# MORE GRAPH ALGORITHMS

David Kauchak
CS 140 – Fall 2024

1

## Admin

Assignment 8 out soon: don't reinvent the wheel

Midterm 2 back today (more on that at the end)

2

## Connectedness

Given an undirected graph, for every node $u \in V$, can we reach all other nodes in the graph?
Algorithm + running time

Run BFS or DFS-Visit (one pass) and mark nodes as we visit them. If we visit all nodes, return true, otherwise false.

Running time: $O(|V| + |E|)$

3

## Strongly connected

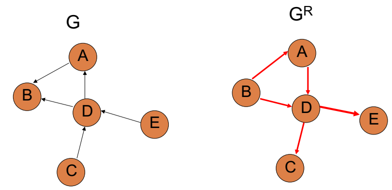
Given a directed graph, can we reach any node v from any other node u?

Can we do the same thing?

4

## Transpose of a graph

Given a graph G, we can calculate the transpose of a graph $G^R$ by reversing the direction of all the edges




Running time to calculate $G^R$? $\theta(|V| + |E|)$

5

## Strongly connected

Strongly-Connected(G)
- Run DFS-Visit or BFS from some node u
- If not all nodes are visited: return false
- Create graph $G^R$
- Run DFS-Visit or BFS on $G^R$ from node u
- If not all nodes are visited: return false
- return true

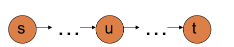
6

## Is it correct?

What do we know after the first pass?
- Starting at u, we can reach every node

What do we know after the second pass?
- All nodes can reach u. Why?
- We can get from u to every node in $G^R$, therefore, if we reverse the edges (i.e. G), then we have a path from every node to u

Which means that any node can reach any other node. Given any two nodes s and t we can create a path through u




7

## Runtime?

Strongly-Connected(G)

| | |
|---|---|
| - Run DFS-Visit or BFS from some node u | $O(\|V\| + \|E\|)$ |
| - If not all nodes are visited: return false | $O(\|V\|)$ |
| - Create graph $G^R$ | $\theta(\|V\| + \|E\|)$ |
| - Run DFS-Visit or BFS on $G^R$ from node u | $O(\|V\| + \|E\|)$ |
| - If not all nodes are visited: return false | $O(\|V\|)$ |
| - return true | |

$O(|V| + |E|)$

8

## Minimum spanning trees

What are they?

What do you remember about them?

What algorithms do you remember?

9

## Minimum spanning trees

What is the lowest weight set of edges that connects all vertices of an undirected graph with positive weights

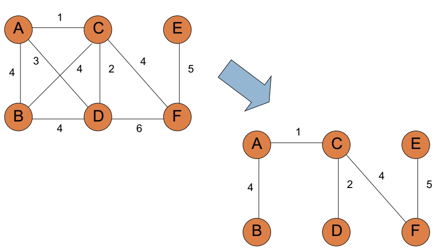
Input: An undirected, positive weight graph, G=(V,E)

Output: A tree T=(V,E') where E' $\subseteq$ E that minimizes

$$weight(T) = \sum_{e \in E'} w_e$$

10

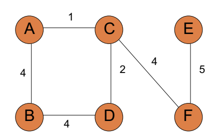
## MST example




11

## MSTs

Can an MST have a cycle?




12

## MSTs

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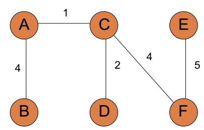



13

## Applications?

Connectivity

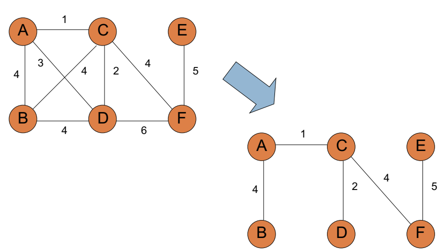
- Networks (e.g. communications)
- Circuit design/wiring

hub/spoke models (e.g. flights, transportation)

Traveling salesman problem?

14

## Algorithm ideas?

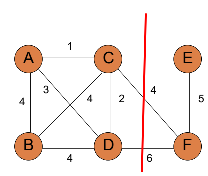



15

## Cuts

A cut is a partitioning of the vertices into two sets S and V-S

An edge “crosses” the cut if it connects a vertex u∈V and v∈V-S

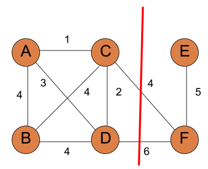



16

## Minimum cut property

Given a partition S, let edge e be the minimum cost edge that **crosses** the partition. *Every* minimum spanning tree contains edge e.

Proof by contradiction?




17

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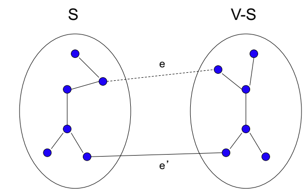


Consider an MST with edge e' that is not the minimum edge

18

## Minimum cut property

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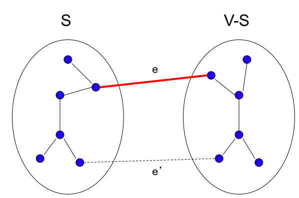
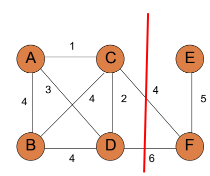



Using e instead of e', still connects the graph, but produces a tree with smaller weights

19

## Minimum cut property

If the minimum cost edge that **crosses** the partition is not unique, then *some* minimum spanning tree contains edge e.




20

## Kruskal's algorithm

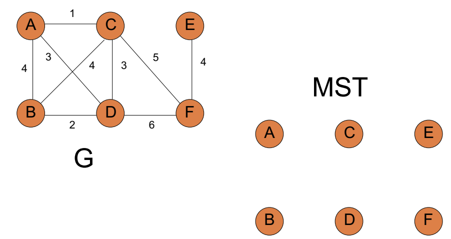
Given a partition S, let edge e be the minimum cost edge that **crosses** the partition. *Every* minimum spanning tree contains edge e.

```
KRUSKAL(G)
for all v ∈ V
        MAKESET(v)
T ← {}
sort the edges of E by weight
for all edges (u, v) ∈ E in increasing order of weight
        if FIND-SET(u) ≠ FIND-SET(v)
                add edge to T
                UNION(FIND-SET(u),FIND-SET(v))
```

21

## Kruskal's algorithm

Add smallest edge that connects two sets not already connected




22

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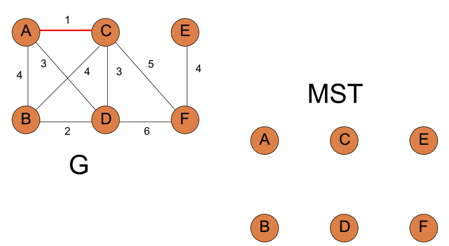


23

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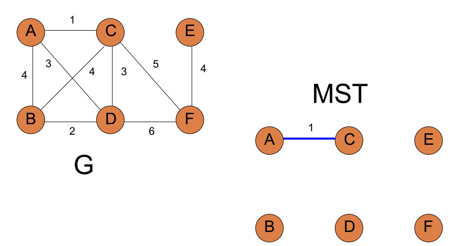
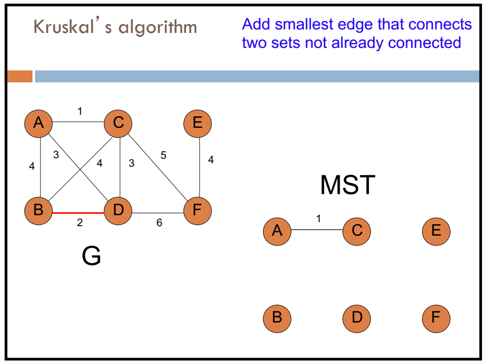
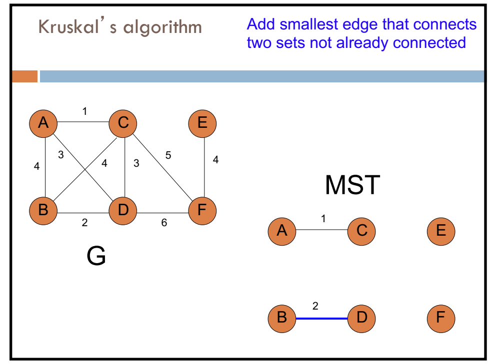



24


Kruskal's algorithm
Add smallest edge that connects
two sets not already connected
G
MST

25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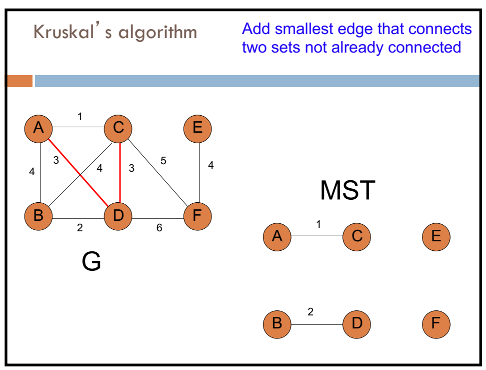
26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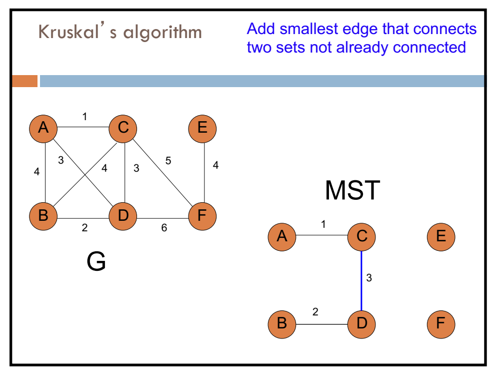
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28

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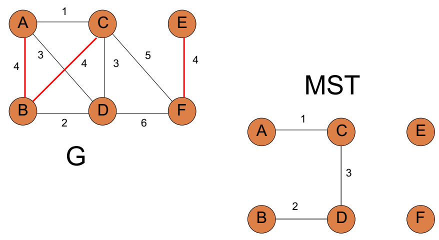


29

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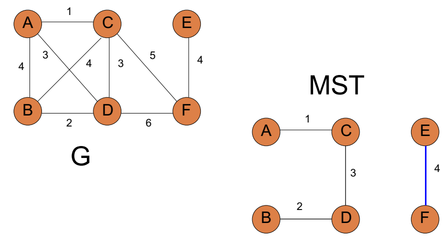


30

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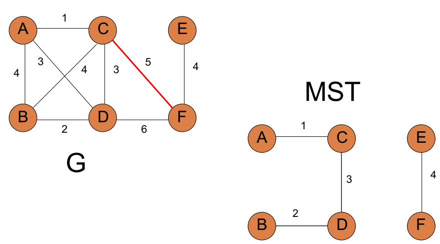


31

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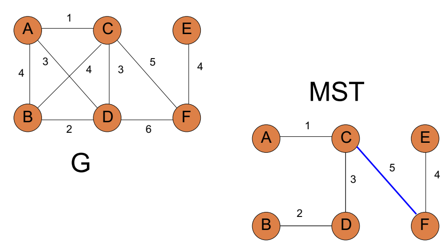


32

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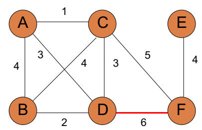
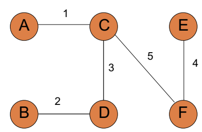



G

MST




33

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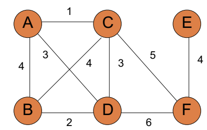
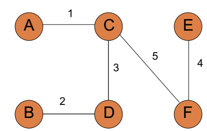



G

Done!

MST




34

## Correctness of Kruskal's

Never adds an edge that connects already connected vertices

Always adds lowest cost edge to connect two sets. By min cut property, that edge must be part of the MST

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```

35

## Running time of Kruskal's

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36

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```

|V| calls to MakeSet

O(|E| log |E|)

2 |E| calls to FindSet

|V| calls to Union

37

## Disjoint set data structures

Represents a collection of one or more sets

Operations:

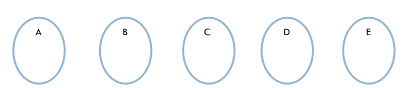
- MakeSet: Add a new value to the collections and make the value it's own set
- FindSet: Given a value, return the set the value is in
- Union: Merge two sets into a single set

38

## Disjoint set data structure

MakeSet(A), MakeSet(B), MakeSet(C), MakeSet(D), MakeSet(E)

Disjoint Set

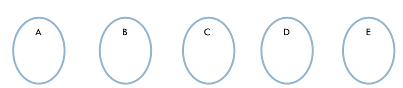



39

## Disjoint set data structure

FindSet(A)?

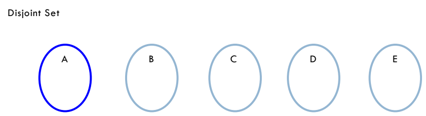
Disjoint Set




40

## Disjoint set data structure

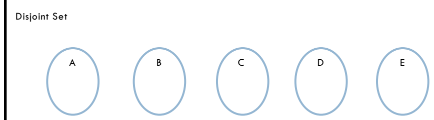
FindSet(A)?




41

## Disjoint set data structure

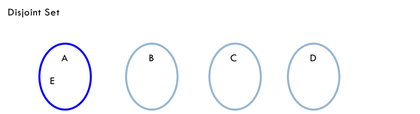
Union(FindSet(A), FindSet(E))




42

## Disjoint set data structure

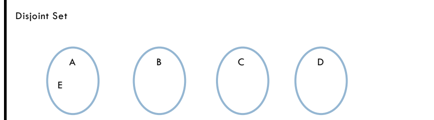
Union(FindSet(A), FindSet(E))




43

## Disjoint set data structure

Union(FindSet(C), FindSet(D))

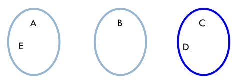



44

## Disjoint set data structure

Union(FindSet(C), FindSet(D))

Disjoint Set

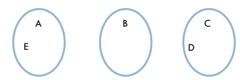



45

## Disjoint set data structure

FindSet(D)?

Disjoint Set

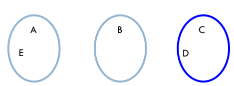



46

## Disjoint set data structure

FindSet(D)?

Disjoint Set

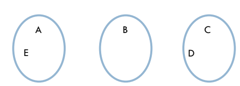



47

## Disjoint set data structure

Union(FindSet(D), FindSet(B))

Disjoint Set




48

## Disjoint set data structure

Union(FindSet(D), FindSet(B))

Disjoint Set


49

## Disjoint set data structure

How would we implement it with a list of linked lists?
MakeSet?
FindSet?
Union?

Disjoint Set


50

## Disjoint set

A B C D

51

## Disjoint set: union

A B C D

52

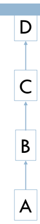
## Disjoint set: union


53

## Disjoint set: union


54

## Disjoint set: union




Running time?

55

## Disjoint set: union


O(1)

56

## Disjoint set: find-set


Search each linked list

57

## Disjoint set: find-set


Running time?

58

## Disjoint set: find-set


O(n) -- n = number of things in set

59

## Running time of Kruskal's

Disjoint set data structure

O(|E| log |E|) +

| | MakeSet (V calls) | FindSet (\|E\| calls) | Union (\|V\| calls) | Total |
|---|---|---|---|---|
| Linked lists | \|V\| | O(\|V\| \|E\|) | \|V\| | O(\|V\|\|E\| + \|E\| log \|E\|)<br>O(\|V\| \|E\|) |
| Linked lists + heuristics | \|V\| | O(\|E\| log \|V\|) | \|V\| | O(\|E\| log \|V\|+ \|E\| log \|E\|)<br>O(\|E\| log \|E\| ) |

60

## Prim's algorithm

Start at some root node and build out the MST by adding the lowest weighted edge at the frontier

```
Prim(G, r)
for all v ∈ V
    key[v] ← ∞
    prev[v] ← null
key[r] ← 0
H ← MakeHeap(key)
while !Empty(H)
    u ← Extract-Min(H)
    visited[u] ← true
    for each edge (u, v) ∈ E
        if !visited[v] and w(u, v) < key(v)
            Decrease-Key(v, w(u, v))
            prev[v] ← u
```

64

## Prim's

```
while !Empty(H)
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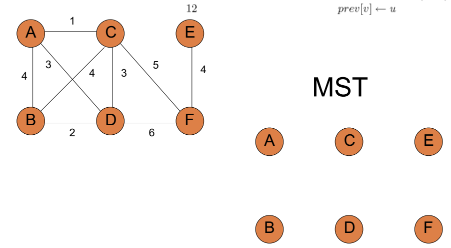


65

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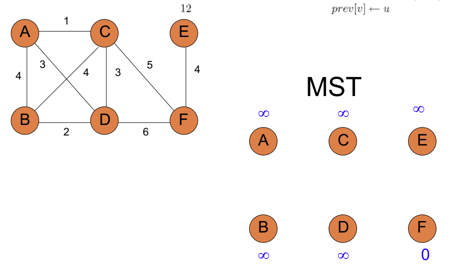


66

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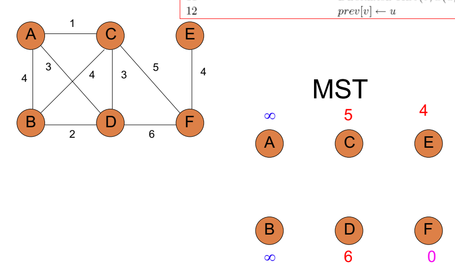
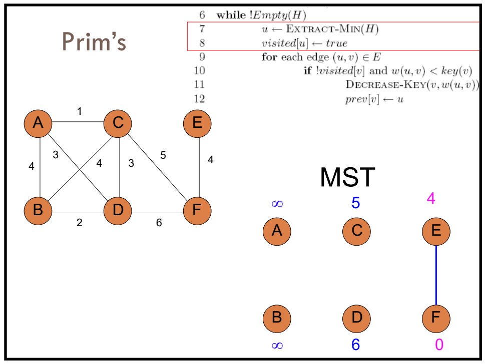



67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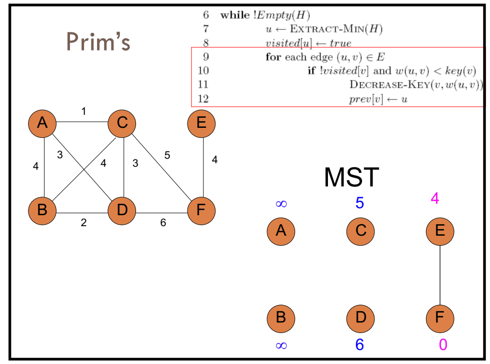
prev[v] ← u
MST

68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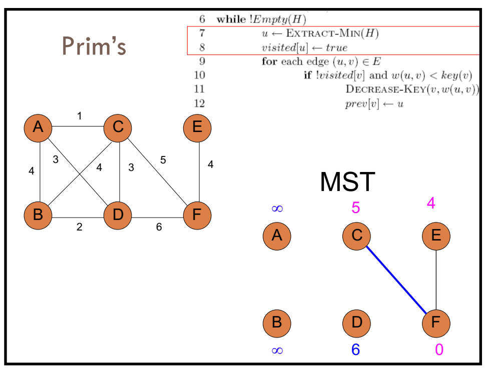
MST

69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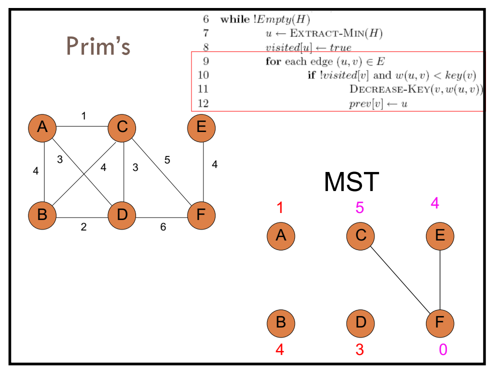
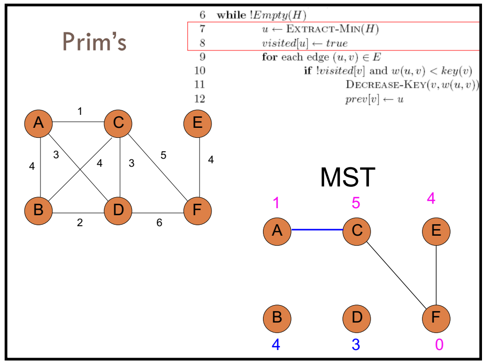
MST

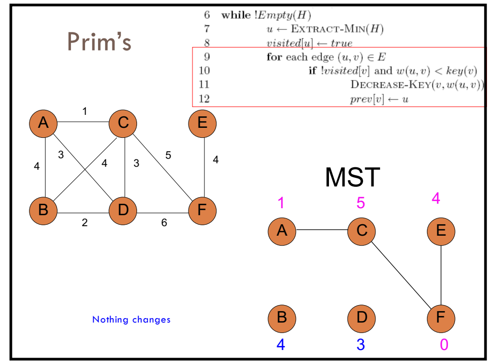
70


Prim's
while !Empty(H)
u ← Extract-Min(H)
visited[u] ← true
for each edge (u, v) ∈ E
if !visited[v] and w(u, v) < key(v)
Decrease-Key(v, w(u, v))
prev[v] ← u
MST

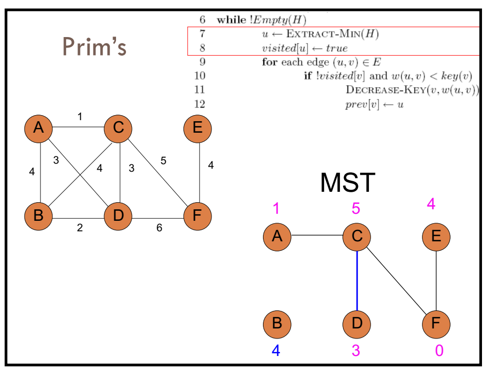
71


Prim's
while !Empty(H)
u ← Extract-Min(H)
visited[u] ← true
for each edge (u, v) ∈ E
if !visited[v] and w(u, v) < key(v)
Decrease-Key(v, w(u, v))
prev[v] ← u
A
C
E
B
D
F
1
4
3
4
3
5
4
2
6
MST
1
5
4
A
C
E
B
D
F
4
3
0

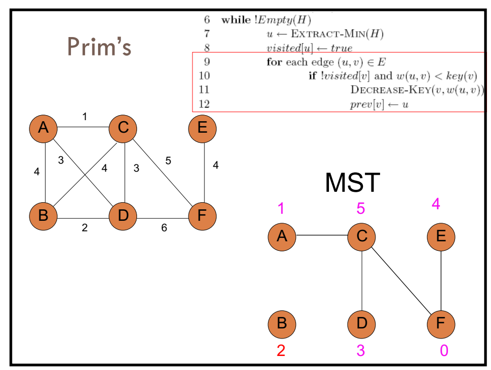
72


Prim's
while !Empty(H)
u ← Extract-Min(H)
visited[u] ← true
for each edge (u, v) ∈ E
if !visited[v] and w(u, v) < key(v)
Decrease-Key(v, w(u, v))
prev[v] ← u
A
C
E
B
D
F
1
4
3
4
3
5
4
2
6
MST
1
5
4
A
C
E
Nothing changes
B
D
F
4
3
0

73


Prim's
while !Empty(H)
u ← Extract-Min(H)
visited[u] ← true
for each edge (u, v) ∈ E
if !visited[v] and w(u, v) < key(v)
Decrease-Key(v, w(u, v))
prev[v] ← u
A
C
E
B
D
F
1
4
3
4
3
5
4
2
6
MST
1
5
4
A
C
E
B
D
F
4
3
0

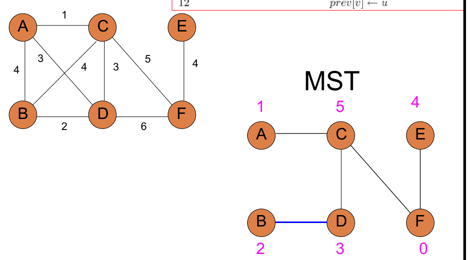
74


Prim's
while !Empty(H)
u ← Extract-Min(H)
visited[u] ← true
for each edge (u, v) ∈ E
if !visited[v] and w(u, v) < key(v)
Decrease-Key(v, w(u, v))
prev[v] ← u
A
C
E
B
D
F
1
4
3
4
3
5
4
2
6
MST
1
5
4
A
C
E
B
D
F
2
3
0

75

## Prim's

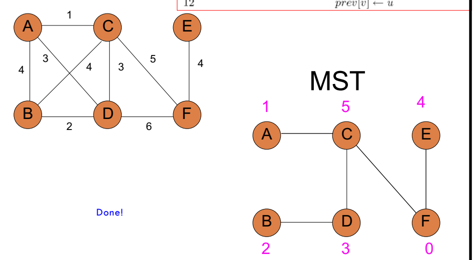
```
while !Empty(H)
    u ← Extract-Min(H)
    visited[u] ← true
    for each edge (u, v) ∈ E
        if !visited[v] and w(u, v) < key(v)
            Decrease-Key(v, w(u, v))
            prev[v] ← u
```




76

## Prim's

```
while !Empty(H)
    u ← Extract-Min(H)
    visited[u] ← true
    for each edge (u, v) ∈ E
        if !visited[v] and w(u, v) < key(v)
            Decrease-Key(v, w(u, v))
            prev[v] ← u
```




77

## Correctness of Prim's?

Can we use the min-cut property?

- Given a partion S, let edge e be the minimum cost edge that **crosses** the partition. *Every* minimum spanning tree contains edge e.

Let S be the set of vertices visited so far

The only time we add a new edge is if it's the lowest weight edge from S to V-S

78

## Running time of Prim's

```
Prim(G, r)
for all v ∈ V
    key[v] ← ∞
    prev[v] ← null
key[r] ← 0
H ← MakeHeap(key)
while !Empty(H)
    u ← Extract-Min(H)
    visited[u] ← true
    for each edge (u, v) ∈ E
        if !visited[v] and w(u, v) < key(v)
            Decrease-Key(v, w(u, v))
            prev[v] ← u
```

79

## Running time of Prim's

```
PRIM(G, r)
for all v ∈ V
        key[v] ← ∞
        prev[v] ← null
key[r] ← 0
H ← MAKEHEAP(key)
while !Empty(H)
        u ← EXTRACT-MIN(H)
        visited[u] ← true
        for each edge (u, v) ∈ E
                if !visited[v] and w(u, v) < key(v)
                        DECREASE-KEY(v, w(u, v))
                        prev[v] ← u
```

$\Theta(|V|)$

1 call to MakeHeap

|V| calls to Extract-Min

|E| calls to Decrease-Key

80

## Running time of Prim's

| | 1 MakeHeap | \|V\| ExtractMin | \|E\| DecreaseKey | Total |
|---|---|---|---|---|
| Array | $\theta(\|V\|)$ | $O(\|V\|^2)$ | $O(\|E\|)$ | $O(\|V\|^2)$ |
| Bin heap | $\theta(\|V\|)$ | $O(\|V\| \log \|V\|)$ | $O(\|E\| \log \|V\|)$ | $O((\|V\|+\|E\|) \log \|V\|)$<br>$O(\|E\| \log \|V\|)$ |
| Fib heap | $\theta(\|V\|)$ | $O(\|V\| \log \|V\|)$ | $O(\|E\|)$ | **$O(\|V\| \log \|V\| + \|E\|)$** |

Kruskal's: $O(|E| \log |E|)$

81