1

Introduction

- A little about me:
  - Carnegie Mellon University → Weizmann Institute
  - Hiking, guitar, volleyball.
- I love max-flows!

2

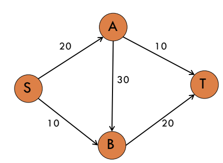
Learning Goals

- What is the max-flow problem?
- Why is it useful?
- What algorithm can we use to solve it?
- What's the connection to cuts?

3

Student networking

You decide to create your own campus network:
- You and three friends string together some network cables
- Because of capacity (due to cable type, distance, computer, etc) you can only send a certain amount of data to each person
- What is the maximum throughput you can you send from S to T?



4

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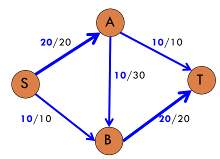
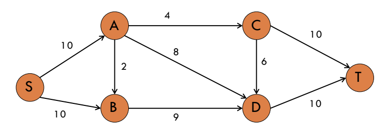


30 units

5

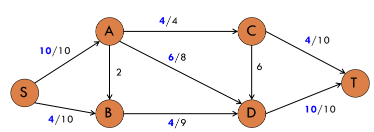
Another flow problem



How much water flow can we continually send from s to t?

6

Another flow problem



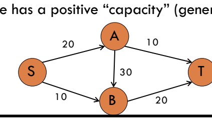
14 units

7

Flow graph/networks

Flow network

- Directed graph (V, E)
- Two distinguished vertices:
  - a source $s \in V$ with no incoming edges
  - a sink/target $t \in V$ with no outgoing edges
- Each edge has a positive "capacity" (generally, assume integers)



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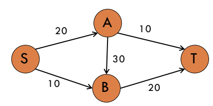
Flow constraints

What is a valid flow?



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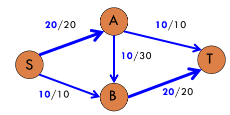
Flow constraints

What is a valid flow?

1. Flow along an edge cannot exceed its capacity

2. In-flow = out-flow for every vertex, except s, t

3. Flows are non-negative



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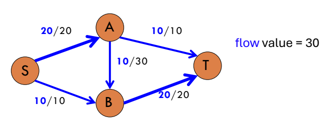
Max flow problem

Given a flow network: *what is the maximum flow we can send from s to t that meets the flow constraints?*



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Applications?

network flow
- water, electricity, sewage, cellular…
- traffic/transportation capacity

bipartite matching

sports elimination

…

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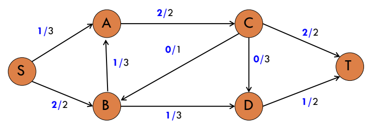
Check in

Pair up with a partner and think about:

- Is this a feasible flow?
- Is there a way to increase the flow?



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Learning Goals

- What is the max-flow problem?
- Why is it useful?
- What algorithm can we use to solve it?
- What's the connection to cuts?

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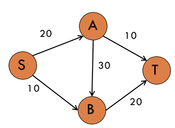
Algorithmic idea

Keep sending flow along "augmenting paths"



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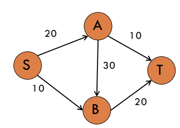
Algorithmic idea

send flow on an augmenting path



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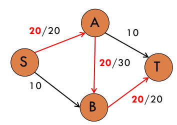
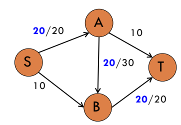
Algorithmic idea

send flow on an augmenting path



20

Algorithmic idea



Now what?

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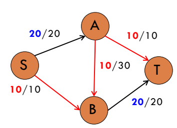
Algorithmic idea

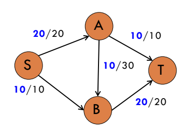


Total flow?

22

Algorithmic idea

An augmenting path may need to undo some flow



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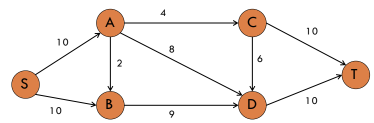
Algorithmic idea



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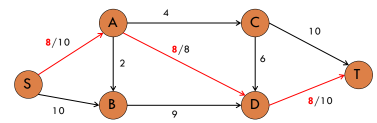
Algorithmic idea

Find some augmenting path



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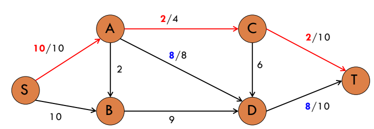
Algorithmic idea

Find some augmenting path



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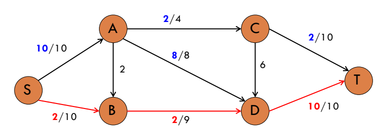
Algorithmic idea

Find some augmenting path



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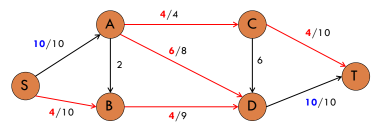
Algorithmic idea

Find some augmenting path



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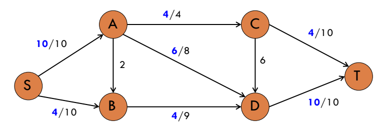
Algorithmic idea



Are we done?

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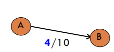
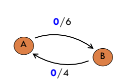
The residual graph

The *residual graph* G_f is constructed from G to help us understand if an augmenting path exists.

For each edge e in the original graph (G), we introduce two new edges.



becomes



G

G_f

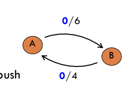
30

The residual graph

The *residual graph* G_f is constructed from G to help us understand if an augmenting path exists.

For each edge e in the original graph (G), we introduce two new edges:

- Forward edge:
  - has capacity = capacity(e)-flow(e)
  - this represents the remaining flow we can still push



- Reverse edge:
  - put an edge in G_f in the *opposite direction* with capacity = flow(e)
  - this represents the flow that we can reverse / undo

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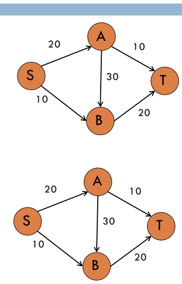
Algorithm idea

G



Gf

Find a path from *s* to *t* in Gf

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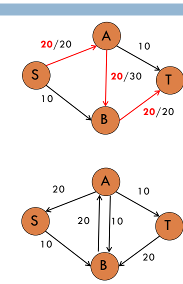
Algorithm idea

G



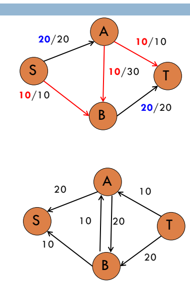
Gf

Find a path from *s* to *t* in Gf

33

Algorithm idea

G



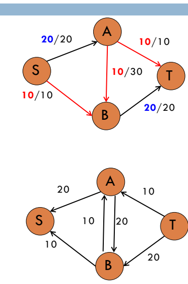
Gf

Find a path from *s* to *t* in Gf

34

Algorithm idea

G



Gf

None exist… done!

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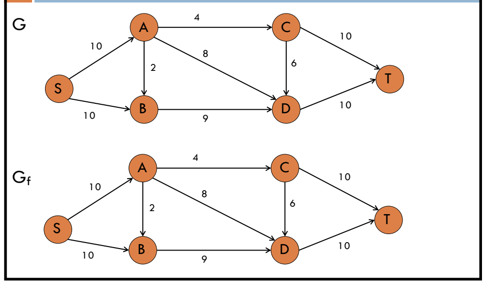
Algorithm idea



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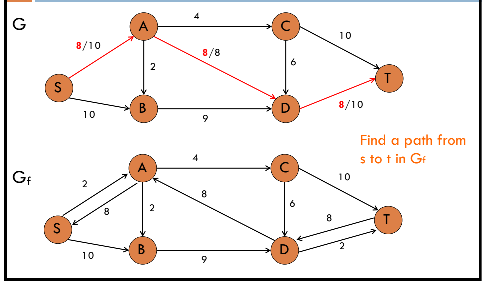
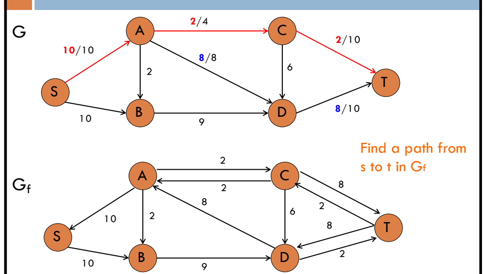
Algorithm idea



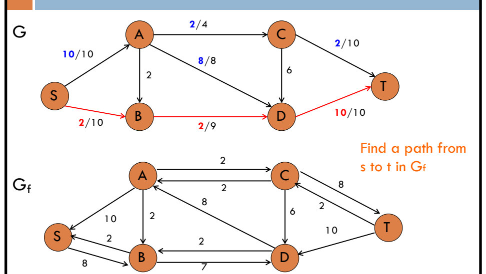
37

Algorithm idea



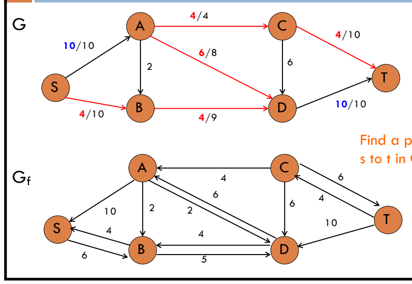
38

Algorithm idea



39

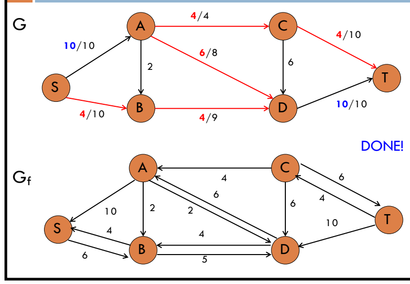
Algorithm idea



Find a path from s to t in G_f

40

Algorithm idea



DONE!

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Ford-Fulkerson

```
Ford-Fulkerson(G, s, t)
  flow = 0 for all edges
  Gf = residualGraph(G)
  while a path exists from s to t in Gf
    send as much flow along the path as possible
    Gf = residualGraph(G)
  return flow
```

Questions?

42

Ford-Fulkerson: runtime?

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Ford-Fulkerson(G, s, t)
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43

Ford-Fulkerson: runtime?

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```

- traverse the graph
- at most add 2 edges per original edge
- θ(V + E)

Can we simplify this expression?

44

Ford-Fulkerson: runtime?

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Ford-Fulkerson(G, s, t)
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- traverse the graph
- at most add 2 edges per original edge
- θ(V + E) = θ(E)

45

Ford-Fulkerson: runtime?

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    send as much flow along path as possible
    Gf = residualGraph(G)
  return flow
```

- BFS or DFS
- O(V + E) = O(E)

46

Ford-Fulkerson: runtime?

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Ford-Fulkerson(G, s, t)
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```

- max-flow!
- increases every iteration
- integer capacities, so integer increases

Can we bound the number of times the loop will execute?

47

Ford-Fulkerson: runtime?

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Ford-Fulkerson(G, s, t)
  flow = 0 for all edges
  Gf = residualGraph(G)
  while a path exists from s to t in Gf
    send as much flow along path as possible
    Gf = residualGraph(G)
  return flow
```

- max-flow!
- increases every iteration
- integer capacities, so integer increases

Overall runtime? O(max-flow * E)

48

Learning Goals

- What is the max-flow problem?
- Why is it useful?
- What algorithm can we use to solve it?
  Ford-Fulkerson algorithm Runs in O(max-flow*E) time
- What's the connection to cuts?

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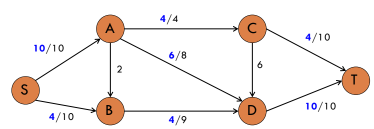
Algorithm idea

Remember this graph?



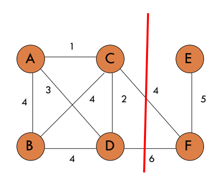
We know there is no augmenting path.

How do we know the flow is maximum? Enter cuts

50

Cuts

A cut is a partitioning of the vertices into two sets S and $T = V - S$



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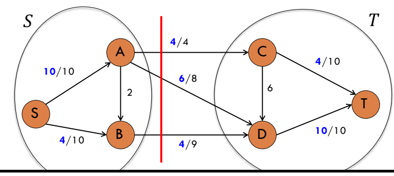
Flow across cuts

In flow graphs, we're interested in cuts that separate s from t, that is $s \in S$ and $t \in T$



52

Flow across cuts

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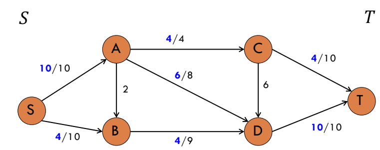
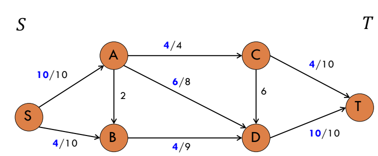


53

Flow across cuts

In flow graphs, we're interested in cuts that separate s from t, that is $s \in S$ and $t \in T$

Is this an $s, t -$ cut?



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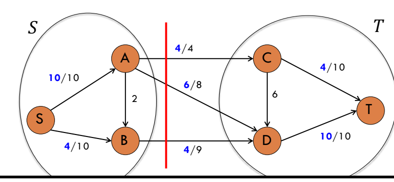
Flow across cuts

Some edges go "across" the cut, some stay within one side.

Two parameters of interest: the net flow and the capacity.



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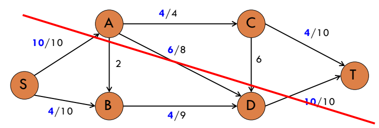
Flow across cuts

The net flow across a cut is the total flow from nodes in S to nodes in T ***minus*** the total from nodes in T to S



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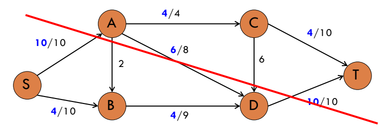
Flow across cuts

The net flow across a cut is the total flow from nodes in S to nodes in T ***minus*** the total from nodes in T to S

What is the net flow across this cut?



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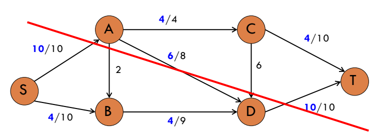
Flow across cuts

The net flow across a cut is the total flow from nodes in S to nodes in T ***minus*** the total from nodes in T to S

10+10-6 = 14



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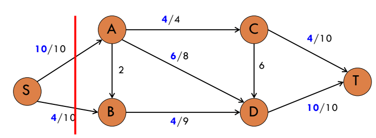
Flow across cuts

The net flow across a cut is the total flow from nodes in S to nodes in T ***minus*** the total from nodes in T to S

4+10 = 14



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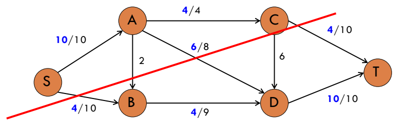
Flow across cuts

The net flow across a cut is the total flow from nodes in S to nodes in T *minus* the total from nodes in T to S

4+6+4 = 14



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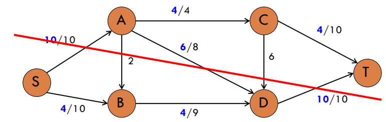
Flow across cuts

The net flow across a cut is the total flow from nodes in S to nodes in T *minus* the total from nodes in T to S

10+10-6 = 14



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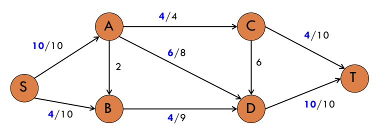
Flow across cuts

The net flow across ANY s,t-cut is the same and is the current flow in the network.

Why might this be?



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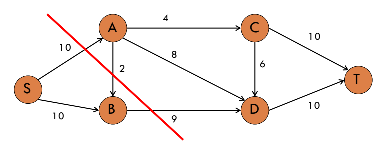
Capacity of a cut

The "**capacity of a cut**" is the maximum net flow that *could* be sent across it.

How do we calculate the capacity?



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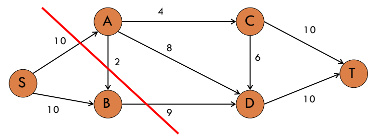
Capacity of a cut

The "**capacity of a cut**" is the maximum net flow that *could* be sent across it.

Capacity is the sum of capacities of edges from S to T



10 + 9 = 19

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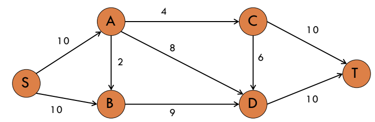
Capacity of a cut

The "**capacity of a cut**" is the maximum net flow that *could* be sent across it.

Capacity is the sum of capacities of edges from S to T



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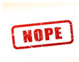
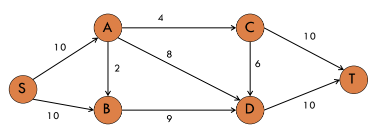
Bounding the flow

What if someone told you:

"The max flow in this network is 21"



"The max flow in this network is 20"



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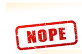
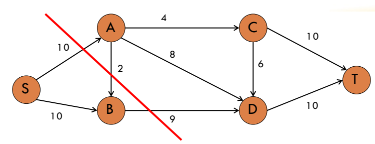
Bounding the flow

What if someone told you:

"The max flow in this network is 21"



"The max flow in this network is 20"



10 + 9 = 19

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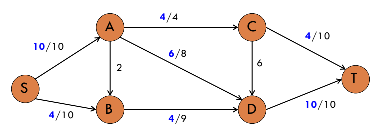
Maximum flow

For any cut where $s \in S$ and $t \in T$
- the net flow is at most the capacity
- the net flow across all cuts is the same

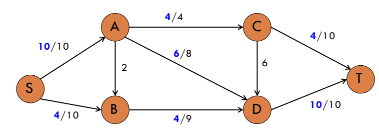


Is this the best we can do?

77

Maximum flow

For any cut where $s \in S$ and $t \in T$
- the net flow is at most the capacity
- the net flow across all cuts is the same



We can do no better than the minimum capacity cut!

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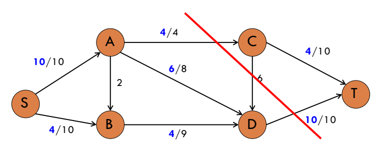
Maximum flow

What is the minimum capacity cut for this graph?

Capacity = 10 + 4



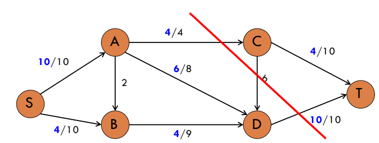
Is this the best we can do?

79

Maximum flow

What is the minimum capacity cut for this graph?

Capacity = 10 + 4



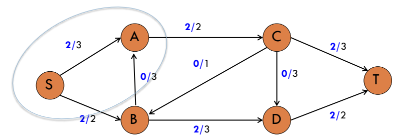
flow = minimum capacity, so we can do no better

80

Check in

The current flow in this network is 4.

- Can you find a cut with net flow equal to 4?
- Can you find a cut with capacity equal to 4?



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Ford-Fulkerson: is it correct?

When the algorithm terminates, is it a maximum flow?

→ Termination condition:
- No s,t-path in the residual graph G_f

→ Look at the nodes S reachable from s in G_f by a path
- This is an s, t - cut!

→ This cut has net flow equal to its capacity

→ This is the maximum flow possible.

□

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Learning Goals

What is the max-flow problem?

Why is it useful?

What algorithm can we use to solve it?

Ford-Fulkerson algorithm Runs in O(max-flow*E) time.

What's the connection to cuts?

In any flow network, the maximum st-flow is equal to the minimum capacity st-cut.

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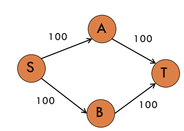
Runtime considerations

Ford –Fulkerson runs in time O(max-flow * E)

Can you construct a graph that *could get* this running time?

Hint:

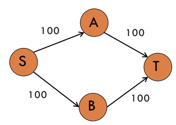


84

O(max-flow * E)

Can you construct a graph that *could get* this running time?

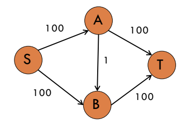
Hint:



85

O(max-flow * E)

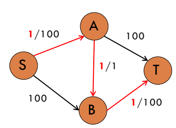
Can you construct a graph that *could get* this running time?



86

O(max-flow * E)

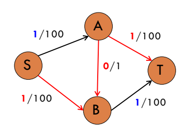
Can you construct a graph that *could get* this running time?



87

O(max-flow * E)

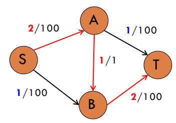
Can you construct a graph that *could get* this running time?



88

O(max-flow * E)

Can you construct a graph that *could get* this running time?



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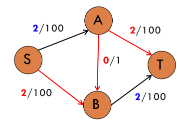
O(max-flow * E)

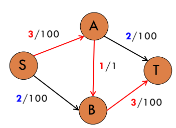
Can you construct a graph that *could get* this running time?



90

O(max-flow * E)

Can you construct a graph that *could get* this running time?



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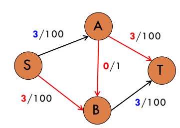
O(max-flow * E)

Can you construct a graph that *could get* this running time?



What is the problem here?
Could we do better?

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Faster variants

Edmonds-Karp
- Select the *shortest path* (in number of edges) from s to t in G_f
  - How can we do this?
    - use BFS for search
- Running time: $O(V E^2)$
  - avoids issues like the one we just saw
  - see the book for the proof
  - or http://www.cs.cornell.edu/courses/CS4820/2011sp/handouts/edmondskarp.pdf

preflow-push (aka push-relabel) algorithms
- $O(V^3)$

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Other variations…

Method	Complexity
Linear programming	
Ford–Fulkerson algorithm	$O(E \max\|f\|)$
Edmonds–Karp algorithm	$O(VE^2)$
Dinitz blocking flow algorithm	$O(V^2E)$
General push-relabel maximum flow algorithm	$O(V^2E)$
Push-relabel algorithm with *FIFO* vertex selection rule	$O(V^3)$
Dinitz blocking flow algorithm with dynamic trees	$O(VE \log(V))$
Push-relabel algorithm with dynamic trees	$O(VE \log(V^2/E))$
Binary blocking flow algorithm [1]	$O(E \min(V^{2/3}, \sqrt{E}) \log(V^2/E) \log U)$
MPM (Malhotra, Pramodh-Kumar and Maheshwari) algorithm	$O(V^3)$

http://en.wikipedia.org/wiki/Maximum_flow

TABLE I. POLYNOMIAL-TIME ALGORITHMS FOR THE MAXIMUM FLOW PROBLEM[a]

Algorithm no.	Date	Discoverer	Running time	References
1	1969	Edmonds and Karp	$O(nm^2)$	[5]
2	1970	Dinic	$O(n^2m)$	[4]
3	1974	Karzanov	$O(n^3)$	[18]
4	1977	Cherkasky	$O(n^2m^{1/2})$	[3]
5	1978	Malhotra, Pramodh Kumar, and Maheshwari	$O(n^3)$	[21]
6	1978	Galil	$O(n^{5/3}m^{2/3})$	[11]
7	1978	Galil and Naamad; Shiloach	$O(nm(\log n)^2)$	[12, 25]
8	1980	Sleator and Tarjan	$O(nm \log n)$	[27, 28]
9	1982	Shiloach and Vishkin	$O(n^3)$	[26]
10	1983	Gabow	$O(nm \log U)$	[10]
11	1984	Tarjan	$O(n^3)$	[31]
12	1985	Goldberg	$O(n^3)$	[14]
13	1986	Goldberg and Tarjan	$O(nm \log(n^2/m))$	[16, 15]
14	1986	Ahuja and Orlin	$O(nm + n^2 \log U)$	[1]

[a] Algorithm 13 is presented in this paper.

http://akira.ruc.dk/~keld/teaching/algoritmedesign_f03/Artikler/08/Goldberg88.pdf

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Network flow properties

If one of these is true then all are true (i.e. each implies the the others):

f is a maximum flow

G_f (residual graph) has no paths from s to t

$|f|$ = minimum capacity cut

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Handout

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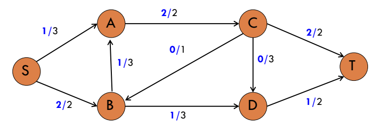
Check in

Pair up with a partner and think about:

- Is this a feasible flow?
- Is it maximum?



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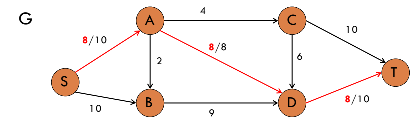
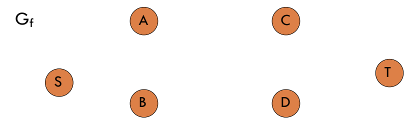


Calculate residual graph

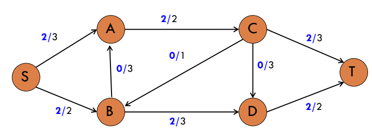


99

Check in

The current flow in this network is 4.

- Can you find a cut with net flow equal to 4?
- Can you find a cut with capacity equal to 4?



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