In-Class Worksheet

CS 181 Advanced Algorithms — Spring 2026

Definition: the integrality gap of a linear program is the ratio between the integer optimal solution value and the fractional optimal solution value:

$$\text{Integrality Gap} = \frac{OPT(IP)}{OPT(LP)}$$

Note that for a minimization problem, this ratio will be at least 1. For a maximization problem, the ratio is at most 1.

Vertex Cover

Given an instance of vertex cover where the optimal solution to the LP relaxation has a lower cost than cheapest integral solution. What does this say about the integrality gap?

What is the integrality gap of the LP relaxation for Vertex Cover on a fully connected (complete) graph on n vertices?

Independent Set

Definition. Given a graph $G = (V, E)$, an *independent set* is a subset of vertices $S \subseteq V$ such that no two vertices in S share an edge. The goal is to find the largest such set.

- Formulate the problem as an integer linear program.
- What is the integrality gap of your formulation?
- Is there a way you can add valid inequalities to strengthen the formulation?

Min-Cost Bipartite Perfect Matching (MCBPM)

Definition. Given a bipartite graph $G = (L, R, E)$ where $|L| = |R|$ and each edge $e \in E$ has an associated cost c_e, a *perfect matching* is a subset of edges M such that every vertex is incident to exactly one edge in M. The goal is to find the perfect matching of minimum total cost.

- Formulate MCBPM as an integer program. (Note: as we have seen, this particular class of integer programs can be solved in polynomial time via the Hungarian method.)

- Can you find an instance of MCBPM with a fractional solution which costs less than the optimal integral solution?