GRAPHS

David Kauchak
CS 140 – Fall 2024

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Admin

Assignment 7

Group 7

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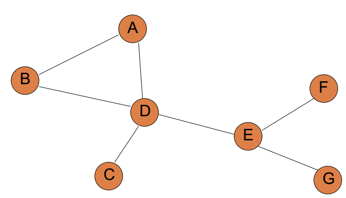
Graphs

What is a graph?



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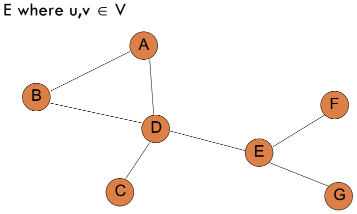
Graphs

A graph is a set of vertices V and a set of edges $(u,v) \in E$ where $u,v \in V$



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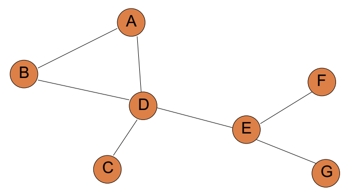
Graphs

How do graphs differ? What are graph characteristics we might care about?



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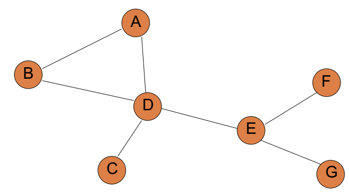
Different types of graphs

Undirected – edges do not have a direction



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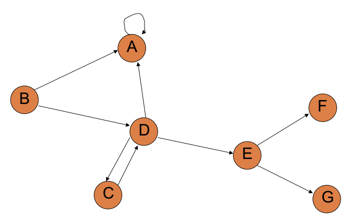
Different types of graphs

Directed – edges **do** have a direction



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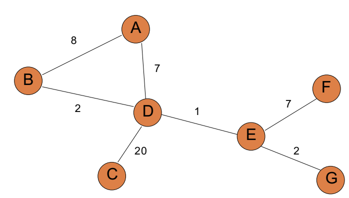
Different types of graphs

Weighted – edges have an associated weight



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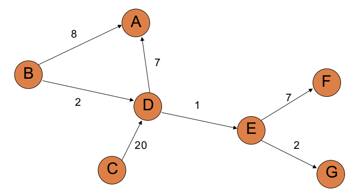
Different types of graphs

Weighted – edges have an associated weight



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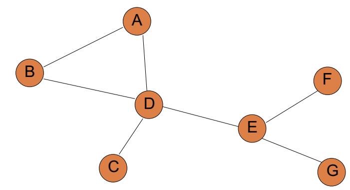
Terminology

Path – A path is a list of vertices $p_1, p_2, \ldots p_k$ where there exists an edge $(p_i, p_{i+1}) \in E$



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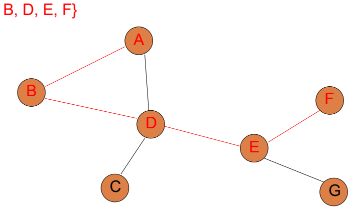
Terminology

Path – A path is a list of vertices $p_1, p_2, \ldots p_k$ where there exists an edge $(p_i, p_{i+1}) \in E$

{A, B, D, E, F}



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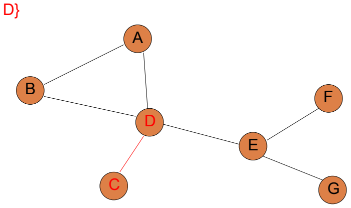
Terminology

Path – A path is a list of vertices $p_1, p_2, \ldots p_k$ where there exists an edge $(p_i, p_{i+1}) \in E$

{C, D}



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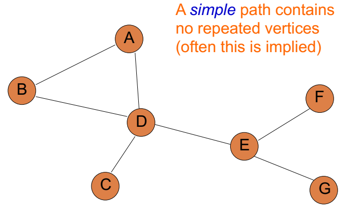
Terminology

Path – A path is a list of vertices $p_1, p_2, \ldots p_k$ where there exists an edge $(p_i, p_{i+1}) \in E$

A *simple* path contains no repeated vertices (often this is implied)



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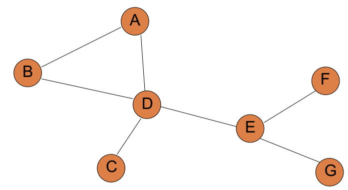
Terminology

Cycle?



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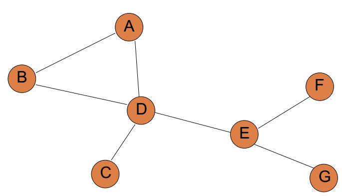
Terminology

Cycle – A subset of the edges that form a path such that the first and last node are the same



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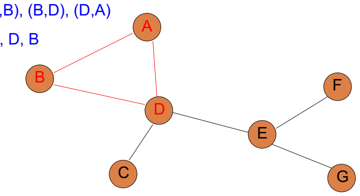
Terminology

Cycle – A subset of the edges that form a path where each edge is traversed once such that the first and last node are the same

Edges: (A,B), (B,D), (D,A)

Path: B, A, D, B



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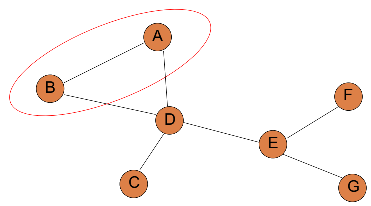
Terminology

Cycle – A subset of the edges that form a path where each edge is traversed once such that the first and last node are the same

cycle?



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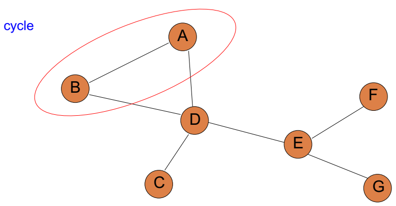
Terminology

Cycle – A subset of the edges that form a path where each edge is traversed once such that the first and last node are the same

not a cycle



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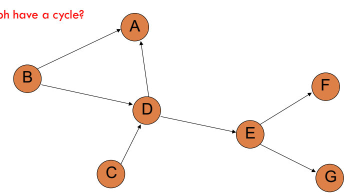
Terminology

Cycle – A subset of the edges that form a path where each edge is traversed once such that the first and last node are the same

Does this graph have a cycle?



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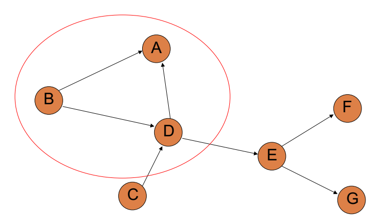
Terminology

Cycle – A subset of the edges that form a path where each edge is traversed once such that the first and last node are the same

not a cycle



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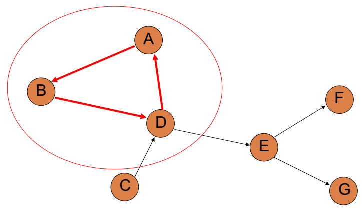
Terminology

Cycle – A path $p_1, p_2, \ldots p_k$ where $p_1 = p_k$

cycle



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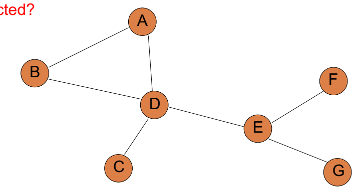
Terminology

Connected – every pair of vertices is connected by a path

Is this graph connected?



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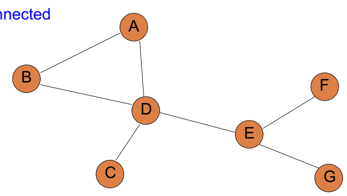
Terminology

Connected – every pair of vertices is connected by a path

connected



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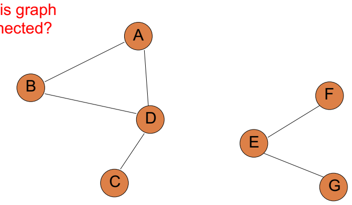
Terminology

Connected – every pair of vertices is connected by a path

Is this graph connected?



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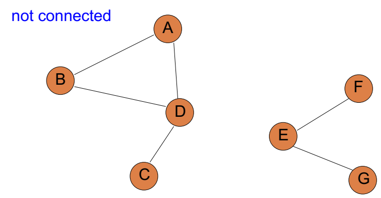
Terminology

Connected – every pair of vertices is connected by a path



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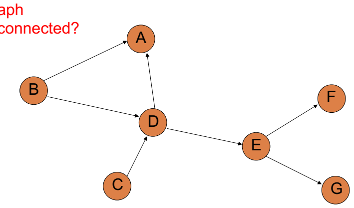
Terminology

Strongly connected (directed graphs) –
Every two vertices are reachable by a path

Is this graph
strongly connected?



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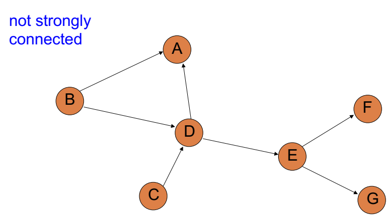
Terminology

Strongly connected (directed graphs) –
Every two vertices are reachable by a path



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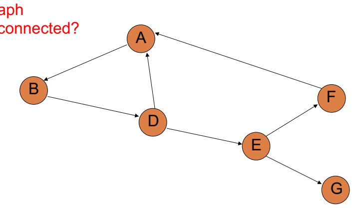
Terminology

Strongly connected (directed graphs) –
Every two vertices are reachable by a path

Is this graph
strongly connected?



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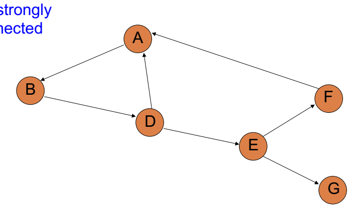
Terminology

Strongly connected (directed graphs) –
Every two vertices are reachable by a path

not strongly
connected



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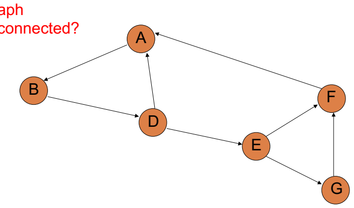
Terminology

Strongly connected (directed graphs) –
Every two vertices are reachable by a path

Is this graph
strongly connected?



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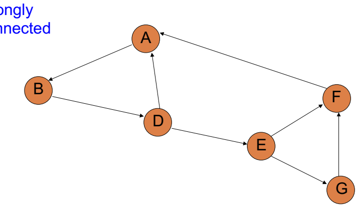
Terminology

Strongly connected (directed graphs) –
Every two vertices are reachable by a path

strongly
connected



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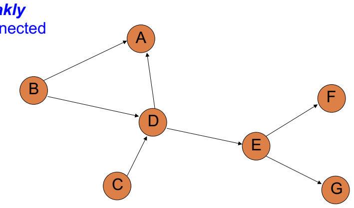
Terminology

Weakly connected (directed graphs) –
graph is connected when considered as undirected graph

weakly
connected



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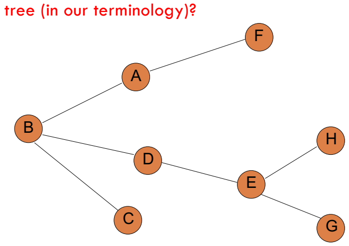
Different types of graphs

What is a tree (in our terminology)?



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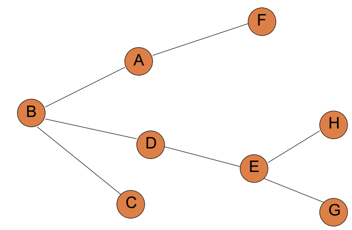
Different types of graphs

Tree – connected, undirected graph without any cycles



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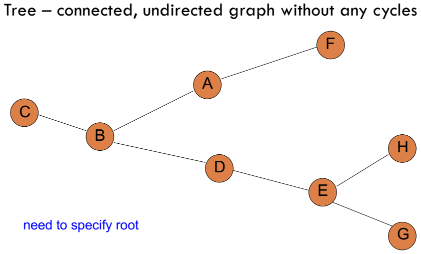
Different types of graphs

Tree – connected, undirected graph without any cycles



need to specify root

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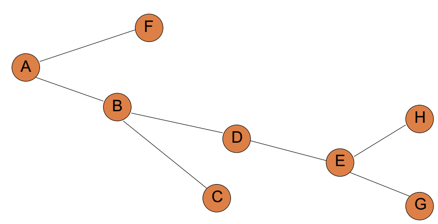
Different types of graphs

Tree – connected, undirected graph without any cycles



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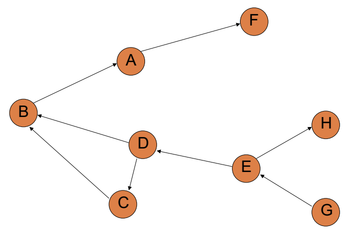
Different types of graphs

DAG – directed, acyclic graph



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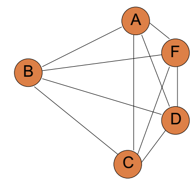
Different types of graphs

Complete graph – an edge exists between every node



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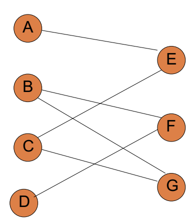
Different types of graphs

Bipartite graph – a graph where every vertex can be partitioned into two sets X and Y such that all edges connect a vertex $u \in X$ and a vertex $v \in Y$



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When do we see graphs in real life problems?

Transportation networks (flights, roads, etc.)

Communication networks

Web

Social networks

Circuit design

Bayesian networks

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Representing graphs

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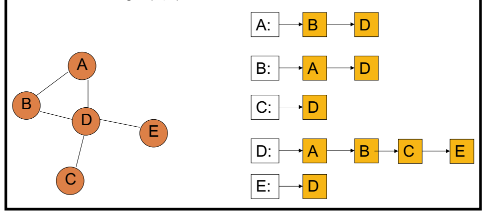
Representing graphs

Adjacency list – Each vertex u ∈ V contains an adjacency list of the set of vertices v such that there exists an edge (u,v) ∈ E



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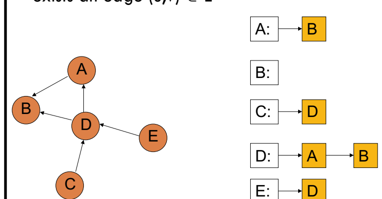
Representing graphs

Adjacency list – Each vertex u ∈ V contains an adjacency list of the set of vertices v such that there exists an edge (u,v) ∈ E



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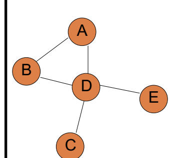
Representing graphs

Adjacency matrix – A |V|x|V| matrix A such that:

$$a_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$



	A	**B**	**C**	**D**	**E**
A	0	1	0	1	0
B	1	0	0	1	0
C	0	0	0	1	0
D	1	1	1	0	1
E	0	0	0	1	0

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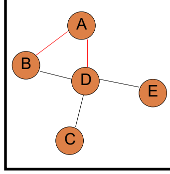
Representing graphs

Adjacency matrix – A |V|x|V| matrix A such that:

$$a_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$



	A	B	C	D	E
A	0	**1**	0	**1**	0
B	1	0	0	1	0
C	0	0	0	1	0
D	1	1	1	0	1
E	0	0	0	1	0

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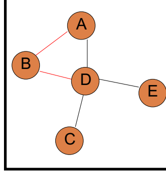
Representing graphs

Adjacency matrix – A |V|x|V| matrix A such that:

$$a_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$



	A	B	C	D	E
A	0	1	0	1	0
B	**1**	0	0	**1**	0
C	0	0	0	1	0
D	1	1	1	0	1
E	0	0	0	1	0

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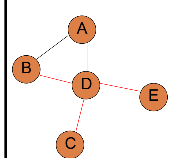
Representing graphs

Adjacency matrix – A |V|x|V| matrix A such that:

$$a_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$



	A	B	C	D	E
A	0	1	0	1	0
B	1	0	0	1	0
C	0	0	0	1	0
D	**1**	**1**	**1**	0	**1**
E	0	0	0	1	0

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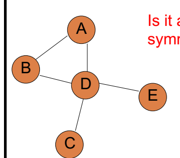
Representing graphs

Adjacency matrix – A |V|x|V| matrix A such that:

$$a_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$



Is it always symmetric?

	A	B	C	D	E
A	0	1	0	1	0
B	1	0	0	1	0
C	0	0	0	1	0
D	1	1	1	0	1
E	0	0	0	1	0

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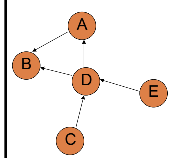
Representing graphs

Adjacency matrix – A |V|x|V| matrix A such that:

$$a_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$



	A	B	C	D	E
A	0	1	0	0	0
B	0	0	0	0	0
C	0	0	0	1	0
D	1	1	0	0	0
E	0	0	0	1	0

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Adjacency list vs. adjacency matrix

Adjacency list

Adjacency matrix

Pros and cons?

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Adjacency list vs. adjacency matrix

Adjacency list

- Sparse graphs (e.g. web)
- Space efficient
- Must traverse the adjacency list to discover is an edge exists

Adjacency matrix

- Dense graphs
- Constant time lookup to discover if an edge exists
- Simple to implement
- For non-weighted graphs, only requires boolean matrix

Can we get the best of both worlds?

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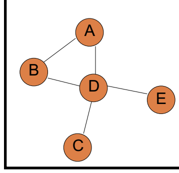
Sparse adjacency matrix

Rather than using an adjacency list, use an adjacency hashtable



A:	hashtable [B,D]
B:	hashtable [A,D]
C:	hashtable [D]
D:	hashtable [A,B,C,E]
E:	hashtable [D]

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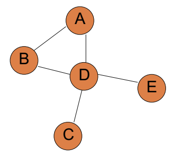
Sparse adjacency matrix

Constant time lookup

Fairly space efficient

Not good for dense graphs, why?



A:	hashtable [B,D]
B:	hashtable [A,D]
C:	hashtable [D]
D:	hashtable [A,B,C,E]
E:	hashtable [D]

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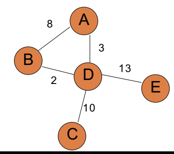
Weighted graphs

Adjacency list
- store the weight as an additional field in the list



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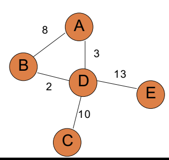
Weighted graphs

Adjacency matrix

$$a_{ij} = \begin{cases} weight & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$



	A	B	C	D	E
A	0	8	0	3	0
B	8	0	0	2	0
C	0	0	0	10	0
D	3	2	10	0	13
E	0	0	0	13	0

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Graph algorithms/questions

Graph traversal (BFS, DFS)

Shortest path from a to b
- unweighted
- weighted positive weights
- negative/positive weights

Minimum spanning trees

Are all nodes in the graph connected?

Is the graph bipartite?

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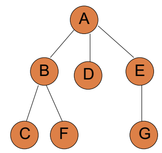
DFS and BFS

How are they implemented?

What would be the result starting at A?
If you ask for the children of a node, they're given in alphabetical order.



Run-time (in terms of V and E):
- adjacency list
- adjacency matrix

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Search implemented

```
TreeBFS(T)
Enqueue(Q, Root(T))
while !Empty(Q)
        v ← Dequeue(Q)
        Visit(v)
        for all c ∈ Children(v)
                Enqueue(Q, c)
```

```
TreeDFS(T)
Push(S, Root(T))
while !Empty(S)
        v ← Pop(S)
        Visit(v)
        for all c ∈ Children(v)
                Push(S, c)
```

```
TreeDFS(v)
    visit(v)
    if not leaf(v)
        for all c in children(x)
            TreeDFS(v)
```

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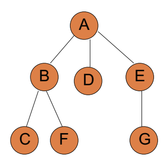
BFS

```
TreeBFS(T)
Enqueue(Q, Root(T))
while !Empty(Q)
        v ← Dequeue(Q)
        Visit(v)
        for all c ∈ Children(v)
                Enqueue(Q, c)
```



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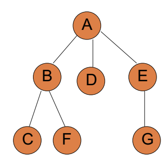
BFS

```
TreeBFS(T)
Enqueue(Q, Root(T))
while !Empty(Q)
        v ← Dequeue(Q)
        Visit(v)
        for all c ∈ Children(v)
                Enqueue(Q, c)
```

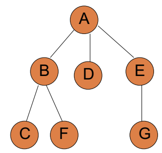


A B D E C F G

60

DFS

```
TreeDFS(T)
  Push(S, Root(T))
  while !Empty(S)
      v ← Pop(S)
      Visit(v)
      for all c ∈ Children(v)
          Push(S, c)
```



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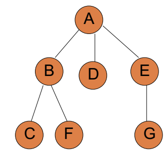
DFS

```
TreeDFS(T)
  Push(S, Root(T))
  while !Empty(S)
      v ← Pop(S)
      Visit(v)
      for all c ∈ Children(v)
          Push(S, c)
```



A E G D B F C

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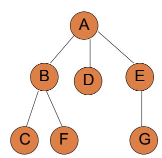
DFS

```
TreeDFS(v)
    visit(v)
    if not leaf(v)
        for all c in children(v)
            TreeDFS(c)
```



What changes?

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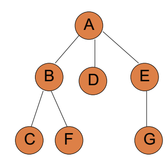
DFS

```
TreeDFS(v)
    visit(v)
    if not leaf(v)
        for all c in children(v)
            TreeDFS(c)
```



A B C F D E G

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Running time of BFS/DFS

Adjacency list
- How many times does it visit each vertex?
- How many times is each edge traversed?
- θ(|V|+|E|) – for trees, assuming a connected graph

Adjacency matrix
- For each vertex visited, how much work is done?
- θ(|V|^2) – for trees, assuming a connected graph

```
TreeBFS(T)
Enqueue(Q, Root(T))
while !Empty(Q)
    v ← Dequeue(Q)
    Visit(v)
    for all c ∈ Children(v)
        Enqueue(Q, c)
```

```
TreeDFS(T)
Push(S, Root(T))
while !Empty(S)
    v ← Pop(S)
    Visit(v)
    for all c ∈ Children(v)
        Push(S, c)
```

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DFS/BFS

Do they visit all of the nodes?

If the graph is connected or strongly connected


```
TreeBFS(T)
Enqueue(Q, Root(T))
while !Empty(Q)
    v ← Dequeue(Q)
    Visit(v)
    for all c ∈ Children(v)
        Enqueue(Q, c)
```

```
TreeDFS(T)
Push(S, Root(T))
while !Empty(S)
    v ← Pop(S)
    Visit(v)
    for all c ∈ Children(v)
        Push(S, c)
```


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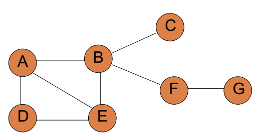
DFS/BFS for graphs

What needs to change for graphs?

Need to make sure we don't visit a node multiple times



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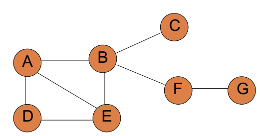
BFS for graphs

What order will BFS visit starting at A (again, assume children are enumerated alphabetically)?



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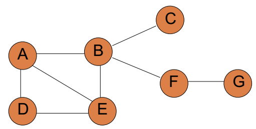
BFS for graphs

What order will BFS visit starting at A (again, assume children are enumerated alphabetically)?

A B D E C F G



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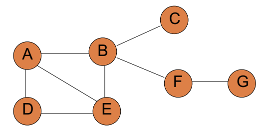
```
BFS(G, s)
for each v ∈ V
        dist[v] = ∞
dist[s] = 0
Enqueue(Q, s)
while !Empty(Q)
        u ← Dequeue(Q)
        Visit(u)
        for each edge (u, v) ∈ E
                if dist[v] = ∞
                        Enqueue(Q, v)
                        dist[v] ← dist[u] + 1
```

distance variable keeps track of how far from the starting node and whether we've seen the node yet



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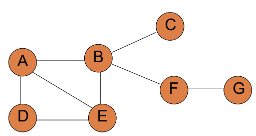
```
BFS(G, s)
for each v ∈ V
        dist[v] = ∞
dist[s] = 0
Enqueue(Q, s)
while !Empty(Q)
        u ← Dequeue(Q)
        Visit(u)
        for each edge (u, v) ∈ E
                if dist[v] = ∞
                        Enqueue(Q, v)
                        dist[v] ← dist[u] + 1
```

```
TreeBFS(T)
Enqueue(Q, Root(T))
while !Empty(Q)
        v ← Dequeue(Q)
        Visit(v)
        for all c ∈ Children(v)
                Enqueue(Q, c)
```



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DFS on graphs

```
DFS(G)
for all v ∈ V
        visited[v] ← false
for all v ∈ V
        if !visited[v]
                DFS-Visit(v)

DFS-Visit(u)
visited[u] ← true
PreVisit(u)
for all edges (u, v) ∈ E
        if !visited[v]
                DFS-Visit(v)
PostVisit(u)
```

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DFS on graphs

```
DFS(G)
for all v ∈ V
    visited[u] ← false
for all v ∈ V
    if !visited[v]
        DFS-Visit(v)

DFS-Visit(u)
visited[u] ← true
PreVisit(u)
for all edges (u, v) ∈ E
    if !visited[v]
        DFS-Visit(v)
PostVisit(u)
```

mark all nodes as not visited

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DFS on graphs

```
DFS(G)
for all v ∈ V
    visited[u] ← false
for all v ∈ V
    if !visited[v]
        DFS-Visit(v)

DFS-Visit(u)
visited[u] ← true
PreVisit(u)
for all edges (u, v) ∈ E
    if !visited[v]
        DFS-Visit(v)
PostVisit(u)
```

until **all** nodes have been visited repeatedly call DFS-Visit

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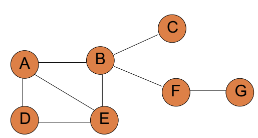
DFS for graphs

What order will DFS visit starting at A (again, assume children are enumerated alphabetically)?

```
DFS(G)
for all v ∈ V
    visited[u] ← false
for all v ∈ V
    if !visited[v]
        DFS-Visit(v)

DFS-Visit(u)
visited[u] ← true
PreVisit(u)
for all edges (u, v) ∈ E
    if !visited[v]
        DFS-Visit(v)
PostVisit(u)
```



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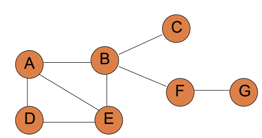
DFS for graphs

What order will DFS visit starting at A (again, assume children are enumerated alphabetically)?

```
DFS(G)
for all v ∈ V
    visited[u] ← false
for all v ∈ V
    if !visited[v]
        DFS-Visit(v)

DFS-Visit(u)
visited[u] ← true
PreVisit(u)
for all edges (u, v) ∈ E
    if !visited[v]
        DFS-Visit(v)
PostVisit(u)
```

A B C E D F G



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What does DFS do?

Finds connected components

Each call to DFS-Visit from DFS starts exploring a new set of connected components

Helps us understand the structure/connectedness of a graph

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Running time of graph BFS/DFS

Nothing changes!

Adjacency list
- $O(|V|+|E|)$

Adjacency matrix
- $O(|V|^2)$

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Connectedness

Given an undirected graph, for every node $u \in V$, can we reach all other nodes in the graph?
Algorithm + running time

Run BFS or DFS-Visit (one pass) and mark nodes as we visit them. If we visit all nodes, return true, otherwise false.

Running time: $O(|V| + |E|)$

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Strongly connected

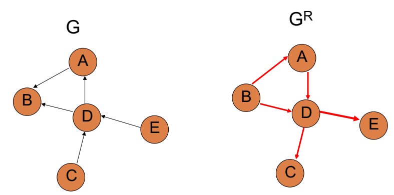
Given a directed graph, can we reach any node v from any other node u?

Can we do the same thing?

80

Transpose of a graph

Given a graph G, we can calculate the transpose of a graph G^R by reversing the direction of all the edges



Running time to calculate G^R? $\theta(|V| + |E|)$

81

Strongly connected

Strongly-Connected(G)
- Run DFS-Visit or BFS from some node u
- If not all nodes are visited: return false
- Create graph G^R
- Run DFS-Visit or BFS on G^R from node u
- If not all nodes are visited: return false
- return true

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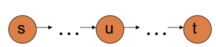
Is it correct?

What do we know after the first pass?
- Starting at u, we can reach every node

What do we know after the second pass?
- All nodes can reach u. Why?
- We can get from u to every node in G^R, therefore, if we reverse the edges (i.e. G), then we have a path from every node to u

Which means that any node can reach any other node. Given any two nodes s and t we can create a path through u



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Runtime?

Strongly-Connected(G)	
- Run DFS-Visit or BFS from some node u	$O(\|V\| + \|E\|)$
- If not all nodes are visited: return false	$O(\|V\|)$
- Create graph G^R	$\theta(\|V\| + \|E\|)$
- Run DFS-Visit or BFS on G^R from node u	$O(\|V\| + \|E\|)$
- If not all nodes are visited: return false	$O(\|V\|)$
- return true	

$O(|V| + |E|)$

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Handout

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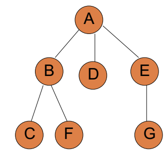
DFS and BFS

How are they implemented?

What would be the result starting at A? If you ask for the children of a node, they're given in alphabetical order.

Run-time (in terms of V and E):
- adjacency list
- adjacency matrix



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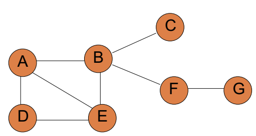
BFS/DFS for graphs

What order will BFS/DFS visit starting at A (again, assume children are enumerated alphabetically)?



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