GREEDY ALGORITHMS

David Kauchak
CS 140 – Fall 2024

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Admin

Assignment 5

Group sessions:
- Thu 4:30-5:30pm (Catherine)
- Thu 6-9pm (Sae) -- show up on the hour for group sessions
- Fri 9:30-10:30am (Taylor)
- Fri 5-6pm (Stanley)

Mentor hours:
- Thu 7-9pm (Sae)
- Sat 10am-12 (Stanley)
- Sat 4-6pm (Taylor)

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A problem

Input: an integer k

Output: integers n_p, n_n, n_d, n_q where
$n_p+5n_n+10n_d+25n_q=k$ and $n_p+n_n+n_d+n_q$ is minimized

What is this problem?
How would you state it in words?

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A problem

Input: an integer k

Output: integers n_p, n_n, n_d, n_q where
$n_p+5n_n+10n_d+25n_q=k$ and $n_p+n_n+n_d+n_q$ is minimized

Provide (U.S.) coins that sum up to *k* such that we minimize the number of coins

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A problem

Input: an integer k

Output: integers n_p, n_n, n_d, n_q where $n_p+5n_n+10n_d+25n_q=k$ and $n_p+n_n+n_d+n_q$ is minimized

Algorithm to solve it?

5

A problem

Input: an integer k

Output: integers n_p, n_n, n_d, n_q where $n_p+5n_n+10n_d+25n_q=k$ and $n_p+n_n+n_d+n_q$ is minimized

$n_q = \lfloor k / 25 \rfloor$ pick as many quarters as we can

Solve:
$n_p + 5n_n + 10n_d = \mathrm{k} - 25\lfloor k / 25 \rfloor$ recurse

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Algorithms vs heuristics

What is the difference between an algorithm and a heuristic?

Algorithm: a set of steps for arriving at the *correct* solution

Heuristic: a set of steps that will arrive at some solution

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Making change!

$n_q = \lfloor k / 25 \rfloor$ pick as many quarters as we can

Solve:
$n_p + 5n_n + 10n_d = \mathrm{k} - 25\lfloor k / 25 \rfloor$ recurse

Algorithm or heuristic?

Need to prove its correct!

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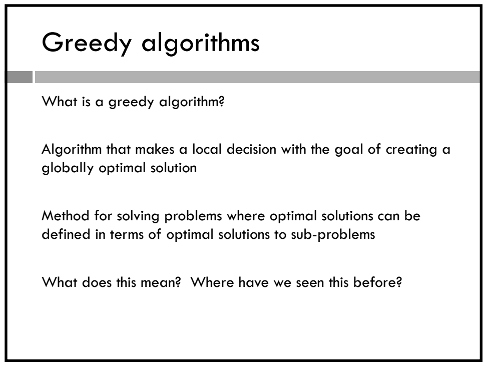

Greedy algorithms

What is a greedy algorithm?

Algorithm that makes a local decision with the goal of creating a globally optimal solution

Method for solving problems where optimal solutions can be defined in terms of optimal solutions to sub-problems

What does this mean? Where have we seen this before?

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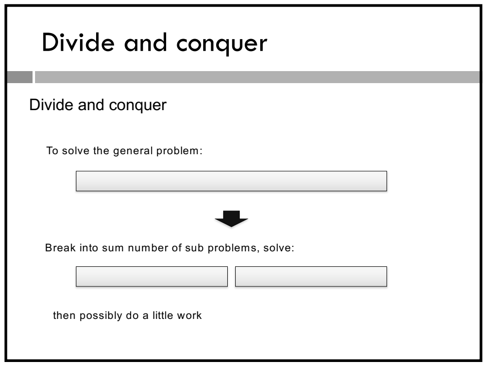

Divide and conquer

Divide and conquer

To solve the general problem:

Break into sum number of sub problems, solve:

then possibly do a little work

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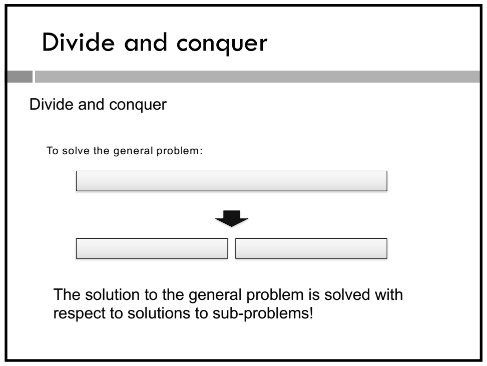

Divide and conquer

Divide and conquer

To solve the general problem:

The solution to the general problem is solved with respect to solutions to sub-problems!

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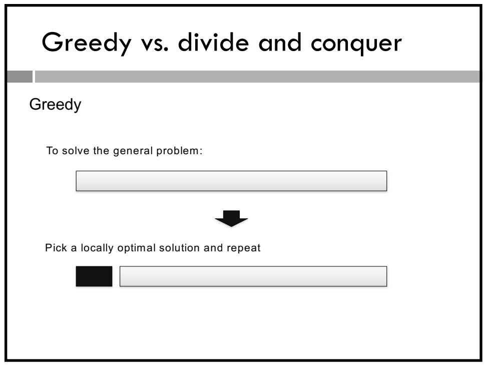

Greedy vs. divide and conquer

Greedy

To solve the general problem:

Pick a locally optimal solution and repeat

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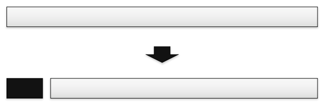
Greedy vs. divide and conquer

Greedy

To solve the general problem:



The solution to the general problem is solved with respect to solutions to sub-problems!

Slightly different than divide and conquer

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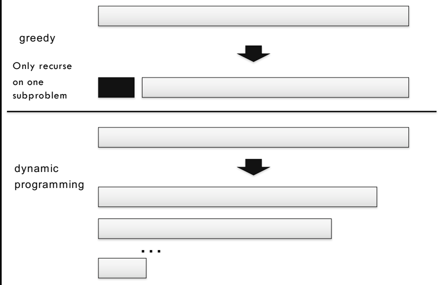
Greedy vs. DP



Need to solve (recurse on) subproblems to figure out optimal answer

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Proving correctness: greedy choice property

Greedy choice property: The greedy choice is contained within some optimal solution

The greedy choice results in an optimal solution

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Making change!

$n_q = \lfloor k / 25 \rfloor$ pick as many quarters as we can

Solve:
$n_p + 5n_n + 10n_d = k - 25\lfloor k / 25 \rfloor$ recurse

$\{c_1, c_2, c_3, \dots, c_m\}$ solution: individual coins selected

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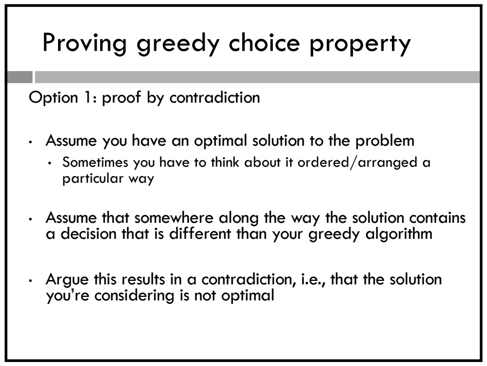

Proving greedy choice property

Option 1: proof by contradiction

- Assume you have an optimal solution to the problem
  - Sometimes you have to think about it ordered/arranged a particular way
- Assume that somewhere along the way the solution contains a decision that is different than your greedy algorithm
- Argue this results in a contradiction, i.e., that the solution you're considering is not optimal

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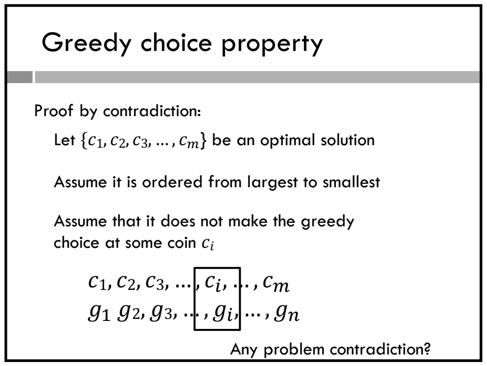

Greedy choice property

Proof by contradiction:

Let $\{c_1, c_2, c_3, \dots, c_m\}$ be an optimal solution

Assume it is ordered from largest to smallest

Assume that it does not make the greedy choice at some coin c_i

$$c_1, c_2, c_3, \dots, \boxed{c_i}, \dots, c_m$$
$$g_1\ g_2, g_3, \dots, \boxed{g_i}, \dots, g_n$$

Any problem contradiction?

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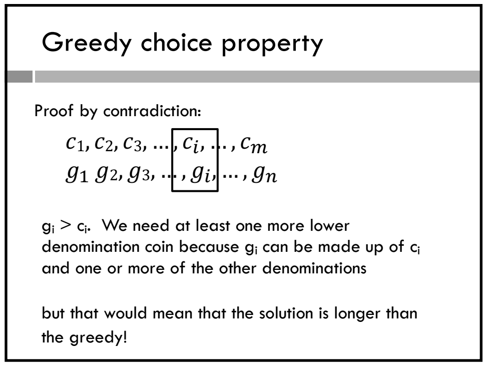

Greedy choice property

Proof by contradiction:

$$c_1, c_2, c_3, \dots, \boxed{c_i}, \dots, c_m$$
$$g_1\ g_2, g_3, \dots, \boxed{g_i}, \dots, g_n$$

$g_i > c_i$. We need at least one more lower denomination coin because g_i can be made up of c_i and one or more of the other denominations

but that would mean that the solution is longer than the greedy!

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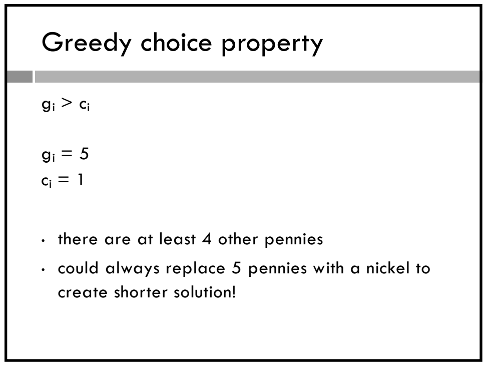

Greedy choice property

$g_i > c_i$

$g_i = 5$
$c_i = 1$

- there are at least 4 other pennies
- could always replace 5 pennies with a nickel to create shorter solution!

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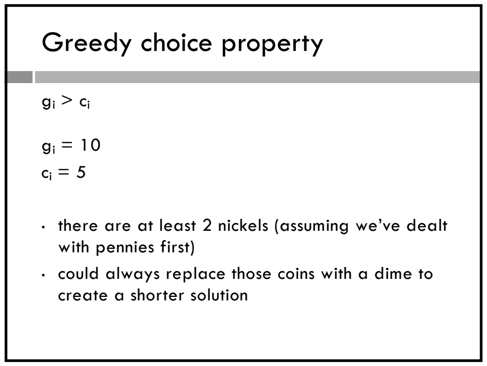

Greedy choice property

$g_i > c_i$

$g_i = 10$
$c_i = 5$

- there are at least 2 nickels (assuming we've dealt with pennies first)
- could always replace those coins with a dime to create a shorter solution

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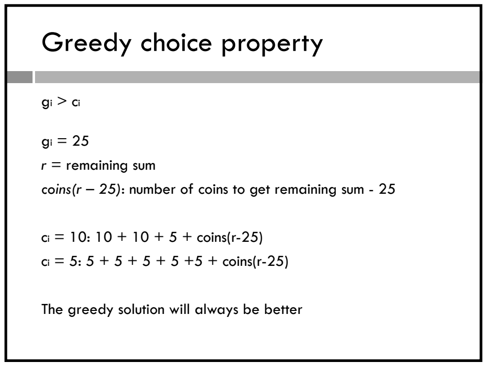

Greedy choice property

$g_i > c_i$

$g_i = 25$
r = remaining sum
coins(r – 25): number of coins to get remaining sum - 25

c_i = 10: 10 + 10 + 5 + coins(r-25)
c_i = 5: 5 + 5 + 5 + 5 +5 + coins(r-25)

The greedy solution will always be better

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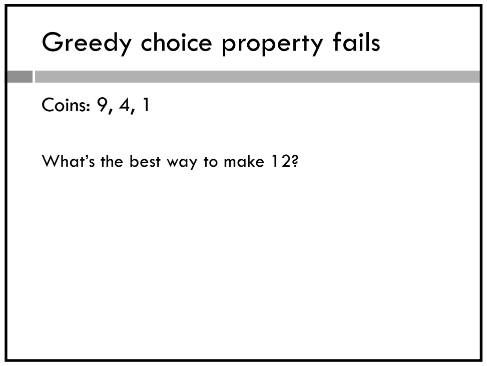

Greedy choice property fails

Coins: 9, 4, 1

What's the best way to make 12?

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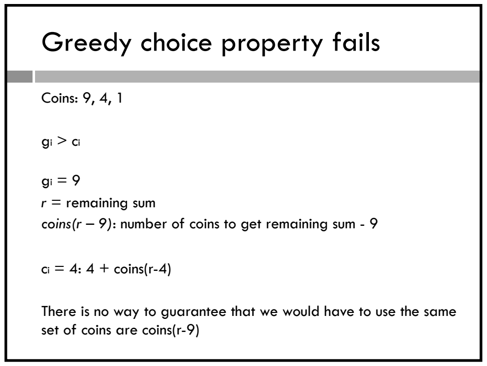

Greedy choice property fails

Coins: 9, 4, 1

$g_i > c_i$

$g_i = 9$
r = remaining sum
coins(r – 9): number of coins to get remaining sum - 9

c_i = 4: 4 + coins(r-4)

There is no way to guarantee that we would have to use the same set of coins are coins(r-9)

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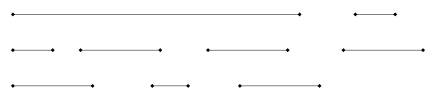
Interval scheduling

Given *n* activities A = [a_1,a_2, .., a_n] where each activity (a_i) has start time s_i and a finish time f_i. Schedule as many as possible of these activities such that they don't conflict.



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Interval scheduling

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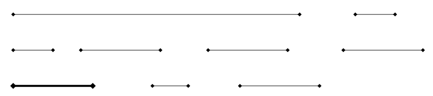

Which activities conflict?

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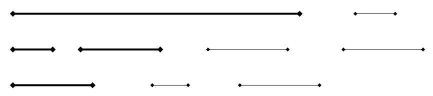
Interval scheduling

Given *n* activities A = [a_1,a_2, .., a_n] where each activity (a_i) has start time s_i and a finish time f_i. Schedule as many as possible of these activities such that they don't conflict.



Which activities conflict?

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Simple recursive solution

Enumerate all possible solutions and find which schedules the most activities

```
IntervalSchedule-Recursive($A$)
if $A = \{\}$
        return 0
else
        $max = -\infty$
        for all $a \in A$
                $A' \leftarrow A$ minus $a$ and all conflicting activites with $a$
                $s =$ IntervalSchedule-Recursive($A'$)
                if $s > max$
                        $max = s$
        return $1 + max$
```

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Simple recursive solution

Is it correct?

- max{all possible solutions}

Running time?

- O(n!)

```
INTERVALSCHEDULE-RECURSIVE($A$)
if $A = \{\}$
        return 0
else
        $max = -\infty$
        for all $a \in A$
                $A' \leftarrow A$ minus $a$ and all conflicting activites with $a$
                $s =$ INTERVALSCHEDULE-RECURSIVE($A'$)
                if $s > max$
                        $max = s$
        return $1 + max$
```

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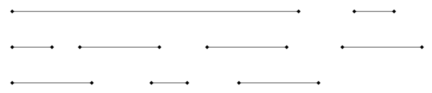
Can we do better?

Dynamic programming

- O(n^2)

Greedy solution – Is there a way to repeatedly make local decisions?

- Key: we'd still like to end up with the *optimal* solution



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Overview of a greedy approach

Greedily pick an activity to schedule

Add that activity to the answer

Remove that activity and all conflicting activities. Call this A'.

Repeat on A' until A' is empty

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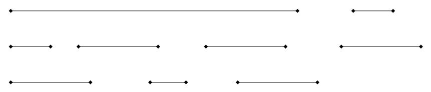
Greedy options



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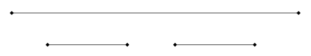
Greedy options

Select the activity that starts the earliest, i.e. argmin$\{s_1, s_2, s_3, \ldots, s_n\}$?



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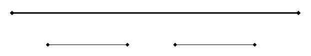
Greedy options

Select the activity that starts the earliest, i.e. argmin$\{s_1, s_2, s_3, \ldots, s_n\}$?



non-optimal

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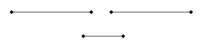
Greedy options

Select the shortest activity, i.e. argmin$\{f_1-s_1, f_2-s_2, f_3-s_3, \ldots, f_n-s_n\}$



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Greedy options

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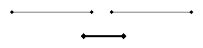

non-optimal

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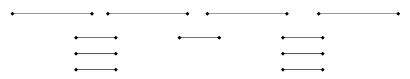
Greedy options

Select the activity with the smallest number of conflicts



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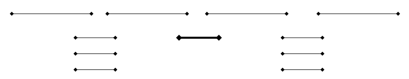
Greedy options

Select the activity with the smallest number of conflicts



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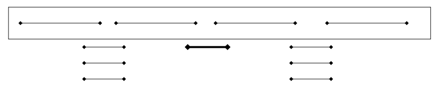
Greedy options

Select the activity with the smallest number of conflicts



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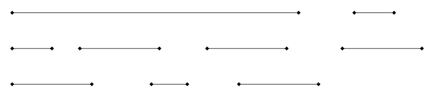
Greedy options

Select the activity that ends the earliest, i.e. argmin{f_1, f_2, f_3, …, f_n}?



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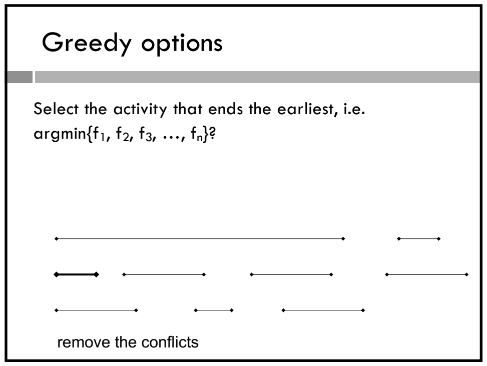

Greedy options

Select the activity that ends the earliest, i.e. argmin$\{f_1, f_2, f_3, \ldots, f_n\}$?

remove the conflicts

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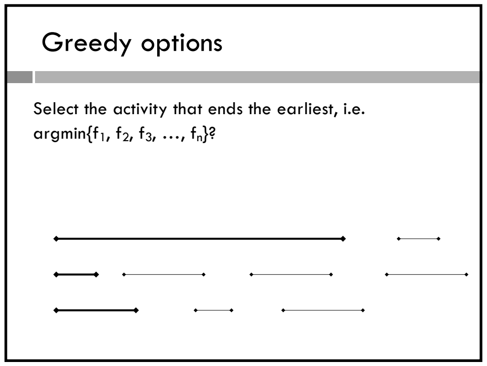

Greedy options

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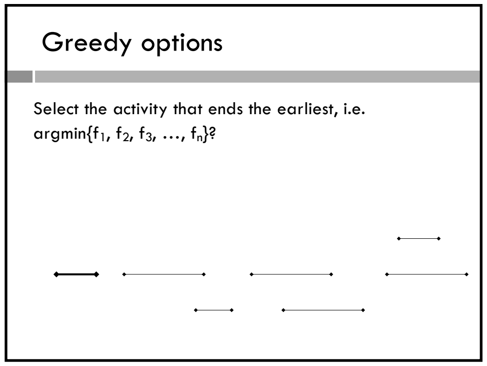

Greedy options

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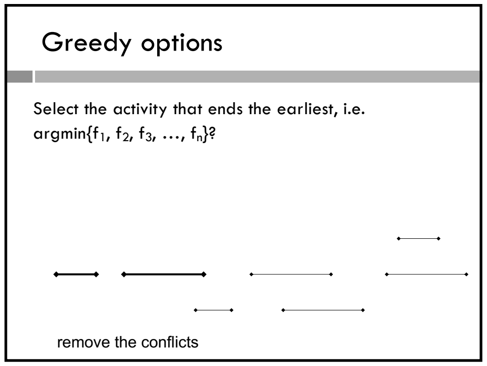
48



Greedy options

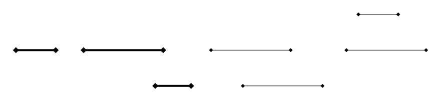
Select the activity that ends the earliest, i.e. argmin$\{f_1, f_2, f_3, \ldots, f_n\}$?

remove the conflicts

49

Greedy options

Select the activity that ends the earliest, i.e. argmin{f_1, f_2, f_3, …, f_n}?



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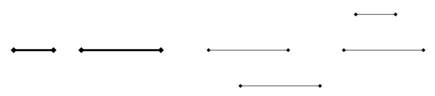
Greedy options

Select the activity that ends the earliest, i.e. argmin{f_1, f_2, f_3, …, f_n}?



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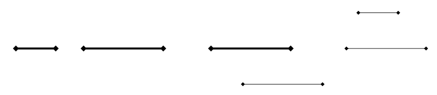
Greedy options

Select the activity that ends the earliest, i.e. argmin{f_1, f_2, f_3, …, f_n}?



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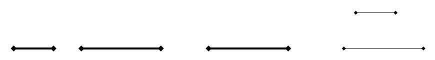
Greedy options

Select the activity that ends the earliest, i.e. argmin{f_1, f_2, f_3, …, f_n}?



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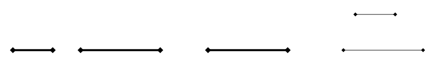
Greedy options

Select the activity that ends the earliest, i.e. argmin{f_1, f_2, f_3, …, f_n}?



Multiple optimal solutions

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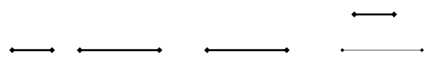
Greedy options

Select the activity that ends the earliest, i.e. argmin{f_1, f_2, f_3, …, f_n}?



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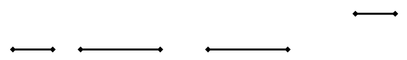
Greedy options

Select the activity that ends the earliest, i.e. argmin{f_1, f_2, f_3, …, f_n}?



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Efficient greedy algorithm

Once you've identified a reasonable greedy heuristic:

- Prove that it always gives the correct answer
- Develop an efficient solution

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Is our greedy approach correct?

Option 1: proof by contradiction

Option 2:"Stays ahead" argument:

show that no matter what other solution someone provides you, the solution provided by your algorithm always "stays ahead", in that no other choice could do better

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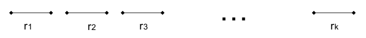
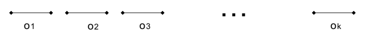
Is our greedy approach correct?

"Stays ahead" argument

Let $r_1, r_2, r_3, \ldots, r_k$ be the solution found by our approach



Let $o_1, o_2, o_3, \ldots, o_k$ be another optimal solution



Show our approach "stays ahead" of any other solution

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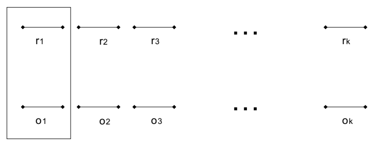
Stays ahead



Compare first activities of each solution

what do we know?

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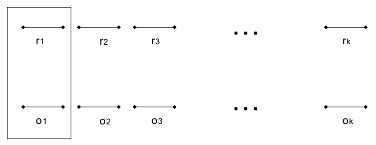
Stays ahead



$\text{finish}(r_1) \leq \text{finish}(o_1)$

what does this imply?

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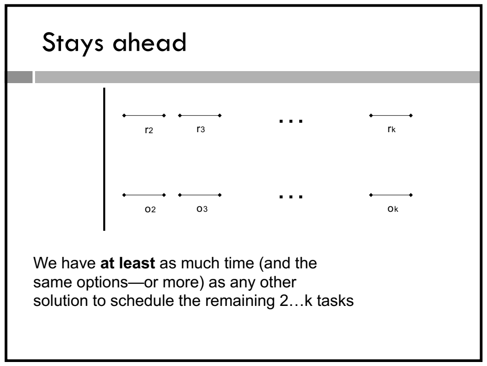
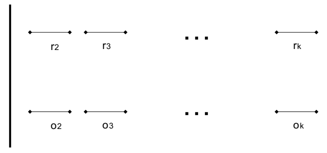


Stays ahead



We have **at least** as much time (and the same options—or more) as any other solution to schedule the remaining 2…k tasks

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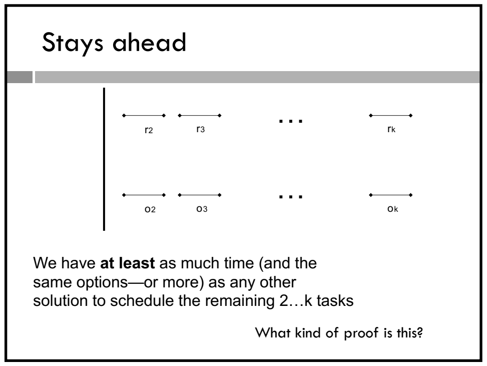
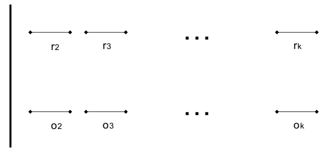


Stays ahead



We have **at least** as much time (and the same options—or more) as any other solution to schedule the remaining 2…k tasks

What kind of proof is this?

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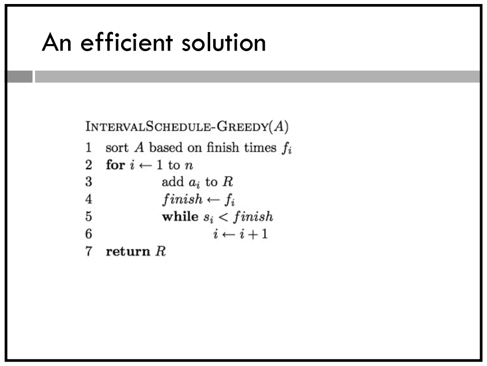

An efficient solution

```
IntervalSchedule-Greedy($A$)
sort $A$ based on finish times $f_i$
for $i \leftarrow 1$ to $n$
        add $a_i$ to $R$
        $finish \leftarrow f_i$
        while $s_i < finish$
                $i \leftarrow i + 1$
return $R$
```

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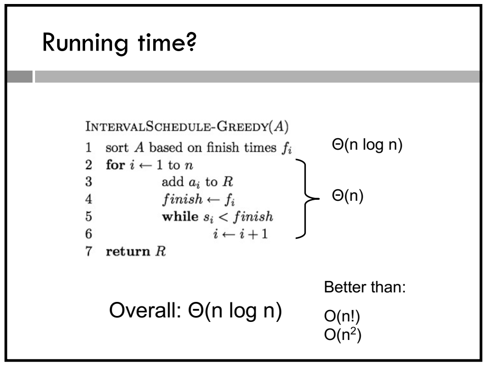

Running time?

```
IntervalSchedule-Greedy($A$)
sort $A$ based on finish times $f_i$        Θ(n log n)
for $i \leftarrow 1$ to $n$                  ⎫
        add $a_i$ to $R$                     ⎪
        $finish \leftarrow f_i$              ⎬ Θ(n)
        while $s_i < finish$                 ⎪
                $i \leftarrow i + 1$         ⎭
return $R$
```

Overall: $\Theta(n \log n)$

Better than:

$O(n!)$
$O(n^2)$

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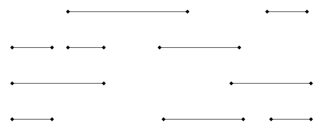
Scheduling *all* intervals

Given *n* activities, we need to schedule **all** activities.
Goal: minimize the number of resources required.



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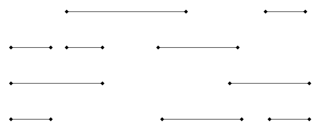
Greedy approach?

The best we could ever do is the maximum number of conflicts for any time period



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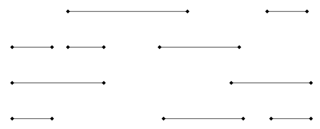
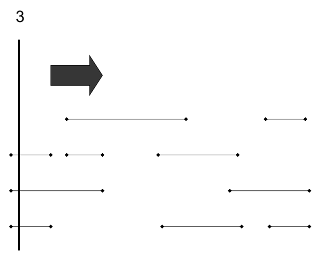
Calculating max conflicts efficiently



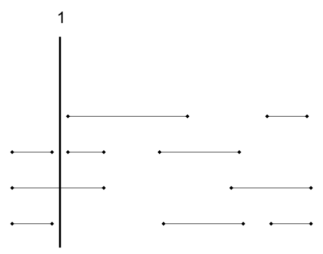
68

Calculating max conflicts efficiently



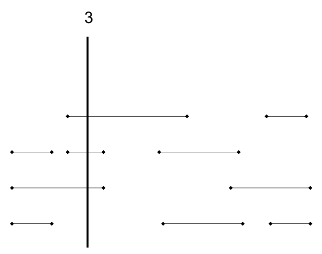
69

Calculating max conflicts efficiently



70

Calculating max conflicts efficiently



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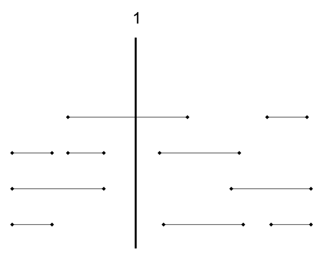
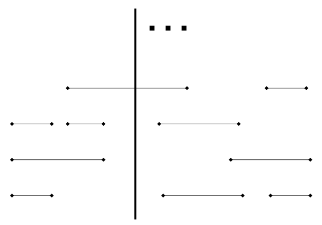
Calculating max conflicts efficiently



72

Calculating max conflicts efficiently



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Calculating max conflicts

```
AllIntervalScheduleCount($A$)
Sort the start and end times, call this $X$
$current \leftarrow 0$
$max \leftarrow 0$
for $i \leftarrow 1$ to $length[X]$
    if $x_i$ is a start node
        $current++$
    else
        $current--$
    if $current > max$
        $max \leftarrow current$
return $max$
```

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Correctness?

We can do no better then the max number of conflicts.
This exactly counts the max number of conflicts.

```
AllIntervalScheduleCount($A$)
Sort the start and end times, call this $X$
$current \leftarrow 0$
$max \leftarrow 0$
for $i \leftarrow 1$ to $length[X]$
    if $x_i$ is a start node
        $current++$
    else
        $current--$
    if $current > max$
        $max \leftarrow current$
return $max$
```

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Runtime?

O(2n log 2n + n) = O(n log n)

```
AllIntervalScheduleCount($A$)
Sort the start and end times, call this $X$
$current \leftarrow 0$
$max \leftarrow 0$
for $i \leftarrow 1$ to $length[X]$
    if $x_i$ is a start node
        $current++$
    else
        $current--$
    if $current > max$
        $max \leftarrow current$
return $max$
```

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Knapsack problems: Greedy or not?

0-1 Knapsack – A thief robbing a store finds n items worth v_1, v_2, .., v_n dollars and weight w_1, w_2, …, w_n pounds, where v_i and w_i are integers. The thief can carry at most W pounds in the knapsack. Which items should the thief take if he wants to maximize value.

Fractional knapsack problem – Same as above, but the thief happens to be at the bulk section of the store and can carry fractional portions of the items. For example, the thief could take 20% of item i for a weight of $0.2w_i$ and a value of $0.2v_i$.

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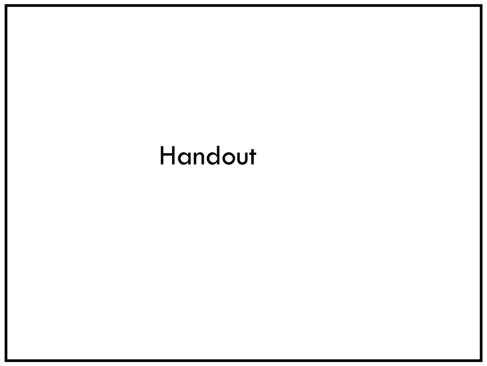

Handout

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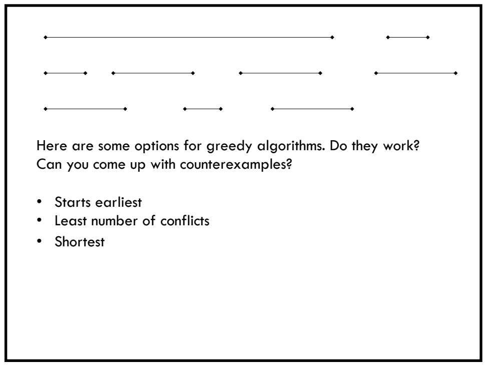

Here are some options for greedy algorithms. Do they work?
Can you come up with counterexamples?

- Starts earliest
- Least number of conflicts
- Shortest

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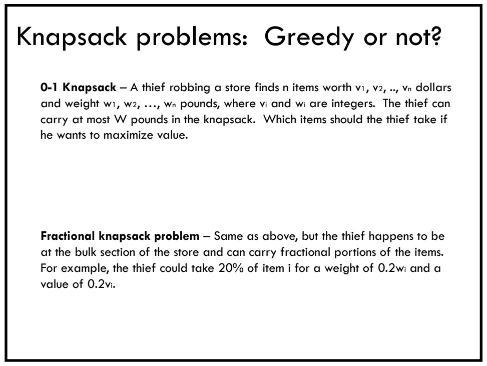

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