurrence relations, (strong) induction

proofs

- logic

- proof techniques so far
  - direct proofs
  - proof of the contrapositive
  - proof by example / disproof by counterexample
  - using cases
  - induction

- today:
  - more induction, including strong induction (for when regular induction is not enough)

Proofs by induction

Definition 5.1: Proof by mathematical induction.

Suppose that we want to prove that $P(n)$ holds for all $n \in \mathbb{Z}^{\geq 0}$. To give a *proof by mathematical induction* of $\forall n \in \mathbb{Z}^{\geq 0} : P(n)$, we prove two things:

1. the *base case:* prove $P(0)$.
2. the *inductive case*: for every $n \geq 1$, prove $P(n-1) \Rightarrow P(n)$.

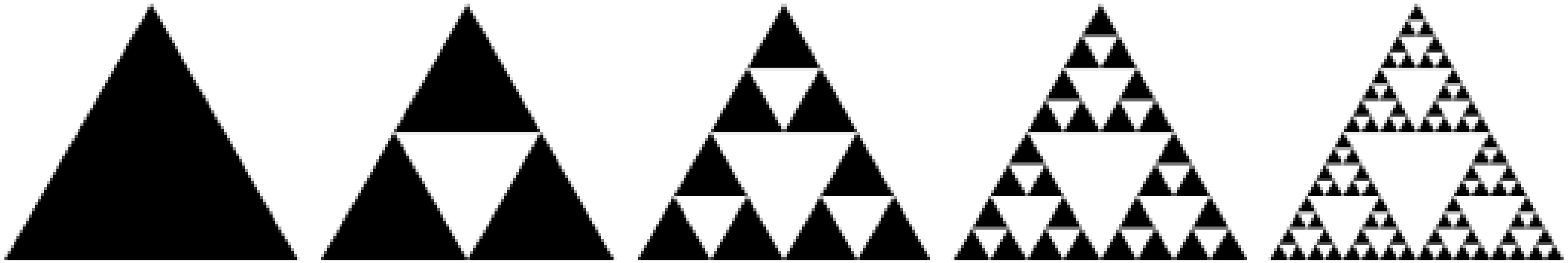
- we prove the claim using a proof by induction on:

- base case:
- inductive hypothesis (IHOP):
- inductive step:

- therefore by the principle of mathematical induction:

Recurrence relations

- Consider Sierpinski's triangle.



- Let T(n) be the number of filled triangles in a Sierpinski's triangle after n iterations, where T(0) is a single filled triangle.
- Observation: $T(n) = 3T(n-1)$ where $T(0)=1$
- Claim: $T(n) = 3^n$

Recurrence relations

- A function that is defined in terms of itself

- How would you prove that A(n) is odd for all N for the following recurrence rela

$$A(n) = A(n-1) + A(n-2) + A(n-3)$$
$$A(0) = 1$$
$$A(1) = 1$$
$$A(2) = 3$$

- base case: A(0) is odd
- IHOP: A(n) is odd
- inductive step: wts A(n+1) is odd
  - A(n+1) = A(n)+A(n-1)+A(n-2)
  - A(n) is odd
  - now what?

Proofs by strong induction

Definition 5.1: Proof by mathematical induction.

Suppose that we want to prove that $P(n)$ holds for all $n \in \mathbb{Z}^{\geq 0}$. To give a *proof by mathematical induction* of $\forall n \in \mathbb{Z}^{\geq 0} : P(n)$, we prove two things:

1. the *base case:* prove $P(0)$.
2. the *inductive case*: for every $n \geq 1$, prove $P(n-1) \Rightarrow P(n)$.

Definition 5.10: Proof by strong induction.

Suppose that we want to prove that $P(n)$ holds for all $n \in \mathbb{Z}^{\geq 0}$. To give a *proof by strong induction* of $\forall n \in \mathbb{Z}^{\geq 0} : P(n)$, we prove the following:

1. the *base case:* prove $P(0)$.
2. the *inductive case*: for every $n \geq 1$, prove $[P(0) \wedge P(1) \wedge \cdots \wedge P(n-1)] \Rightarrow P(n)$.

Proofs by strong induction

Definition 5.10: Proof by strong induction.

Suppose that we want to prove that $P(n)$ holds for all $n \in \mathbb{Z}^{\geq 0}$. To give a *proof by strong induction* of $\forall n \in \mathbb{Z}^{\geq 0} : P(n)$, we prove the following:

1. the *base case:* prove $P(0)$.
2. the *inductive case*: for every $n \geq 1$, prove $[P(0) \wedge P(1) \wedge \cdots \wedge P(n-1)] \Rightarrow P(n)$.

- we prove the claim using a proof by strong induction on:
- base case(s):
- inductive hypothesis (IHOP):
- inductive step:
  - wts:
- therefore by the principle of mathematical induction:

may need more than one base case; need for every n where inductive step doesn't hold

IHOP: assume true for all values up to n-1

inductive step: wts true for n

Proofs by strong induction

$$A(n) = A(n-1) + A(n-2) + A(n-3)$$
$$A(0) = 1$$
$$A(1) = 1$$
$$A(2) = 3$$

claim: A(n) is odd for all N

- we prove the claim using a proof by strong induction on:
- base case(s):
- inductive hypothesis (IHOP):
- inductive step:
  - wts:
- therefore by the principle of mathematical induction:

Proofs by strong induction

$$A(n) = A(n-1) + A(n-2) + A(n-3)$$
$$A(0) = 1$$
$$A(1) = 1$$
$$A(2) = 3$$

Claim: A(n) is odd for all N

- we prove the claim using a proof by strong induction on n
- base case(s): A(0), A(1), and A(2) are all odd.
- inductive hypothesis (IHOP): A(x) is odd for all x<y
- inductive step: we want to show that A(y) is odd
  - by the IHOP we know that A(y-1), A(y-2), A(y-3) are all odd, so there exist integers a,b,c such that …
  - this means a+b+c is odd and therefore A(y) is also odd.
- therefore by the principle of mathematical induction: A(n) is odd

Strong vs. regular (weak) induction

- Anything that can be proven using regular induction can also be shown using strong induction.
- However, if you can prove something using regular induction, you should.